Int. J. Anal. Appl. (2024), 22:211

International Journal of Analysis and Applications

Uncertainty Principles for the Weinstein Wavelet Transform

Amel Touati, Imen Kallel, Ahmed Saoudi*

Department of Mathematics, College of Science, Northern Border University, Arar, Saudi Arabia

**Corresponding author:* ahmed.saoudi@ipeim.rnu.tn

Abstract. In the present paper we explore the localization properties of the Weinstein continuous wavelet transform via entropy and we introduce a version of L^p local uncertainty inequalities.

1. Introduction

Uncertainty principles are fundamental concepts in physics and signal processing that describe limitations on the precision with which certain pairs of properties of a system can be simultaneously known. The most famous example of an uncertainty principle is the Heisenberg Uncertainty Principle in quantum mechanics, but similar principles exist in various fields. These principles arise from the mathematical relationships between conjugate variables or transform pairs, such as time and frequency or position and momentum.

In the context of transform, an uncertainty principle refers to the trade-off between the precision in time and frequency localization of a signal. This principle arises due to the nature of analysis, where a signal's time and frequency characteristics are analyzed simultaneously. The uncertainty principle in transform emphasizes the inherent compromise between time and frequency localization, playing a pivotal role in selecting functions and scales that suit the characteristics of the signals under analysis. It enables efficient representation and extraction of information from signals with diverse time and frequency components.

The uncertainty principle for wavelet transforms is an extension of the classical uncertainty principle in signal processing, which states that a function cannot be both time-localized and frequency-localized beyond a certain limit. In other words, there is a trade-off between how precisely a signal can be localized in time and how precisely it can be localized in frequency.

Received: Sep. 23, 2024.

2020 *Mathematics Subject Classification.* Primary 47G10; Secondary 42B10, 42C40.

Key words and phrases. Weinstein wavelet transform; entropy; Heisenberg's type inequality.

https://doi.org/10.28924/2291-8639-22-2024-211

ISSN: 2291-8639

Our main objective, in the present paper is to investigate the localization properties of the Weinstein continuous wavelet transform using entropy, and we present a version of local L^p uncertainty inequalities.

Recently, many authors have been studying the behavior of the Weinstein operator in relation to different problems already studied in classical Fourier transform. For instance, we refer the reader to see Wigner and Weyl transform [10,16], wavelet transform [12,13], pseudo differential operators [1,20], inequalities, uncertainty principles [4,8,9,11,15] and others [2,17,19]. In the same context of investigating uncertainty principles, the second author has studied the Hardy theorem for the linear canonical Dunkl transform [14] and several uncertainty inequalities of the linear canonical Fourier-Bessel wavelet transform [6].

The layout of this article is as follows. Section 2 is dedicated to providing a concise summary of the Weinstein continuous wavelet transform and its basic properties. In section 3, we explore the localization properties of the Weinstein continuous wavelet transform via entropy. Finally, we introduce a version of L^p local uncertainty inequalities of the Weinstein continuous wavelet transform in Section 4.

2. Preliminaires

2.1. **Weinstein transform.** The Weinstein operator Δ_α^n defined on $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0,\infty)$, by

$$\Delta_\alpha^n = \Delta_n + \mathscr{B}_\alpha,\ \alpha > -1/2,$$

where Δ_n is the Laplacian operator on $\mathbb{R}^n$ and $\mathscr{B}_\alpha$ is the Bessel operator for the last variable given on $(0,\infty)$ by

$$\mathscr{B}_\alpha g = \frac{\partial^2 g}{\partial x_{n+1}^2} + \frac{2\alpha+1}{x_{n+1}}\frac{\partial g}{\partial x_{n+1}}.$$

For all $\zeta = (\zeta_1, ..., \zeta_{n+1}) \in \mathbb{C}^{n+1}$, the below system

$$\frac{\partial^2 g}{\partial x_j^2}(x) = -\zeta_j^2 g(x), \quad \text{if } 1 \le j \le n$$

$$L_\alpha g(x) = -\zeta_{n+1}^2 g(x),$$

$$g(0) = 1, \quad \frac{\partial g}{\partial x_{n+1}}(0) = 0, \quad \frac{\partial g}{\partial x_j}(0) = -i\zeta_j, \quad \text{if } 1 \le j \le n$$

has a unique solution denoted by $\Phi_\alpha(\zeta, .)$, and given by

$$\Phi_\alpha(\zeta, x) = e^{-i<x',\zeta'>} j_\alpha(x_{n+1}\zeta_{n+1}) \tag{2.1}$$

where $\zeta = (\zeta', \zeta_{n+1}),\ x = (x', x_{n+1})$ and j_α is the normalized Bessel function given by

$$j_\alpha(x) = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \Gamma(\alpha+k+1)}.$$

$(\zeta, x) \mapsto \Phi_\alpha(\zeta, x)$ is the Weinstein kernel and satisfies for all $(\zeta, x) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$$\left|\Phi_\alpha(\zeta, x)\right| \le 1. \tag{2.2}$$

Along this article, we note by $L^p_\alpha(\mathbb{R}^{n+1}_+),\ 1 \le p \le \infty$, the space of all measurable functions g on $\mathbb{R}^{n+1}_+$ such that

$$\left\|g\right\|_{\alpha,p} = \left(\int_{\mathbb{R}^{n+1}_+} \left|g(x)\right|^p d\sigma_\alpha(x)\right)^{1/p} < \infty,\ p \in [1,\infty),$$

$$\left\|g\right\|_{\alpha,\infty} = \operatorname{ess}\sup_{x\in\mathbb{R}^{n+1}_+} \left|g(x)\right| < \infty,$$

where $d\sigma_\alpha(x)$ denote measure on $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times (0,\infty)$ defined by

$$d\sigma_\alpha(x) = \frac{x_{n+1}^{2\alpha+1}}{(2\pi)^{\frac{d}{2}} 2^\alpha \alpha^2(\alpha+1)} dx.$$

If $g \in L^1_\alpha(\mathbb{R}^{n+1}_+)$ is radial function then $\tilde{g}$ defined on $\mathbb{R}_+$ by $g(x) = \tilde{g}(|x|)$, for all $x \in \mathbb{R}^{n+1}_+$, is integrable function with respect to $r^{2\alpha+n+1}dr$, and we have the equality

$$a_\alpha \int_0^\infty \tilde{g}(r) r^{2\alpha+n+1} dr = \int_{\mathbb{R}^{n+1}_+} g(x) d\sigma_\alpha(x), \tag{2.3}$$

where a_α is a constant given by

$$a_\alpha = \frac{1}{2^{\alpha+\frac{d}{2}}\Gamma(\alpha+\frac{d}{2}+1)}. \tag{2.4}$$

The Weinstein transform is defined for $g \in L^1_\alpha(\mathbb{R}^{n+1}_+)$ by

$$\forall \zeta \in \mathbb{R}^{n+1}_+, \quad \mathcal{F}_\alpha(g)(\zeta) = \int_{\mathbb{R}^{n+1}_+} g(x)\Phi_\alpha(x,\zeta) d\sigma_\alpha(x).$$

We present the following properties, which will be useful throughout the remainder of this paper (see [11, 12])

- If $g \in L^1_\alpha(\mathbb{R}^{n+1}_+)$, then $\mathcal{F}_\alpha(g)$ is continuous on $\mathbb{R}^{n+1}_+$ such that
$$\left\|\mathcal{F}_\alpha(g)\right\|_{\alpha,\infty} \le \left\|g\right\|_{\alpha,1}. \tag{2.5}$$
- For all $g \in L^2_\alpha(\mathbb{R}^{n+1}_+)$, we have
$$\left\|\mathcal{F}_\alpha(g)\right\|_{\alpha,2} = \left\|g\right\|_{\alpha,2}. \tag{2.6}$$
- For all $g \in L^p_\alpha(\mathbb{R}^{n+1}_+)$, $1 \le p \le 2$, the function $\mathcal{F}_\alpha(g)$ belongs to $L^q_\alpha(\mathbb{R}^{n+1}_+)$, where $q = p/(p-1)$, and we have
$$\left\|\mathcal{F}_\alpha(g)\right\|_{\alpha,q} \le \left\|g\right\|_{\alpha,p}. \tag{2.7}$$
- If $g \in L^p_\alpha(\mathbb{R}^{n+1}_+)$ for all $1 \le p \le 2$, then $\mathcal{F}_\alpha(g) \in L^q_\alpha(\mathbb{R}^{n+1}_+), q = p/(p-1)$, and we have
$$\left\|\mathcal{F}_\alpha g\right\|_{\alpha,q} \le \left\|g\right\|_{\alpha,p}. \tag{2.8}$$

For $g \in \mathcal{S}_*(\mathbb{R}^{n+1})$ and $y \in \mathbb{R}^{n+1}_+$ the Weinstein translation $\tau^\alpha_x g$ is defined by [7]

$$\mathcal{F}_\alpha(\tau^\alpha_x g)(y) = \Phi_\alpha(x,y)\mathcal{F}_\alpha(g)(y). \tag{2.9}$$

Proposition 2.1. *(see [7]) The translation operator $\tau^\alpha_x,\ x \in \mathbb{R}^{n+1}_+$ satisfies the following properties.*

(1) *For $g \in \mathbb{C}_*(\mathbb{R}^{n+1})$, we have for all $x, y \in \mathbb{R}_+^{n+1}$*

$$\tau_x^\alpha g(y) = \tau_y^\alpha g(x) \text{ and } \tau_0^\alpha g = g. \tag{2.10}$$

(2) *Let $g \in L_\alpha^p(\mathbb{R}_+^{n+1})$, $1 \leq p \leq \infty$ and $x \in \mathbb{R}_+^{n+1}$. Then $\tau_x^\alpha g$ belongs to $L_\alpha^p(\mathbb{R}_+^{n+1})$ and we have*

$$\left\|\tau_x^\alpha g\right\|_{\alpha,p} \leq \left\|g\right\|_{\alpha,p}. \tag{2.11}$$

For $g,\ h \in L_\alpha^1(\mathbb{R}_+^{n+1})$, the Weinstein convolution product $g * h$ is given by (see [7])

$$g * h(x) = \int_{\mathbb{R}_+^{n+1}} \tau_x^\alpha g(-y) v(y) d\sigma_\alpha(y). \tag{2.12}$$

Proposition 2.2. (1) *For all $g, h \in L_\alpha^1(\mathbb{R}_+^{n+1})$, (resp. $g, h \in \mathcal{S}_*(\mathbb{R}^{n+1})$), then $g * h \in L_\alpha^1(\mathbb{R}_+^{n+1})$, (resp. $g * h \in \mathcal{S}_*(\mathbb{R}^{n+1})$) and we have*

$$\mathcal{F}_\alpha(g * h) = \mathcal{F}_\alpha(g)\mathcal{F}_\alpha(h). \tag{2.13}$$

(2) *Let $p, q, r \in [1, \infty]$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then for all $g \in L_\alpha^p(\mathbb{R}_+^{n+1})$ and $h \in L_\alpha^q(\mathbb{R}_+^{n+1})$ the function $g * h$ belongs to $L_\alpha^r(\mathbb{R}_+^{n+1})$ and we have*

$$\left\|g * h\right\|_{\alpha,r} \leq \left\|g\right\|_{\alpha,p} \|v\|_{\alpha,q}. \tag{2.14}$$

(3) *Let $g, h \in L_\alpha^2(\mathbb{R}_+^{n+1})$. Then*

$$g * h = \mathcal{F}_\alpha^{-1}\left(\mathcal{F}_\alpha(g)\mathcal{F}_\alpha(h)\right). \tag{2.15}$$

(4) *Let $g, h \in L_\alpha^2(\mathbb{R}_+^{n+1})$. Then $g * h$ belongs to $L_\alpha^2(\mathbb{R}_+^{n+1})$ if and only if $\mathcal{F}_\alpha(g)\mathcal{F}_\alpha(h)$ belongs to $L_\alpha^2(\mathbb{R}_+^{n+1})$ and we have*

$$\mathcal{F}_\alpha(g * h) = \mathcal{F}_\alpha(g)\mathcal{F}_\alpha(h). \tag{2.16}$$

(5) *Let $g, h \in L_\alpha^2(\mathbb{R}_+^{n+1})$. Then*

$$\|g * h\|_{\alpha,2} = \|\mathcal{F}_\alpha(g)\mathcal{F}_\alpha(h)\|_{\alpha,2}, \tag{2.17}$$

where both sides are finite or infinite.

2.2. **Weinstein continuous wavelet transform.** Along this paper, we denote by $\mathcal{Y} = \left\{(t, x) : x \in \mathbb{R}_+^{n+1} \text{ and } t > 0\right\}$ and $L_\alpha^p(\mathcal{Y})$, $p \in [1, \infty]$ the space of measurable functions g on $\mathcal{Y}$ such that

$$\begin{aligned} \|g\|_{L_\alpha^p(\mathcal{Y})} &= \left(\int_{\mathcal{Y}} |g(t, x)|^p d\sigma_\alpha(t, x)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|g\|_{L_\alpha^\infty(\mathcal{Y})} &= \operatorname*{ess\,sup}_{(t,x)\in\mathcal{Y}} |g(t, x)| < \infty, \end{aligned}$$

where $\sigma_\alpha(t, x)$ is the measure on $\mathcal{Y}$ defined by:

$$d\sigma_\alpha(t, x) = \frac{d\sigma_\alpha(x)da}{t^{\beta+1}}.$$

For the simplicity of the parameters, let us consider in the rest of the paper

$$\mathbf{fi} = 2\alpha + d + 2. \tag{2.18}$$

Let $t > 0$, we define the dilatation operator δ_t of a measurable function g as below

$$\forall x \in \mathbb{R}^{n+1}_+, \quad \delta_t(x) = \frac{1}{t^{\beta/2}} g\left(\frac{x}{t}\right), \tag{2.19}$$

which satisfies the below properties:

Proposition 2.3. (1) *For every $s, t \in (0, \infty)$, we have*

$$\delta_s \delta_t = \delta_{st}. \tag{2.20}$$

(2) *If $t > 0$ and g a function in $L^2_\alpha(\mathbb{R}^{n+1}_+)$, then $\delta_t g$ is also in $L^2_\alpha(\mathbb{R}^{n+1}_+)$ and we have*

$$\left\|\delta_t g\right\|_{\alpha,2} = \left\|g\right\|_{\alpha,2}, \tag{2.21}$$

and

$$\mathcal{F}_\alpha(\delta_t w)(\xi) = t^{\beta/2} \mathcal{F}_\alpha(w)(t\xi). \tag{2.22}$$

(3) *For all $t > 0$, we have*

$$\forall g, h \in L^2_\alpha(\mathbb{R}^{n+1}_+), \quad \langle \delta_t g, h\rangle_{\alpha,2} = \langle g, \delta_{\frac{1}{t}} h\rangle_{\alpha,2}. \tag{2.23}$$

Definition 2.1. *[3] A Weinstein wavelet on $\mathbb{R}^{n+1}_+$ is a measurable function g on $\mathbb{R}^{n+1}_+$ satisfying for almost all $\xi \in \mathbb{R}^{n+1}_+$, the admissibility condition*

$$0 < C_g = \int_0^\infty |\mathcal{F}_\alpha(h)(t\xi)|^2 \frac{dt}{t} < \infty. \tag{2.24}$$

Let $t > 0$ and g be a Weinstein wavelet on $L^2_\alpha(\mathbb{R}^{n+1}_+)$. Let us consider the Weinstein-type family $g_{t,y}$, $y \in \mathbb{R}^{n+1}_+$ of function on $\mathbb{R}^{n+1}_+$ in $L^2_\alpha(\mathbb{R}^{n+1}_+)$ defined by:

$$g_{t,y}(x) = \tau^\alpha_y(\delta_t g)(x), \quad \forall x \in \mathbb{R}^{n+1}_+, \tag{2.25}$$

where, τ^α_y, $y \in \mathbb{R}^{n+1}_+$ is the Weinstein translation operator given by (2.9).

Definition 2.2. *[5] Let g be a Weinstein wavelet on $\mathbb{R}^{n+1}_+$ in $L^2_\alpha(\mathbb{R}^{n+1}_+)$. The Weinstein continuous wavelet transform Ψ_g on $\mathbb{R}^{n+1}_+$ is defined for regular functions h on $\mathbb{R}^{n+1}_+$ by*

$$\forall (t, y) \in \mathcal{Y}, \quad \Psi_g(h)(t, y) = \int_{\mathbb{R}^{n+1}_+} h(x)\overline{g_{t,y}(x)} d\sigma_\alpha(x) = \langle h, \tau^\alpha_y(\delta_t g)\rangle_{\alpha,2}. \tag{2.26}$$

Weinstein continuous wavelet transform can be written as below:

$$\Psi_g(h)(t, y) = \check{h} * \overline{\delta_t g}(y). \tag{2.27}$$

Lemma 2.1. *Let g be a Weinstein wavelet on $\mathbb{R}^{n+1}_+$ in $L^2_\alpha(\mathbb{R}^{n+1}_+)$. For all $h \in L^2_\alpha(\mathbb{R}^{n+1}_+)$ and for all $\rho > 0m$ we have*

$$\forall (t, y) \in \mathcal{Y}, \quad \Psi_g(\delta_\rho h)(t, y) = \Psi_g(h)(\frac{t}{\rho}, \frac{y}{\rho}). \tag{2.28}$$

Lemma 2.2. *(see [13]) Let g be a Weinstein wavelet on $\mathbb{R}^{n+1}_+$ in $L^2_\alpha(\mathbb{R}^{n+1}_+)$. Then we have for all $h \in L^2_\alpha(\mathbb{R}^{n+1}_+)$*

$$||\Psi_g(h)||_{L^\infty_\alpha(\mathcal{Y})} \leq ||h||_{\alpha,2}||g||_{\alpha,2}. \tag{2.29}$$

Using the Riesz-Thorin interpolation theorem, we derive the following result.

Lemma 2.3. *Let g be a Weinstein wavelet on $\mathbb{R}^{n+1}_+$ in $L^2_\alpha(\mathbb{R}^{n+1}_+)$, $h \in L^2_\alpha(\mathbb{R}^{n+1}_+)$, and $2 \leq p \leq \infty$, the we have*

$$\|\Psi_g(h)\|_{L^p_\alpha(\mathcal{Y})} \leq \left(C_g\right)^{\frac{1}{p}} \left\|g\right\|_{\alpha,2}^{\frac{p-2}{p}} \|h\|_{\alpha,2}. \tag{2.30}$$

3. HUP via Weinstein continuous wavelet transform Entropy

A probability density function $\mathcal{D}$ on the space $\mathcal{Y}$ is a measurable, non-negative function on $\mathcal{Y}$ that satisfies the normalization condition:

$$\int_{\mathcal{Y}} \mathcal{D}(t,y)d\sigma_\alpha(t,y) = 1.$$

Shannon's definition [18] allows us to express the Weinstein continuous wavelet transform entropy of a probability density function $\mathcal{D}$ on the space $\mathcal{Y}$ as

$$\mathscr{E}_\alpha(\mathcal{D}) := -\int_E \ln(\mathcal{D}(t,y))\mathcal{D}(t,y)d\sigma_\alpha(t,y),$$

assuming that the integral on the right-hand side is well-defined.

The primary aim of this section is to explore the localization properties of the Weinstein continuous wavelet transform entropy within the space $\mathcal{Y}$.

Proposition 3.1. *Let g and h be two functions in $L^2_\alpha(\mathbb{R}^{n+1}_+)$ such that h is nonzero function. Then we have the following logarithmic inequality:*

$$\mathscr{E}_\alpha(|\Psi_g(h)|^2) \geq -2C_g \|h\|^2_{\alpha,2} \ln\left(\left\|g\right\|_{\alpha,2} \|h\|_{\alpha,2}\right). \tag{3.1}$$

Proof. We assume that $\|g\|_{\alpha,2}\|h\|_{\alpha,2} = 1$, then by using the inequality (2.29), we get

$$\forall (t,y) \in \mathcal{Y}, \quad \left|\Psi_g(h)(t,y)\right| \leq \left\|g\right\|_{\alpha,2} \|h\|_{\alpha,2} = 1.$$

In particular $\mathscr{E}_\alpha(|\Psi_g(h)|^2) \geq 0$. If the entropy $\mathscr{E}_\alpha(|\Psi_g(h)|^2) = \infty$, then the inequality (3.1) holds. Now, we suppose that the entropy $\mathscr{E}_\alpha(|\Psi_g(h)|^2) < \infty$. Now, we assume that g and h be two functions in $L^2_\alpha(\mathbb{R}^{n+1}_+)$ such that g is nonzero function and let

$$v = \frac{g}{\left\|g\right\|_{\alpha,2}} \quad \text{and} \quad w = \frac{h}{\|h\|_{\alpha,2}}.$$

Hence, v and w belong to $L^2_\alpha(\mathbb{R}^{n+1}_+)$ and $\|v\|_{\alpha,2}\|w\|_{\alpha,2} = 1$, and we have

$$\mathscr{E}_\alpha(|\Psi_v(w)|^2) \geq 0.$$

However,

$$\Psi_v(w) = \frac{1}{\left\|g\right\|_{\alpha,2} \|h\|_{\alpha,2}} \Psi_g(h),$$

and

$$\mathscr{E}_\alpha(|\Psi_v(w)|^2) = \frac{1}{\left\|g\right\|^2_{\alpha,2} \|h\|^2_{\alpha,2}} \mathscr{E}_\alpha(|\Psi_g(h)|^2) + \frac{2C_g}{\left\|g\right\|^2_{\alpha,2}} \ln\left(\left\|g\right\|_{\alpha,2} \|h\|_{\alpha,2}\right).$$

Therefore, it follows that

$$\mathscr{E}_\alpha(|\Psi_g(h)|^2) \geq -2C_g \|h\|^2_{\alpha,2} \ln\left(\|g\|_{\alpha,2} \|h\|_{\alpha,2}\right).$$

□

By utilizing the entropy of the Weinstein continuous wavelet transform, we derive a Heisenberg uncertainty principle for Ψ_g. Now, we state the main result of this section.

Theorem 3.1. *Let a and b two positive real numbers. Then, there exists a positive constant $K_{a,b}(\alpha)$ such that for all function g and h in $L^2_\alpha(\mathbb{R}^{n+1}_+)$, we have the following inequality:*

$$\begin{aligned} \|h\|^2_{\alpha,2} \leq \frac{1}{C_g K_{a,b}(\alpha)} &\left(\int_{\mathcal{Y}} |y|^a \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y)\right)^{\frac{b}{a+b}} \\ &\times \left(\int_{\mathcal{Y}} t^{-b} \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y)\right)^{\frac{a}{a+b}} \end{aligned} \tag{3.2}$$

where

$$K_{a,b}(\alpha) = \frac{\beta}{a^{\frac{b}{a+b}} b^{\frac{a}{a+b}}} e^{u(a,b)}, \tag{3.3}$$

with

$$u(a,b) = ab\frac{\ln\left(\frac{ab}{a_\alpha \Gamma(\beta/a)\Gamma(\beta/b)}\right)}{\beta(a+b)} - 1, \tag{3.4}$$

here a_α is given by the identity (2.4).

Proof. Assume that $\|g\|_{\alpha,2}\|h\|_{\alpha,2} = 1$. For every positive real numbers a, b, c, we put the function $\psi^c_{a,b}$ defined on $\mathcal{Y}$ by:

$$\psi^c_{a,b}(t,y) = \frac{abe^{-\frac{|y|^a + t^{-b}}{c}}}{a_\alpha \Gamma(\beta/a)\Gamma(\beta/b) c^{\frac{\beta(a+b)}{ab}}}.$$

Thus, by simple calculus, it becomes that

$$\int_{\mathcal{Y}} \psi^c_{a,b}(t,y) d\sigma_\alpha(t,y) = 1,$$

in particular the measure $dm^{c,\alpha}_{a,b}(t,y) = \psi^c_{a,b}(t,y) d\sigma_\alpha(t,y)$ is a probability measure on the space $\mathcal{Y}$. According to the convexity of the function $w(t) = t\ln(t)$ over $(0, \infty)$ and by using Jensen's inequality for convex functions, we obtain

$$\int_{\mathcal{Y}} \frac{\left|\Psi_g(h)(t,y)\right|^2}{\psi^c_{a,b}(t,y)} \ln\left(\frac{\left|\Psi_g(h)(t,y)\right|^2}{\psi^c_{a,b}(t,y)}\right) dm^{c,\alpha}_{a,b}(t,y) \geq 0,$$

which implies, in terms of Weinstein continuous wavelet transform entropy, that for any positive real numbers a, b, c, we have the following inequality

$$\mathscr{E}_\alpha(|\Psi_g(h)|^2) + \ln\left(\frac{ab}{a_\alpha\Gamma(\beta/a)\Gamma(\beta/b)}\right) C_g \|h\|_{\alpha,2}^2 \leq \ln\left(c^{\frac{\beta(a+b)}{ab}}\right) C_g \|h\|_{\alpha,2}^2 + \frac{1}{c}\int_{\mathcal{Y}} (|y|^a + t^{-b})|\Psi_g(h)(t,y)|^2 d\sigma_\alpha(t,y).$$

Next, according to Proposition 3.1, we get

$$c\left[\ln\left(\frac{ab}{a_\alpha\Gamma(\beta/a)\Gamma(\beta/b)}\right) - \ln\left(c^{\frac{\beta(a+b)}{ab}}\right)\right] \left\|\Psi_g(h)\right\|_{L^2_\alpha(\mathcal{Y})} \leq \int_{\mathcal{Y}} (|y|^a + t^{-b})|\Psi_g(h)(t,y)|^2 d\sigma_\alpha(t,y).$$

However, the below expression

$$c\left[\ln\left(\frac{ab}{a_\alpha\Gamma(\beta/a)\Gamma(\beta/b)}\right) - \ln\left(c^{\frac{\beta(a+b)}{ab}}\right)\right] \left\|\Psi_g(h)\right\|_{L^2_\alpha(\mathcal{Y})},$$

reaches its maximum value at $c_0 = e^{u(a,b)}$. Therefore, we have

$$C_{a,b}(\alpha) C_g \|h\|_{\alpha,2}^2 \leq \int_{\mathcal{Y}} (|y|^a + t^{-b})|\Psi_g(h)(t,y)|^2 d\sigma_\alpha(t,y),$$

where

$$C_{a,b}(\alpha) = \frac{\beta(a+b)}{ab} e^{u(a,b)}.$$

Now, by substituting g by $g/\left\|g\right\|_{\alpha,2}$ and h by $h/\|h\|_{\alpha,2}$. Thus, for all g and h in $L^2_\alpha(\mathbb{R}^{n+1}_+)$, we have

$$C_{a,b}(\alpha) C_g \|h\|_{\alpha,2}^2 \leq \int_{\mathcal{Y}} |y|^a \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y) \times \int_{\mathcal{Y}} t^{-b} \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y).$$

In other hand, we have for all $\rho > 0$, the dilated function $\delta_{\frac{1}{\rho}} h$ belongs to $L^2_\alpha(\mathbb{R}^{n+1}_+)$. Moreover, $\delta_{\frac{1}{\rho}} h$ is a nonzero function. Hence, according to the above inequality, we obtain

$$C_{a,b}(\alpha) C_g \|\delta_{\frac{1}{\rho}} h\|_{\alpha,2}^2 \leq \int_{\mathcal{Y}} |y|^a \left|\Psi_g(\delta_{\frac{1}{\rho}} h)(t,y)\right|^2 d\sigma_\alpha(t,y) \times \int_{\mathcal{Y}} t^{-b} \left|\Psi_g(\delta_{\frac{1}{\rho}} h)(t,y)\right|^2 d\sigma_\alpha(t,y).$$

Furthermore, we have

$$\|\delta_{\frac{1}{\rho}} h\|_{\alpha,2}^2 = \|h\|_{\alpha,2}^2.$$

Thus, according to relation (2.27), we get

$$C_{a,b}(\alpha)C_g \|h\|_{\alpha,2}^2 \leq \rho^{-a} \int_{\mathcal{Y}} |y|^a \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y)$$
$$\times \rho^b \int_{\mathcal{Y}} t^{-b} \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y).$$

In particular, the inequality is valid at the critical point

$$\rho = \left(\frac{a \int_{\mathcal{Y}} |y|^a \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y)}{b \int_{\mathcal{Y}} t^{-b} \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y)} \right)^{\frac{1}{a+b}}.$$

This implies that

$$K_{a,b}(\alpha)C_g \|h\|_{\alpha,2}^2 \leq \left(\int_{\mathcal{Y}} |y|^a \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y) \right)^{\frac{b}{a+b}}$$
$$\times \left(\int_{\mathcal{Y}} t^{-b} \left|\Psi_g(h)(t,y)\right|^2 d\sigma_\alpha(t,y) \right)^{\frac{a}{a+b}},$$

where

$$K_{a,b}(\alpha) = C_{a,b}(\alpha) \frac{a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}}{a+b} = \frac{\beta}{a^{\frac{b}{a+b}} b^{\frac{a}{a+b}}} e^{u(a,b)}.$$

□

Remark 3.1. *In particular case when $a = b = 2$, we have*

$$\||y|\Psi_g(h)\|_{L^2_\alpha(\mathcal{Y})} \|t^{-1}\Psi_g(h)\|_{L^2_\alpha(\mathcal{Y})} \geq \frac{\beta}{2e} \left(\frac{4}{\Gamma^2(\beta/2)} \right)^{\frac{1}{\beta}} C_g \|h\|_{\alpha,2}^2.$$

4. L^p UP for Weinstein continuous wavelet transform

For $s > 0$, let us consider the following function

$$\mathcal{E}_s(t,y) = e^{-s\|(\frac{1}{t},y)\|^2}, \quad \forall (t,y) \in \mathcal{Y}.$$

A straightforward calculation allows us to obtain the following lemma.

Lemma 4.1. *For $1 \leq q < \infty$ and $s > 0$, there exists a positive constant K, such that the following inequality holds:*

$$\|\mathcal{E}_s\|_{L^q_\alpha(\mathcal{Y})} = K e^{-\frac{\beta}{q}}.$$

Proposition 4.1. *Let $1 < p \leq 2$ and $0 < r < \beta/2p$. Then, there exists a positive constant K such that for all $h \in L^2_\alpha(\mathbb{R}^{n+1}_+)$ and $s > 0$, the following inequality holds:*

$$\|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} \leq K \left(C_g\right)^{\frac{1}{p'}} \left\|g\right\|_{\alpha,2}^{\frac{p'-2}{p'}} s^{-2r} \left(\||y|^r h\|_{\alpha,2} + \||y|^r h\|_{\alpha,2p} \right), \tag{4.1}$$

where p' is the conjugate of p.

Proof. The result is trivial if the expression $\||y|^r h\|_{\alpha,2} + \||y|^r h\|_{\alpha,2p}$ is infinite. Next, we suppose that is finite.

For $\epsilon > 0$, we denote by $B(0,\epsilon)$ the ball of $\mathbb{R}^{n+1}_+$ with center zero and radius $\epsilon > 0$, $h_\epsilon = h\chi_{B(0,\epsilon)}$ and $h^\epsilon = h - h_\epsilon$.

Now, according to inequality (2.30) in Lemma 2.3, we obtain

$$|h^\epsilon(y)| \leq \epsilon^{-r} \left||y|^r h(y)\right|.$$

Hence, we get

$$\begin{aligned} \|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h\chi_{B^c(0,\epsilon)})\|_{L^{p'}_\alpha(\mathcal{Y})} &\leq \|e^{-s\|(\frac{1}{t},y)\|^2}\|_{L^\infty_\alpha(\mathcal{Y})}\|\Psi_g(h\chi_{B^c(0,\epsilon)})\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\leq \left(C_g\right)^{\frac{1}{p'}} \left\|g\right\|_{\alpha,2}^{\frac{p'-2}{p'}} \|h\chi_{B^c(0,\epsilon)}\|_{\alpha,2} \\ &\leq \left(C_g\right)^{\frac{1}{p'}} \left\|g\right\|_{\alpha,2}^{\frac{p'-2}{p'}} \epsilon^{-r} \left\||y|^r h\right\|_{\alpha,2}. \end{aligned}$$

On other hand, using inequality (2.29) and Hölder's inequality, we obtain

$$\begin{aligned} &\|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h\chi_{B(0,\epsilon)})\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\quad\leq \|e^{-s\|(\frac{1}{t},y)\|^2}\|_{L^{p'}_\alpha(\mathcal{Y})}\|\Psi_g(h\chi_{B(0,\epsilon)})\|_{L^\infty_\alpha(\mathcal{Y})} \\ &\quad\leq \left\|g\right\|_{\alpha,2} \|h\chi_{B(0,\epsilon)}\|_{\alpha,2}\|e^{-s\|(\frac{1}{t},y)\|^2}\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\quad\leq \left\|g\right\|_{\alpha,2} \||y|^{-r}\chi_{B(0,\epsilon)}\|_{\alpha,2p'}\||y|^r h\|_{\alpha,2p}\|e^{-s\|(\frac{1}{t},y)\|^2}\|_{L^{p'}_\alpha(\mathcal{Y})}. \end{aligned}$$

A straightforward calculation shows that there exists a positive constant K, such that

$$\||y|^{-r}\chi_{B(0,\epsilon)}\|_{\alpha,2p'} = K\epsilon^{-r+\frac{\beta}{2p'}}.$$

So it follows that

$$\begin{aligned} &\|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\quad\leq \|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h_\epsilon)\|_{L^{p'}_\alpha(\mathcal{Y})} + \|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h^\epsilon)\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\quad\leq K\epsilon^{-r} \left\|g\right\|_{\alpha,2} \left(\left(C_g \left\|g\right\|_{\alpha,2}^{-2}\right)^{\frac{1}{p'}} \left\||y|^r h\right\|_{\alpha,2} + \epsilon^{\frac{\beta}{2p'}} \|\mathcal{E}_s\|_{L^{p'}_\alpha(\mathcal{Y})}\||y|^r h\|_{\alpha,2p}\right). \end{aligned}$$

Finally, by choosing

$$\epsilon = \left(C_g \left\|g\right\|_{\alpha,2}^{-2}\right)^{\frac{2}{\beta}} s^2,$$

we get the desired result.

□

Theorem 4.1. *Let $1 < p \leq 2$, $0 < r < \beta/2p'$ and $\epsilon > 0$. Then, there exists a positive constant K such that for all $h \in L^2_\alpha(\mathbb{R}^{n+1}_+)$, the following inequality holds:*

$$\begin{aligned} &\|\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\leq K\left(C_g^{\frac{1}{p'}}\left\|g\right\|_{\alpha,2}^{\frac{p'-2}{p'}}\right)^{\frac{\epsilon}{r+\epsilon}}\left(\left\||y|^r h\right\|_{\alpha,2}+\||y|^r h\|_{\alpha,2p}\right)^{\frac{\epsilon}{r+\epsilon}}\||(\frac{1}{t},y)|^{4\epsilon}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})}^{\frac{r}{r+\epsilon}}. \end{aligned} \tag{4.2}$$

Proof. Let $1 < p \leq 2$ and $0 < r < \beta/2p$. Firstly we suppose that $\epsilon \leq 1/2$. According to Lemma 4.1, we have for all $s > 0$

$$\begin{aligned} \|\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} &\leq \|e^{-s\|(\frac{1}{t},y)\|^2}\Psi_g(h))\|_{L^{p'}_\alpha(\mathcal{Y})}+\|(1-e^{-s\|(\frac{1}{t},y)\|^2})\Psi_g(h))\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &\leq K\left(C_g\right)^{\frac{1}{p'}}\left\|g\right\|_{\alpha,2}^{\frac{p'-2}{p'}}s^{-2r}\left(\||y|^r h\|_{\alpha,2}+\||y|^r h\|_{\alpha,2p}\right) \\ &+ \|(1-e^{-s\|(\frac{1}{t},y)\|^2})\Psi_g(h))\|_{L^{p'}_\alpha(\mathcal{Y})}. \end{aligned}$$

Moreover we have

$$\begin{aligned} &\|(1-e^{-s\|(\frac{1}{t},y)\|^2})\Psi_g(h))\|_{L^{p'}_\alpha(\mathcal{Y})} \\ &= s^{2\epsilon}\|(t|(\frac{1}{t},y)|^2)^{-2\epsilon}(1-e^{-s\|(\frac{1}{t},y)\|^2})|(\frac{1}{t},y)|^{4\epsilon}\Psi_g(h))\|_{L^{p'}_\alpha(\mathcal{Y})}. \end{aligned}$$

Now according to the boundedness of $u \mapsto (1-e^{-u})u^{-2\epsilon}$, for all $u \geq 0$, when $\epsilon \leq 1/2$, we deduce that

$$\begin{aligned} \|\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} &\leq K\left(C_g\right)^{\frac{1}{p'}}\left\|g\right\|_{\alpha,2}^{\frac{p'-2}{p'}}s^{-2r}\left(\||y|^r h\|_{\alpha,2}+\||y|^r h\|_{\alpha,2p}\right) \\ &+ Ks^{-2\epsilon}\||(\frac{1}{t},y)|^{4\epsilon}\Psi_g(h))\|_{L^{p'}_\alpha(\mathcal{Y})}, \end{aligned}$$

by optimizing with respect to s, we obtain equation (4.2) for all $0 < r < \beta/2p'$ and $\epsilon \leq 1/2$.

Now, we suppose that $\epsilon > 1/2$ and let $\epsilon' \leq 1/2$. For all $u > 0$ we have $u^{4\epsilon'} \leq 1 + u^{4\epsilon}$. In particular if

$$u = \frac{|(\frac{1}{t},y)|}{\eta}, \quad \text{for all} \quad \eta > 0,$$

we have following inequality

$$\left(\frac{|(\frac{1}{t},y)|}{\eta}\right)^{4\epsilon'} \leq 1 + \left(\frac{|(\frac{1}{t},y)|}{\eta}\right)^{4\epsilon}.$$

Hence, it follows that

$$\||(\frac{1}{t},y)|^{4\epsilon'}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} \leq \eta^{4\epsilon'}\|\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} + \eta^{4(\epsilon'-\epsilon)}\||(\frac{1}{t},y)|^{4\epsilon}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})}.$$

By optimizing with respect to η, we obtain

$$\||\left(\frac{1}{t},y\right)|^{4\epsilon'}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})} \leq \|\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})}^{\frac{\epsilon-\epsilon'}{\epsilon}}\||\left(\frac{1}{t},y\right)|^{4\epsilon}\Psi_g(h)\|_{L^{p'}_\alpha(\mathcal{Y})}^{\frac{\epsilon'}{\epsilon}}.$$

Combining this with equation (4.2) for ϵ', we obtain the result for $\epsilon > \frac{1}{2}$. □

Note that a particular case for $p = 2$ in the previous Theorem, gives the following result.

Corollary 4.1. *Let* $0 < r < \beta/4$ *and* $\epsilon > 0$. *Then, there exists a positive constant* K *such that for all* $h \in L^2_\alpha(\mathbb{R}^{n+1}_+)$, *the following inequality holds:*

$$\|h\|_{\alpha,2} \le K\left(C_g^{\frac{1}{2}}\left\|g\right\|_{\alpha,2}\right)^{\frac{\epsilon}{r+\epsilon}}\left(\left\||y|^r h\right\|_{\alpha,2} + \||y|^r h\|_{\alpha,4}\right)^{\frac{\epsilon}{r+\epsilon}} \||(\frac{1}{t},y)|^{4\epsilon}\Psi_g(h)\|^{\frac{r}{r+\epsilon}}_{L^2_\alpha(\mathcal{Y})}. \tag{4.3}$$

Acknowledgments

The authors gratefully acknowledge the approval and the support of this research study by the grant no. SCIA-2023-12-2095 from the Deanship of Scientific Research at Northern Border University, Arar, Saudi Arabia.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H.B. Mohamed, Y. Bettaibi, Pseudo-Differential Operators in the Generalized Weinstein Setting, Rend. Circ. Mat. Palermo, II. Ser. 72 (2023), 3345–3361. https://doi.org/10.1007/s12215-022-00827-7.

[2] C. Chettaoui, H.B. Mohamed, Bochner–Hecke Theorems in the Generalized Weinstein Theory Setting, Complex Anal. Oper. Theory 17 (2023), 38. https://doi.org/10.1007/s11785-023-01342-y.

[3] A. Gasmi, H. Ben Mohamed, N. Bettaibi, Inversion of Weinstein intertwining operator and its dual using Weinstein wavelets, An. Univ. "Ovidius" Constanța, Ser. Mat. 24 (2016), 289–307. https://doi.org/10.1515/auom-2016-0016.

[4] I. Kallel, A. Saoudi, Uncertainty Principle for the Weinstein-Gabor Transforms, Int. J. Anal. Appl. 22 (2024), 94. https://doi.org/10.28924/2291-8639-22-2024-94.

[5] H. Mejjaoli, A. Ould Ahmed Salem, New Results on the Continuous Weinstein Wavelet Transform, J. Inequal. Appl. 2017 (2017), 270. https://doi.org/10.1186/s13660-017-1534-5.

[6] H.B. Mohamed, A. Saoudi, Linear Canonical Fourier–Bessel Wavelet Transform: Properties and Inequalities, Integr. Transforms Spec. Funct. 35 (2024), 270–290. https://doi.org/10.1080/10652469.2024.2317724.

[7] Z.B. Nahia, Fonctions Harmoniques et Proprietés de la Moyenne Associées à l'Opérateur de Weinstein, Thesis, Department of Mathematics, Faculty of Sciences of Tunis, Tunisia, 1995.

[8] N.B. Salem, Inequalities Related to Spherical Harmonics Associated With the Weinstein Operator, Integr. Transforms Spec. Funct. 34 (2022), 41–64. https://doi.org/10.1080/10652469.2022.2087063.

[9] N.B. Salem, A.R. Nasr, Heisenberg-Type Inequalities for the Weinstein Operator, Integr. Transforms Spec. Funct. 26 (2015), 700–718. https://doi.org/10.1080/10652469.2015.1038531.

[10] A. Saoudi, On the Weinstein–Wigner Transform and Weinstein–Weyl Transform, J. Pseudo-Differ. Oper. Appl. 11 (2020), 1–14. https://doi.org/10.1007/s11868-019-00313-2.

[11] A. Saoudi, A Variation of L^p Uncertainty Principles in Weinstein Setting, Indian J. Pure Appl. Math. 51 (2020), 1697–1712. https://doi.org/10.1007/s13226-020-0490-9.

[12] A. Saoudi, Two-Wavelet Theory in Weinstein Setting, Int. J. Wavelets Multiresolut. Inf. Process. 20 (2022), 2250020. https://doi.org/10.1142/S0219691322500205.

[13] A. Saoudi, Time-Scale Localization Operators in the Weinstein Setting, Results Math. 78 (2022), 14. https://doi.org/10.1007/s00025-022-01792-4.

[14] A. Saoudi, Hardy Type Theorems for Linear Canonical Dunkl Transform, Complex Anal. Oper. Theory 18 (2024), 57. https://doi.org/10.1007/s11785-023-01478-x.

[15] A. SAOUDI, I. KALLEL, A Variation of L^p Local Uncertainty Principles for Weinstein Transform, Proc. Rom. Acad. Ser. A - Math. Phys. Tech. Sci. Inf. Sci. 25 (2024), 3–10. https://doi.org/10.59277/pra-ser.a.25.1.01.

[16] A. Saoudi, B. Nefzi, Boundedness and Compactness of Localization Operators for Weinstein–Wigner Transform, J. Pseudo-Differ. Oper. Appl. 11 (2020), 675–702. https://doi.org/10.1007/s11868-020-00328-0.

[17] M. Sartaj, S.K. Upadhyay, Symmetrically Global Pseudo-Differential Operators Involving the Weinstein Transform, J. Pseudo-Differ. Oper. Appl. 14 (2023), 51. https://doi.org/10.1007/s11868-023-00543-5.

[18] D. Slepian, Prolate Spheroidal Wave Functions, Fourier Analysis and Uncertainty - IV: Extensions to Many Dimensions; Generalized Prolate Spheroidal Functions, Bell Syst. Techn. J. 43 (1964), 3009–3057. https://doi.org/10.1002/j.1538-7305.1964.tb01037.x.

[19] H.M. Srivastava, S. Yadav, S.K. Upadhyay, The Weinstein Transform Associated With a Family of Generalized Distributions, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 117 (2023), 132. https://doi.org/10.1007/s13398-023-01461-3.

[20] S.K. Upadhyay, M. Sartaj, An Integral Representation of Pseudo-Differential Operators Involving Weinstein Transform, J. Pseudo-Differ. Oper. Appl. 13 (2022), 33. https://doi.org/10.1007/s11868-022-00442-1.