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Structural Derivations in Hilbert Algebras by Endomorphisms

Aiyared Iampan[1,*], Neelamegarajan Rajesh[2], C. Arivazhagi[3], R. Vennila[4]

[1] *Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand*

[2] *Department of Mathematics, Rajah Serfoji Government College (affiliated to Bharathidasan University), Thanjavur-613005, Tamilnadu, India*

[3] *Department of Mathematics, Government Arts and Science College, Peravurani-614804, Tamilnadu, India*

[4] *7405 Goreway Drive, Mississauga L4T0A3, Canada*

[*] *Corresponding author: aiyared.ia@up.ac.th*

Abstract. This work introduces the concept of f-derivations in Hilbert algebras, exploring its theoretical foundation alongside a series of illustrative examples. We examine fundamental properties associated with f-derivations through rigorous analysis, shedding light on their algebraic structure and behaviour. In particular, we demonstrate that the kernel $\mathrm{Ker}_{d_f}(A)$ constitutes a near filter (subalgebra), while the fixed set $\mathrm{Fix}_{d_f}(f)$ forms a subalgebra within the Hilbert algebra A. These results provide new insights into the interaction between derivations and substructures in Hilbert algebras, offering potential avenues for further exploration in algebraic logic and related fields.

1. Introduction

The notion of Hilbert algebras was first introduced by Henkin in the early 1950s as a formal framework for studying implications within intuitionistic and other non-classical logics [9]. These algebras provided an algebraic approach to capture the behaviour of implication in systems that deviate from classical logic, thereby offering new tools for understanding logical reasoning in such settings. By the 1960s, the foundational importance of Hilbert algebras was solidified, largely due to the work of Diego [7], who established that Hilbert algebras form a locally finite variety. Diego's results played a pivotal role in embedding these algebras within the broader landscape of algebraic logic, offering a rigorous foundation for their study.

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Building on these early developments, subsequent research expanded the algebraic and logical dimensions of Hilbert algebras. Busneag [4,5] and Jun [13] explored the role of filters in these algebras, demonstrating that they form deductive systems and play a crucial role in the logical structure of Hilbert algebras. Their contributions revealed deeper connections between the algebraic properties of these systems and their logical interpretations. Dudek [8] also expanded the use of Hilbert algebras by adding the idea of fuzzification and studying how subalgebras and deductive systems work in fuzzy logic settings. This approach enriched the study of Hilbert algebras, broadening their relevance to areas dealing with uncertainty and graded truth values.

The study of derivations has seen significant advancements in recent years, with notable contributions across various algebraic structures. In 2021, Muangkarn et al. [16] investigated f_q-derivations, while Bantaojai et al. [3] examined derivations induced by endomorphisms in B-algebras, revealing new perspectives on the interaction between derivations and algebraic morphisms. Building on this work, in 2022, Bantaojai et al. [1,2] extended their study to derivations on d-algebras and B-algebras, further enriching the theoretical landscape. During the same period, Muangkarn et al. [15,17] focused on derivations induced by endomorphisms in BG-algebras and d-algebras, offering new insights into the structural implications of these operations. Additionally, Iampan et al. [10,18,19] contributed to this growing body of research by exploring derivations on UP-algebras, highlighting the versatility of derivation theory across diverse algebraic systems. These studies collectively enhance our understanding of the role of derivations in algebraic structures, providing a foundation for future explorations in both classical and non-classical algebraic settings.

This paper delves into the introduction and exploration of (l,r)-f-derivations, (r,l)-f-derivations, and f-derivations within the framework of Hilbert algebras. We investigate the fundamental properties of these derivations, providing new insights into their structural roles and interactions. In particular, we demonstrate that the kernel of an f-derivation, denoted $\mathrm{Ker}_{d_f}(A)$, constitutes a near filter, effectively functioning as a subalgebra of the Hilbert algebra A. Additionally, we show that the fixed set $\mathrm{Fix}_{d_f}(f)$ similarly forms a subalgebra. These findings not only enhance the understanding of the algebraic structure of Hilbert algebras but also open new directions for further research in the field of algebraic logic and its applications.

2. Preliminaries

Let's review the idea of Hilbert algebras as Diego first introduced it in 1966 [7] before we start.

Definition 2.1. *[7] A Hilbert algebra is a triplet with the formula $A = (A, \cdot, 1)$, where A is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of A that is true according to the axioms stated below:*

$$(\forall x, y \in A)(x \cdot (y \cdot x) = 1) \tag{2.1}$$

$$(\forall x, y, z \in A)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1) \tag{2.2}$$

$$(\forall x, y \in A)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y) \tag{2.3}$$

In [8], the following conclusion was established.

Lemma 2.1. *Let $A = (A, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in A)(x \cdot x = 1)$,
(2) $(\forall x \in A)(1 \cdot x = x)$,
(3) $(\forall x \in A)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$,
(5) $(\forall x, y, z \in A)((x \cdot z) \cdot ((z \cdot y) \cdot (x \cdot y)) = 1)$.

In a Hilbert algebra $A = (A, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on A with 1 as the largest element.

Definition 2.2. *[14] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called a subalgebra of A if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 2.3. *[6] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called an ideal of A if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in A)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in A)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

Definition 2.4. *[11] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called a near filter of A if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in A)(y \in D \Rightarrow x \cdot y \in D)$.

Definition 2.5. *[11] A nonempty subset D of a Hilbert algebra $A = (A, \cdot, 1)$ is called a filter of A if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in A)(x \cdot y, x \in D \Rightarrow y \in D)$.

Definition 2.6. *Let $A = (A, \cdot, 1_A)$ and $B = (B, \star, 1_B)$ be Hilbert algebras. A function $f : A \to B$ is called a homomorphism if $f(x \cdot y) = f(x) \star f(y)$ for all $x, y \in A$. Now, $f(1_A) = f(1_A \cdot 1_A) = f(1_A) \star f(1_A) = 1_B$. A homomorphism $f : A \to A$ is said to be an endomorphism.*

For any x, y in a Hilbert algebra $A = (A, \cdot, 1)$, we define $x \vee y$ by $(y \cdot x) \cdot x$. By Lemma 2.1 (4), we can prove that $x \vee y$ is an upper bound of x and y. That is,

$$(\forall x, y \in A)(x \cdot (x \vee y) = 1), \tag{2.4}$$

$$(\forall x, y \in A)(y \cdot (x \vee y) = 1). \tag{2.5}$$

A Hilbert algebra $A = (A, \cdot, 1)$ is said to be $\vee$-*commutative* [12] if for all $x, y \in A$, $(y \cdot x) \cdot x = (x \cdot y) \cdot y$, that is, $x \vee y = y \vee x$. From [12], we know that

$$(\forall x \in A)(x \vee x = x), \tag{2.6}$$

$$(\forall x \in A)(x \vee 1 = 1 \vee x = 1). \tag{2.7}$$

3. Main Results

In this section, we introduce and explore the concepts of (l, r)-f-derivations, (r, l)-f-derivations, and general f-derivations within the framework of Hilbert algebras. We examine their foundational properties, establishing a theoretical basis for their further study. Additionally, we define two significant subsets associated with an f-derivation d_f of a Hilbert algebra A: the kernel, $\text{Ker}_{d_f}(A)$, and the fixed set, $\text{Fix}_{d_f}(f)$. We then investigate the key properties of these subsets, highlighting their roles within the broader algebraic structure. The insights gained from these analyses contribute to a deeper understanding of the interaction between derivations and substructures in Hilbert algebras.

Definition 3.1. *Let f be an endomorphism of a Hilbert algebra $A = (A, \cdot, 1)$. A self-map $d_f : A \to A$ is called an (l, r)-f-derivation of A if it satisfies the identity*

$$(\forall x, y \in A)(d_f(x \cdot y) = (d_f(x) \cdot f(y)) \vee (f(x) \cdot d_f(y))).$$

Similarly, a self-map $d_f : A \to A$ is called an (r, l)-f-derivation of A if it satisfies the identity

$$(\forall x, y \in A)(d_f(x \cdot y) = (f(x) \cdot d_f(y)) \vee (d_f(x) \cdot f(y))).$$

Moreover, if d_f is both an (l, r)-f-derivation and an (r, l)-f-derivation of A, it is called an f-derivation of A.

Example 3.1. *Let $A = \{1, 2, 3, 4\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	2	3	1

Then $(A, \cdot, 1)$ is a Hilbert algebra. We define an endomorphism f on A as follows:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 4 \end{pmatrix}$$

Define a self-map $d_f : A \to A$ as follows:

$$d_f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 1 \end{pmatrix}$$

Hence, d_f is an f-derivation of A.

Definition 3.2. *An (l, r)-f-derivation (resp., (r, l)-f-derivation, f-derivation) d_f of a Hilbert algebra $A = (A, \cdot, 1)$ is called regular if $d_f(1) = 1$.*

Theorem 3.1. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) every (l, r)-f-derivation of A is regular,

(2) every (r, l)-f-derivation of A is regular.

Proof. (1) Assume that d_f is an (l, r)-f-derivation of A. Then

$$\begin{aligned} d_f(1) &= d_f(1 \cdot 1) && \text{Lemma 2.1 (1)} \\ &= (d_f(1) \cdot f(1)) \vee (f(1) \cdot d_f(1)) \\ &= (d_f(1) \cdot 1) \vee (1 \cdot d_f(1)) \\ &= 1 \vee d_f(1) && \text{Lemma 2.1 (2) and (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, d_f is regular.

(2) Assume that d_f is an (r, l)-f-derivation of A. Then

$$\begin{aligned} d_f(1) &= d_f(1 \cdot 1) && \text{Lemma 2.1 (1)} \\ &= (f(1) \cdot d_f(1)) \vee (d_f(1) \cdot f(1)) \\ &= (1 \cdot d_f(1)) \vee (d_f(1) \cdot 1) \\ &= d_f(1) \vee 1 && \text{Lemma 2.1 (2) and (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, d_f is regular. □

Corollary 3.1. *Every f-derivation of a Hilbert algebra A is regular.*

Theorem 3.2. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l, r)-f-derivation of A, then $d_f(x) = f(x) \vee d_f(x)$ for all $x \in A$,

(2) if d_f is an (r, l)-f-derivation of A, then $d_f(x) = d_f(x) \vee f(x)$ for all $x \in A$.

Proof. (1) Assume that d_f is an (l, r)-f-derivation of A. Then, for all $x \in A$,

$$\begin{aligned} d_f(x) &= d_f(1 \cdot x) && \text{Lemma 2.1 (2)} \\ &= (d_f(1) \cdot f(x)) \vee (f(1) \cdot d_f(x)) \\ &= (1 \cdot f(x)) \vee (1 \cdot d_f(x)) && \text{regular} \\ &= f(x) \vee d_f(x). && \text{Lemma 2.1 (2)} \end{aligned}$$

(2) Assume that d_f is an (r, l)-f-derivation of A. Then, for all $x \in A$,

$$\begin{aligned} d_f(x) &= d_f(1 \cdot x) && \text{Lemma 2.1 (2)} \\ &= (f(1) \cdot d_f(x)) \vee (d_f(1) \cdot f(x)) \\ &= (1 \cdot d_f(x)) \vee (1 \cdot f(x)) && \text{regular} \\ &= d_f(x) \vee f(x). && \text{Lemma 2.1 (2)} \end{aligned}$$

□

Corollary 3.2. *If d_f is an f-derivation of a Hilbert algebra A, then $d_f(x) \vee f(x) = f(x) \vee d_f(x)$ for all $x \in A$.*

Proposition 3.1. *Let d_f be an (l,r)-f-derivation of a Hilbert algebra $A = (A, \cdot, 1)$. Then the following properties hold: for any $x, y \in A$,*

(1) $f(x) \leq d_f(x)$,
(2) $d_f(x) \cdot f(y) \leq d_f(x \cdot y)$,
(3) $f(x) \cdot d_f(y) \leq d_f(x \cdot y)$,
(4) if $f(d_f(x)) = d_f(x)$ or $d_f(d_f(x)) = f(x)$, then $d_f(x \cdot d_f(x)) = 1$,
(5) if $f(d_f(x)) = d_f(x)$ or $d_f(d_f(x)) = f(x)$, then $d_f(d_f(x) \cdot x) = 1$,
(6) if $f(f(x)) = d_f(x)$ or $d_f(f(x)) = f(x)$, then $d_f(f(x) \cdot x) = 1$,
(7) if $f(f(x)) = d_f(x)$ or $d_f(f(x)) = f(x)$, then $d_f(f(x) \cdot x) = 1$.

Proof. (1) For all $x \in A$,

$$\begin{aligned} f(x) \cdot d_f(x) &= f(x) \cdot (f(x) \vee d_f(x)) && \text{Theorem 3.2 (1)} \\ &= 1. && (2.4) \end{aligned}$$

Hence, $f(x) \leq d_f(x)$ for all $x \in A$.

(2) For all $x, y \in A$,

$$\begin{aligned} (d_f(x) \cdot f(y)) \cdot d_f(x \cdot y) &= (d_f(x) \cdot f(y)) \cdot ((d_f(x) \cdot f(y)) \vee (f(x) \cdot d_f(y))) \\ &= 1. && (2.4) \end{aligned}$$

Hence, $d_f(x) \cdot f(y) \leq d_f(x \cdot y)$ for all $x, y \in A$.

(3) For all $x, y \in A$,

$$\begin{aligned} (f(x) \cdot d_f(y)) \cdot d_f(x \cdot y) &= (f(x) \cdot d_f(y)) \cdot ((d_f(x) \cdot f(y)) \vee (f(x) \cdot d_f(y))) \\ &= 1. && (2.5) \end{aligned}$$

Hence, $f(x) \cdot d_f(y) \leq d_f(x \cdot y)$ for all $x, y \in A$.

(4) For all $x \in A$, if $f(d_f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(x \cdot d_f(x)) &= (d_f(x) \cdot f(d_f(x))) \vee (f(x) \cdot d_f(d_f(x))) \\ &= (d_f(x) \cdot d_f(x)) \vee (f(x) \cdot d_f(d_f(x))) \\ &= 1 \vee (f(x) \cdot d_f(d_f(x))) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(d_f(x)) = f(x)$, then

$$\begin{aligned} d_f(x \cdot d_f(x)) &= (d_f(x) \cdot f(d_f(x))) \vee (f(x) \cdot d_f(d_f(x))) \\ &= (d_f(x) \cdot f(d_f(x))) \vee (f(x) \cdot f(x)) \\ &= (d_f(x) \cdot f(d_f(x))) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

(5) For all $x \in A$, if $f(d_f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(d_f(x) \cdot x) &= (d_f(d_f(x)) \cdot f(x)) \vee (f(d_f(x)) \cdot d_f(x)) \\ &= (d_f(d_f(x)) \cdot f(x)) \vee (d_f(x) \cdot d_f(x)) \\ &= (d_f(d_f(x)) \cdot f(x)) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(d_f(x)) = f(x)$, then

$$\begin{aligned} d_f(d_f(x) \cdot x) &= (d_f(d_f(x)) \cdot f(x)) \vee (f(d_f(x)) \cdot d_f(x)) \\ &= (f(x) \cdot f(x)) \vee (f(d_f(x)) \cdot d_f(x)) \\ &= 1 \vee (f(d_f(x)) \cdot d_f(x)) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

(6) For all $x \in A$, if $f(f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(x \cdot f(x)) &= (d_f(x) \cdot f(f(x))) \vee (f(x) \cdot d_f(f(x))) \\ &= (d_f(x) \cdot d_f(x)) \vee (f(x) \cdot d_f(f(x))) \\ &= 1 \vee (f(x) \cdot d_f(f(x))) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(f(x)) = f(x)$, then

$$\begin{aligned} d_f(x \cdot f(x)) &= (d_f(x) \cdot f(f(x))) \vee (f(x) \cdot d_f(f(x))) \\ &= (d_f(x) \cdot f(f(x))) \vee (f(x) \cdot f(x)) \\ &= (d_f(x) \cdot f(f(x))) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

(7) For all $x \in A$, if $f(f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(f(x) \cdot x) &= (d_f(f(x)) \cdot f(x)) \vee (f(f(x)) \cdot d_f(x)) \\ &= (d_f(f(x)) \cdot f(x)) \vee (d_f(x) \cdot d_f(x)) \\ &= (d_f(f(x)) \cdot f(x)) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(f(x)) = f(x)$, then

$$\begin{aligned} d_f(f(x) \cdot x) &= (d_f(f(x)) \cdot f(x)) \vee (f(f(x)) \cdot d_f(x)) \\ &= (f(x) \cdot f(x)) \vee (f(f(x)) \cdot d_f(x)) \\ &= 1 \vee (f(f(x)) \cdot d_f(x)) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

□

Proposition 3.2. *Let d_f be an (r,l)-f-derivation of a Hilbert algebra $A = (A, \cdot, 1)$. Then the following properties hold: for any $x, y \in A$,*

(1) $f(x) \leq d_f(x)$,
(2) $f(x) \cdot d_f(y) \leq d_f(x \cdot y)$,
(3) $d_f(x) \cdot f(y) \leq d_f(x \cdot y)$,
(4) if $f(d_f(x)) = d_f(x)$ or $d_f(d_f(x)) = f(x)$, then $d_f(x \cdot d_f(x)) = 1$,
(5) if $f(d_f(x)) = d_f(x)$ or $d_f(d_f(x)) = f(x)$, then $d_f(d_f(x) \cdot x) = 1$,
(6) if $f(f(x)) = d_f(x)$ or $d_f(f(x)) = f(x)$, then $d_f(x \cdot f(x)) = 1$,
(7) if $f(f(x)) = d_f(x)$ or $d_f(f(x)) = f(x)$, then $d_f(f(x) \cdot x) = 1$.

Proof. (1) For all $x \in A$,

$$\begin{aligned} f(x) \cdot d_f(x) &= f(x) \cdot (d_f(x) \vee f(x)) && \text{Theorem 3.2 (2)} \\ &= 1. && (2.5) \end{aligned}$$

Hence, $f(x) \leq d_f(x)$ for all $x \in A$.

(2) For all $x, y \in A$,

$$\begin{aligned} (f(x) \cdot d_f(y)) \cdot d_f(x \cdot y) &= (f(x) \cdot d_f(y)) \cdot ((f(x) \cdot d_f(y)) \vee (d_f(x) \cdot f(y))) \\ &= 1. && (2.4) \end{aligned}$$

Hence, $f(x) \cdot d_f(y) \leq d_f(x \cdot y)$ for all $x, y \in A$.

(3) For all $x, y \in A$,

$$\begin{aligned} (d_f(x) \cdot f(y)) \cdot d_f(x \cdot y) &= (d_f(x) \cdot f(y)) \cdot ((f(x) \cdot d_f(y)) \vee (d_f(x) \cdot f(y))) \\ &= 1. && (2.5) \end{aligned}$$

Hence, $d_f(x) \cdot f(y) \leq d_f(x \cdot y)$ for all $x, y \in A$.

(4) For all $x \in A$, if $f(d_f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(x \cdot d_f(x)) &= (f(x) \cdot d_f(d_f(x))) \vee (d_f(x) \cdot f(d_f(x))) \\ &= (f(x) \cdot d_f(d_f(x))) \vee (d_f(x) \cdot d_f(x)) \\ &= (f(x) \cdot d_f(d_f(x))) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(d_f(x)) = f(x)$, then

$$\begin{aligned} d_f(x \cdot d_f(x)) &= (f(x) \cdot d_f(d_f(x))) \vee (d_f(x) \cdot f(d_f(x))) \\ &= (f(x) \cdot f(x)) \vee (d_f(x) \cdot f(d_f(x))) \\ &= 1 \vee (d_f(x) \cdot f(d_f(x))) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

(5) For all $x \in A$, if $f(d_f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(d_f(x) \cdot x) &= (f(d_f(x)) \cdot d_f(x)) \vee (d_f(d_f(x)) \cdot f(x)) \\ &= (d_f(x) \cdot d_f(x)) \vee (d_f(d_f(x)) \cdot f(x)) \\ &= 1 \vee (d_f(d_f(x)) \cdot f(x)) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(d_f(x)) = f(x)$, then

$$\begin{aligned} d_f(d_f(x) \cdot x) &= (f(d_f(x)) \cdot d_f(x)) \vee (d_f(d_f(x)) \cdot f(x)) \\ &= (f(d_f(x)) \cdot d_f(x)) \vee (f(x) \cdot f(x)) \\ &= (f(d_f(x)) \cdot d_f(x)) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

(6) For all $x \in A$, if $f(f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(x \cdot f(x)) &= (f(x) \cdot d_f(f(x))) \vee (d_f(x) \cdot f(f(x))) \\ &= (f(x) \cdot d_f(f(x))) \vee (d_f(x) \cdot d_f(x)) \\ &= (f(x) \cdot d_f(f(x))) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(f(x)) = f(x)$, then

$$\begin{aligned} d_f(x \cdot f(x)) &= (f(x) \cdot d_f(f(x))) \vee (d_f(x) \cdot f(f(x)) \\ &= (f(x) \cdot f(x)) \vee (d_f(x) \cdot f(f(x))) \\ &= 1 \vee (d_f(x) \cdot f(f(x))) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

(7) For all $x \in A$, if $f(f(x)) = d_f(x)$, then

$$\begin{aligned} d_f(f(x) \cdot x) &= (f(f(x)) \cdot d_f(x)) \vee (d_f(f(x)) \cdot f(x)) \\ &= (d_f(x) \cdot d_f(x)) \vee (d_f(f(x)) \cdot f(x)) \\ &= 1 \vee (d_f(f(x)) \cdot f(x)) && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

For all $x \in A$, if $d_f(f(x)) = f(x)$, then

$$\begin{aligned} d_f(f(x)\cdot x) &= (f(f(x))\cdot d_f(x)) \vee (d_f(f(x))\cdot f(x)) \\ &= (f(f(x))\cdot d_f(x)) \vee (f(x)\cdot f(x)) \\ &= (f(f(x))\cdot d_f(x)) \vee 1 && \text{Lemma 2.1 (1)} \\ &= 1. && (2.7) \end{aligned}$$

□

Definition 3.3. *A non-empty subset B of a Hilbert algebra $A = (A, \cdot, 1)$ is called f-invariant (with respect to an (l,r)-f-derivation (resp., (r,l)-f-derivation, f-derivation)) d_f of A if $d_f(B) \subseteq B$.*

Theorem 3.3. *Every filter of a Hilbert algebra $A = (A, \cdot, 1)$ containing the endomorphic image of f is f-invariant with respect to any (l,r)-f-derivation of A.*

Proof. Let B be a filter of A. Let $y \in d_f(B)$. Then $y = d_f(x)$ for some $x \in B$. It follows from Proposition 3.1 (1) that $f(x)\cdot y = f(x)\cdot d_f(x) = 1 \in B$. Since $f(A) \subseteq B$, we have $f(x) \in B$. Since B is a filter of A, we have $y \in B$. Thus, $d_f(B) \subseteq B$. Hence, B is f-invariant. □

Definition 3.4. *Let d_f be an (l,r)-f-derivation (resp., (r,l)-f-derivation, f-derivation) of a Hilbert algebra $A = (A, \cdot, 1)$. We define the kernel $\mathrm{Ker}_{d_f}(A)$ of A as follows:*

$$\mathrm{Ker}_{d_f}(A) = \{x \in A : d_f(x) = 1\}$$

Example 3.2. *Let $A = \{1, 2, 3, 4\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

Then $(A, \cdot, 1)$ is a Hilbert algebra. We define an endomorphism f on A as follows:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 1 & 4 \end{pmatrix}$$

Define a self-map $d_f : A \to A$ as follows:

$$d_f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

Hence, d_f is an f-derivation of A and so $\mathrm{Ker}_{d_f}(A) = \{1, 3\}$.

Theorem 3.4. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l,r)-f-derivation of A, then $y \vee x \in \mathrm{Ker}_{d_f}(A)$ for all $y \in \mathrm{Ker}_{d_f}(A)$ and $x \in A$,
(2) if d_f is an (r,l)-f-derivation of A, then $y \vee x \in \mathrm{Ker}_{d_f}(A)$ for all $y \in \mathrm{Ker}_{d_f}(A)$ and $x \in A$.

Proof. (1) Assume that d_f is an (l,r)-f-derivation of A. Let $y \in \mathrm{Ker}_{d_f}(A)$ and $x \in A$. Then $d_f(y) = 1$. Thus,

$$\begin{aligned} d_f(y \vee x) &= d_f((x \cdot y) \cdot y) \\ &= (d_f(x \cdot y) \cdot f(y)) \vee (f(x \cdot y) \cdot d_f(y)) \\ &= (d_f(x \cdot y) \cdot f(y)) \vee (f(x \cdot y) \cdot 1) \\ &= (d_f(x \cdot y) \cdot f(y)) \vee 1 && \text{Lemma 2.1 (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, $y \vee x \in \mathrm{Ker}_{d_f}(A)$.

(2) Assume that d_f is an (r,l)-f-derivation of A. Let $y \in \mathrm{Ker}_{d_f}(A)$ and $x \in A$. Then $d_f(y) = 1$. Thus,

$$\begin{aligned} d_f(y \vee x) &= d_f((x \cdot y) \cdot y) \\ &= (f(x \cdot y) \cdot d_f(y)) \vee (d_f(x \cdot y) \cdot f(y)) \\ &= (f(x \cdot y) \cdot 1) \vee (d_f(x \cdot y) \cdot f(y)) \\ &= 1 \vee (d_f(x \cdot y) \cdot f(y)) && \text{Lemma 2.1 (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, $y \vee x \in \mathrm{Ker}_{d_f}(A)$. □

Corollary 3.3. *If d_f is an f-derivation of a Hilbert algebra A, then $y \vee x \in \mathrm{Ker}_{d_f}(A)$ for all $y \in \mathrm{Ker}_{d_f}(A)$ and $x \in A$.*

Theorem 3.5. *In a $\vee$-commutative Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l,r)-f-derivation of A and for any $x, y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_{d_f}(A)$, then $x \in \mathrm{Ker}_{d_f}(A)$,

(2) if d_f is an (r,l)-f-derivation of A and for any $x, y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_{d_f}(A)$, then $x \in \mathrm{Ker}_{d_f}(A)$.

Proof. (1) Assume that d_f is an (l,r)-f-derivation of A. Let $x, y \in A$ be such that $y \leq x$ and $y \in \mathrm{Ker}_{d_f}(A)$. Then $y \cdot x = 1$ and $d_f(y) = 1$. Thus,

$$\begin{aligned} d_f(x) &= d_f(1 \cdot x) && \text{Lemma 2.1 (2)} \\ &= d_f((y \cdot x) \cdot x) \\ &= d_f((x \cdot y) \cdot y) && \vee\text{-commutative} \\ &= (d_f(x \cdot y) \cdot f(y)) \vee (f(x \cdot y) \cdot d_f(y)) \\ &= (d_f(x \cdot y) \cdot f(y)) \vee (f(x \cdot y) \cdot 1) \\ &= (d_f(x \cdot y) \cdot f(y)) \vee 1 && \text{Lemma 2.1 (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, $x \in \mathrm{Ker}_{d_f}(A)$.

(2) Assume that d_f is an (r,l)-f-derivation of A. Let $x, y \in A$ be such that $y \leq x$ and $y \in \mathrm{Ker}_{d_f}(A)$. Then $y \cdot x = 1$ and $d_f(y) = 1$. Thus,

$$\begin{aligned}
d_f(x) &= d_f(1 \cdot x) && \text{Lemma 2.1 (2)} \\
&= d_f((y \cdot x) \cdot x) \\
&= d_f((x \cdot y) \cdot y) && \vee\text{-commutative} \\
&= (f(x \cdot y) \cdot d_f(y)) \vee (d_f(x \cdot y) \cdot f(y)) \\
&= (f(x \cdot y) \cdot 1) \vee (d_f(x \cdot y) \cdot f(y)) \\
&= 1 \vee (d_f(x \cdot y) \cdot f(y)) && \text{Lemma 2.1 (3)} \\
&= 1. && (2.7)
\end{aligned}$$

Hence, $x \in \mathrm{Ker}_{d_f}(A)$. □

Corollary 3.4. *If d_f is an f-derivation of a $\vee$-commutative Hilbert algebra A and for any $x, y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_{d_f}(A)$, then $x \in \mathrm{Ker}_{d_f}(A)$.*

Theorem 3.6. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l,r)-f-derivation of A, then $y \cdot x \in \mathrm{Ker}_{d_f}(A)$ for all $x \in \mathrm{Ker}_{d_f}(A)$ and $y \in A$,
(2) if d_f is an (r,l)-f-derivation of A, then $y \cdot x \in \mathrm{Ker}_{d_f}(A)$ for all $x \in \mathrm{Ker}_{d_f}(A)$ and $y \in A$.

Proof. (1) Assume that d_f is an (l,r)-f-derivation of A. Let $x \in \mathrm{Ker}_{d_f}(A)$ and $y \in A$. Then $d_f(x) = 1$. Thus,

$$\begin{aligned}
d_f(y \cdot x) &= (d_f(y) \cdot f(x)) \vee (f(y) \cdot d_f(x)) \\
&= (d_f(y) \cdot f(x)) \vee (f(y) \cdot 1) \\
&= (d_f(y) \cdot f(x)) \vee 1 && \text{Lemma 2.1 (3)} \\
&= 1. && (2.7)
\end{aligned}$$

Hence, $y \cdot x \in \mathrm{Ker}_{d_f}(A)$.

(2) Assume that d_f is an (r,l)-f-derivation of A. Let $x \in \mathrm{Ker}_{d_f}(A)$ and $y \in A$. Then $d_f(x) = 1$. Thus,

$$\begin{aligned}
d_f(y \cdot x) &= (f(y) \cdot d_f(x)) \vee (d_f(y) \cdot f(x)) \\
&= (f(y) \cdot 1) \vee (d_f(y) \cdot f(x)) \\
&= 1 \vee (d_f(y) \cdot f(x)) && \text{Lemma 2.1 (3)} \\
&= 1. && (2.7)
\end{aligned}$$

Hence, $y \cdot x \in \mathrm{Ker}_{d_f}(A)$. □

Corollary 3.5. *If d_f is an f-derivation of a Hilbert algebra $A = (A, \cdot, 1)$, then $y \cdot x \in \mathrm{Ker}_{d_f}(A)$ for all $x \in \mathrm{Ker}_{d_f}(A)$ and $y \in A$.*

Theorem 3.7. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l, r)-f-derivation of A, then $\mathrm{Ker}_{d_f}(A)$ is a near filter (subalgebra) of A,

(2) if d_f is an (r, l)-f-derivation of A, then $\mathrm{Ker}_{d_f}(A)$ is a near filter (subalgebra) of A.

Proof. (1) Assume that d_f is an (l, r)-f-derivation of A. Since d_f is regular, we have $d_f(1) = 1$ and so $1 \in \mathrm{Ker}_{d_f}(A) \neq \emptyset$. Let $x \in A$ and $y \in \mathrm{Ker}_{d_f}(A)$. Then $d_f(y) = 1$. Thus,

$$\begin{aligned} d_f(x \cdot y) &= (d_f(x) \cdot f(y)) \vee (f(x) \cdot d_f(y)) \\ &= (d_f(x) \cdot f(y)) \vee (f(x) \cdot 1) \\ &= f(y) \vee 1 && \text{Lemma 2.1 (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Ker}_{d_f}(A)$, so $\mathrm{Ker}_{d_f}(A)$ is a near filter of A.

(2) Assume that d_f is an (r, l)-f-derivation of A. Since d_f is regular, we have $d_f(1) = 1$ and so $1 \in \mathrm{Ker}_{d_f}(A) \neq \emptyset$. Let $x \in A$ and $y \in \mathrm{Ker}_{d_f}(A)$. Then $d_f(y) = 1$. Thus,

$$\begin{aligned} d_f(x \cdot y) &= (f(x) \cdot d_f(y)) \vee (d_f(x) \cdot f(y)) \\ &= (f(x) \cdot 1) \vee (d_f(x) \cdot f(y)) \\ &= 1 \vee f(y) && \text{Lemma 2.1 (3)} \\ &= 1. && (2.7) \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Ker}_{d_f}(A)$, so $\mathrm{Ker}_{d_f}(A)$ is a near filter of A. □

Corollary 3.6. *If d_f is an f-derivation of a Hilbert algebra A, then $\mathrm{Ker}_{d_f}(A)$ is a near filter (subalgebra) of A.*

Definition 3.5. *Let d_f be an (l, r)-f-derivation (resp., (r, l)-f-derivation, f-derivation) of a Hilbert algebra $A = (A, \cdot, 1)$. We define the fixed set $\mathrm{Fix}_{d_f}(f)$ of A as follows:*

$$\mathrm{Fix}_{d_f}(f) = \{x \in A : d_f(x) = f(x)\}$$

Example 3.3. *Let $A = \{1, 2, 3, 4\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	1	3	1

Then $(A, \cdot, 1)$ is a Hilbert algebra. We define an endomorphism f on A as follows:

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 4 \end{pmatrix}$$

Define a self-map $d_f : A \to A$ as follows:

$$d_f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Hence, d_f is an f-derivation of A and so $\mathrm{Fix}_{d_f}(f) = \{1, 2, 3\}$.

Theorem 3.8. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l, r)-f-derivation of A, then $\mathrm{Fix}_{d_f}(f)$ is a subalgebra of A,
(2) if d_f is an (r, l)-f-derivation of A, then $\mathrm{Fix}_{d_f}(f)$ is a subalgebra of A.

Proof. (1) Assume that d_f is an (l, r)-f-derivation of A. Since d_f is regular, we have $d_f(1) = 1 = f(1)$ and so $1 \in \mathrm{Fix}_{d_f}(f) \neq \emptyset$. Let $x, y \in \mathrm{Fix}_{d_f}(f)$. Then $d_f(x) = f(x)$ and $d_f(y) = f(y)$. Thus,

$$\begin{aligned} d_f(x \cdot y) &= (d_f(x) \cdot f(y)) \vee (f(x) \cdot d_f(y)) \\ &= (f(x) \cdot f(y)) \vee (f(x) \cdot f(y)) \\ &= f(x) \cdot f(y) && (2.6) \\ &= f(x \cdot y). \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Fix}_{d_f}(f)$, so $\mathrm{Fix}_{d_f}(f)$ is a subalgebra of A.

(2) Assume that d_f is an (r, l)-f-derivation of A. Since d_f is regular, we have $d_f(1) = 1 = f(1)$ and so $1 \in \mathrm{Fix}_{d_f}(f) \neq \emptyset$. Let $x, y \in \mathrm{Fix}_{d_f}(f)$. Then $d_f(x) = f(x)$ and $d_f(y) = f(y)$. Thus,

$$\begin{aligned} d_f(x \cdot y) &= (f(x) \cdot d_f(y)) \vee (d_f(x) \cdot f(y)) \\ &= (f(x) \cdot f(y)) \vee (f(x) \cdot f(y)) \\ &= f(x) \cdot f(y) && (2.6) \\ &= f(x \cdot y). \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Fix}_{d_f}(f)$, so $\mathrm{Fix}_{d_f}(f)$ is a subalgebra of A. □

Corollary 3.7. *If d_f is an f-derivation of a Hilbert algebra A, then $\mathrm{Fix}_{d_f}(f)$ is a subalgebra of A.*

Theorem 3.9. *In a Hilbert algebra $A = (A, \cdot, 1)$, the following statements hold:*

(1) if d_f is an (l, r)-f-derivation of A, then $x \vee y \in \mathrm{Fix}_{d_f}(f)$ for all $x, y \in \mathrm{Fix}_{d_f}(f)$,
(2) if d_f is an (r, l)-f-derivation of A, then $x \vee y \in \mathrm{Fix}_{d_f}(f)$ for all $x, y \in \mathrm{Fix}_{d_f}(f)$.

Proof. (1) Assume that d_f is an (l, r)-f-derivation of A. Let $x, y \in \mathrm{Fix}_{d_f}(f)$. Then $d_f(x) = f(x)$ and $d_f(y) = f(y)$. By Theorem 3.8 (1), we get $d_f(y \cdot x) = f(y \cdot x)$. Thus,

$$\begin{aligned}
d_f(x \vee y) &= d_f((y \cdot x) \cdot x) \\
&= (d_f(y \cdot x) \cdot f(x)) \vee (f(y \cdot x) \cdot d_f(x)) \\
&= (f(y \cdot x) \cdot f(x)) \vee (f(y \cdot x) \cdot f(x)) \\
&= f(y \cdot x) \cdot f(x) \qquad (2.6) \\
&= f((y \cdot x) \cdot x) \\
&= f(x \vee y).
\end{aligned}$$

Hence, $x \vee y \in \mathrm{Fix}_{d_f}(f)$.

(2) Assume that d_f is an (r,l)-f-derivation of A. Let $x, y \in \mathrm{Fix}_{d_f}(f)$. Then $d_f(x) = f(x)$ and $d_f(y) = f(y)$. By Theorem 3.8 (2), we get $d_f(y \cdot x) = f(y \cdot x)$. Thus,

$$\begin{aligned}
d_f(x \vee y) &= d_f((y \cdot x) \cdot x) \\
&= (f(y \cdot x) \cdot d_f(x)) \vee (d_f(y \cdot x) \cdot f(x)) \\
&= (f(y \cdot x) \cdot f(x)) \vee (f(y \cdot x) \cdot f(x)) \\
&= f(y \cdot x) \cdot f(x) \qquad (2.6) \\
&= f((y \cdot x) \cdot x) \\
&= f(x \vee y).
\end{aligned}$$

Hence, $x \vee y \in \mathrm{Fix}_{d_f}(f)$. □

Corollary 3.8. *If d_f is an f-derivation of a Hilbert algebra A, then $x \vee y \in \mathrm{Fix}_{d_f}(f)$ for all $x, y \in \mathrm{Fix}_{d_f}(f)$.*

4. Conclusion

This study introduces the notions of (l,r)-f-derivations, (r,l)-f-derivations, and f-derivations in Hilbert algebras, offering a comprehensive theoretical framework supported by key examples. We investigate the fundamental properties of f-derivations through detailed analysis, revealing significant aspects of their algebraic structure and behaviour. Notably, we establish that the kernel of an f-derivation, $\mathrm{Ker}_{d_f}(A)$, forms a near filter, while the fixed set, $\mathrm{Fix}_{d_f}(f)$, is identified as a subalgebra within the Hilbert algebra A. These findings illuminate the intricate relationships between derivations and substructures in Hilbert algebras, providing valuable insights that open new directions for research in algebraic logic and the broader study of non-classical algebraic systems.

In the future, the ideas presented in this paper can be extended and applied to studying other algebraic systems. By leveraging the foundational concepts of f-derivations and their properties, it is possible to explore analogous structures in different algebraic frameworks. Such applications could provide deeper insights into various mathematical systems, paving the way for further advancements in algebraic theory, non-classical logic, and beyond.

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