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Decision-Making of a New Type of Stochastic Space and Its Associated Operator Ideal

Salah H. Alshabhi[1], **Achyutananda Baruah**[2], **Mustafa M. Mohammed**[1], **Mona Magzoub**[3], **Khdija O. Taha**[4], **Runda A. A. Bashir**[1,*], **Sakeena E. M. Hamed**[5], **Binod Chandra Tripathy**[6], **Awad A. Bakery**[1,7]

[1]*University of Jeddah, Applied College at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*
[2]*Department of Mathematics, North Gauhati College, College Nagar, Guwahati-781031, India*
[3]*University of Jeddah, Applied college at Alkamil, Department of Mathematics, Jeddah, Saudi Arabia*
[4]*Department of Mathematics, College of Science, Qassim University, Saudi Arabia*
[5]*University of Jeddah, College of Business at Khulis, Jeddah, Saudi Arabia*
[6]*Department of Mathematics, Tripura University, Agartala - 799022, Tripura, India*
[7]*Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Abbassia, Egypt*

[*]*Corresponding author:* rabasher@uj.edu.sa

Abstract. We develop and examine the pre-modular space of null variable exponent-weighted backward generalized difference gai sequences of fuzzy functions in this paper. These sequences of fuzzy functions are important contributions to the concept of modular spaces because they have exponent weighting. Using extended $s-$fuzzy functions as well as this sequence space of fuzzy functions, it has been possible to accomplish an idealization of the mappings. We have presented some topological and geometric properties of this new space, as well as the ideal mappings that correspond to them.

1. Introduction

The mappings' ideal theory is well regarded in functional analysis. The closed mappings' ideals are certain to play an important function in the principle of Banach lattices. Fixed point theory, Banach space geometry, normal series theory, approximation theory, ideal transformations, etc. all use mappings' ideal. Using s-numbers is an essential technique. Pietsch [1–4] developed and studied the theory of s-numbers of linear bounded mappings between Banach spaces. He offered and explained some topological and geometric structures of the quasi ideals of ℓ_p type mappings.

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Then, Constantin [5], generalized the class of ℓ_p type mappings to the class of ces_p type mappings. Makarov and Faried [6], showed some inclusion relations of ℓ_p type mappings. As a generalization of ℓ_p type mappings, Stolz mappings and mappings' ideal were examined by Tita [7,8]. In [9], Maji and Srivastava studied the class $A_p^{(s)}$ of s-type ces_p mappings using s-number sequence and Cesàro sequence spaces and they introduced a new class$A_{p,q}^{(s)}$ of s-type $ces(p,q)$ mappings by weighted ces_p with $1 < p < \infty$. In [10], the class of s-type $Z(u,v;\ell_p)$ mappings was defined and some of their properties were explained. Yaying et al. [11], defined and studied χ_r^η, whose its r-Cesàro matrix in ℓ_η, with $r \in (0,1]$ and $1 \leq \eta \leq \infty$. They explained the quasi Banach ideal of type χ_r^η, with $r \in (0,1]$ and $1 < \eta < \infty$. Pre-quasi mappings' ideals are more extensive than quasi mappings' ideals, according to Faried and Bakery [12].

After Zadeh [13] established the concept of fuzzy sets and fuzzy set operations, many researchers adopted the concept of fuzziness in cybernetics and artificial intelligence as well as in expert systems and fuzzy control. Javed et al. [14] investigated the Banach contraction in R-fuzzy b-metric spaces and discussed some related fixed point results to ensure a fixed point's existence and uniqueness. A nontrivial example is given to illustrate the feasibility of the proposed methods. They offered an application to solve the first kind of Fredholm-type integral equation. In [15], Rehman and Aydi proved some common fixed point theorems for mappings involving generalized rational-type fuzzy cone-contraction conditions in fuzzy cone metric spaces. They gave a common solution of two definite Fredholm integral equations. The concept of orthogonal partial b-metric spaces was pioneered by Javed et al. [16]. They presented a unique fixed point for some orthogonal contractive mappings with some examples and an application. Humaira et al. [17], discussed the existence theorem for a unique solution to a coupled system of impulsive fractional differential equations in complex-valued fuzzy metric spaces and the fuzzy version of some fixed point results by using the definition and presented some properties of a complex-valued fuzzy metric space with some applications. In this study, Rome et al. [18] looked into the concept of extended fuzzy rectangular b-metric space. They explained that some fixed point results in the literature could be generalized by α-admittance in this space. They used this to show solutions for a group of integral equations. Many researchers in sequence spaces and summability theory studied fuzzy sequence spaces and their properties. Different classes of sequences of fuzzy real numbers have been discussed by Nanda [19], Nuray and Savas [20], Matloka [21], Altinok et al. [22], Colak et al. [23], Hazarika and Savas [24] and many others. In [20], the Nakano sequences of fuzzy integers were defined and analyzed. Tripathy and Baruah [25], introduced and examined some properties of a new type of difference sequence spaces of fuzzy real numbers. Subramanian and Misra [26,27], defined and studied the generalized double difference of Gai Sequence Spaces and the generalized semi-normed difference of double gai sequence spaces defined by a modulus function. In [28], Subramanian et al. introduced and offered some properties of the generalized difference gai sequences of fuzzy numbers defined by Orlicz functions. Bakery and Mohamed [29], introduced the certain space of sequences of fuzzy numbers, in short (cssf), under a certain function to be pre-quasi (cssf). This space and $s-$ numbers have been used to describe the structure of the ideal operators. They defined and studied the weighted Nakano sequence spaces of fuzzy functions. They constructed the ideal generated by extended $s-$fuzzy functions and the sequence spaces of fuzzy functions. They presented some topological and geometric structures of this class of ideal and multiplication mappings acting on this sequence space of fuzzy functions. Moreover, the existence of Caristi's fixed point was examined. Many fixed point theorems are effective when applied to a given space because they either enlarge the self-mapping acting on it or expand the space itself. In this paper, we have defined and studied the pre-modular space of null variable exponent-weighted backward generalized difference gai sequences spaces of fuzzy numbers, which are important extensions of the concept of modular spaces. The Fatou property of various pre-quasi norms on this new space has been investigated. Extended $s-$fuzzy functions and this sequence space of fuzzy functions have been used to create the mappings' ideal. The topological and geometric characteristics of mappings' ideal are offered.

2. Definitions and Preliminaries:

Remember that Matloka [21], introduced bounded and convergent fuzzy numbers, investigated some of their properties, and demonstrated that any convergent fuzzy number sequence is bounded. Nanda [19], researched fuzzy number sequences and demonstrated that the set of all convergent fuzzy number sequences forms a complete metric space. Kumar et al. [30], presented the concept limit points and cluster points of sequences of fuzzy numbers. If Ω is the set of all closed and bounded intervals on the real line $\Re$. Assume $f = [f_1, f_2]$ and $g = [g_1, g_2]$ in Ω, let

$$f \leq g \text{ if and only if } f_1 \leq g_1 \text{ and } f_2 \leq g_2.$$

Define a metric ρ on Ω by

$$\rho(f, g) = \max\{|f_1 - g_1|, |f_2 - g_2|\}.$$

Matloka [21] proved that ρ is a metric on Ω and (Ω, ρ) is a complete metric space. The relation $\leq$ is a partial order on Ω.

Definition 2.1. *A fuzzy number f is a fuzzy subset of $\Re$ i.e., a mapping $f : \Re \to [0, 1]$ that verifies the four conditions:*

- **(a):** *f is fuzzy convex, i.e., for $x, y \in \Re$ and $\alpha \in [0, 1]$, $f(\alpha x + (1-\alpha)y) \geq \min\{f(x), f(y)\}$;*
- **(b):** *f is normal, i.e., there is $y_0 \in \Re$ such that $f(y_0) = 1$;*
- **(c):** *f is an upper-semi continuous, i.e., for all $\alpha > 0$, $f^{-1}([0, x+\alpha))$ for all $x \in [0, 1]$ is open in the usual topology of $\Re$;*
- **(d):** *the closure of $f^0 := \{y \in \Re : \ f(y) > 0\}$ is compact.*

The β-level set of a fuzzy real number f, $\ 0 < \beta < 1$, denoted by f^β, is defined as

$$f^\beta = \{y \in \Re : \ f(y) \geq \beta\}.$$

The set of all upper semi-continuous, normal, convex fuzzy number, and f^β is compact, is marked by $\Re([0,1])$. The set $\Re$ can be embedded in $\Re([0,1])$, if we define $r \in \Re([0,1])$ by

$$\bar{r}(t) = \begin{cases} 1, & t = r \\ 0, & t \neq r. \end{cases}$$

The additive identity and multiplicative identity in $\Re[0,1]$ are denoted by $\bar{0}$ and $\bar{1}$, respectively. Assume $f, g \in \Re[0,1]$ and the β-level sets are $[f]^\beta = [f_1^\beta, f_2^\beta]$, $[g]^\beta = [g_1^\beta, g_2^\beta]$, $\beta \in [0,1]$. A partial ordering for any $f, g \in \Re[0,1]$ as follows: $f \leq g$ if and only if $f^\beta \leq g^\beta$, for all $\beta \in [0,1]$.
Assume $\overline{\rho} : \Re[0,1] \times \Re[0,1] \to \Re^+ \cup \{0\}$ is defined by $\overline{\rho}(f,g) = \sup_{0 \leq \beta \leq 1} \rho(f^\beta, g^\beta)$.

Recall that:

1. $(\Re[0,1], \overline{\rho})$ is a complete metric space.
2. $\overline{\rho}(f+k, g+k) = \overline{\rho}(f,g)$ for all $f, g, k \in \Re[0,1]$.
3. $\overline{\rho}(f+k, g+l) \leq \overline{\rho}(f,g) + \overline{\rho}(k,l)$.
4. $\overline{\rho}(\xi f, \xi g) = |\xi| \overline{\rho}(f,g)$, for all $\xi \in \Re$.

By c_0, ℓ_∞ and ℓ_r, we denote the space of null, bounded and r-absolutely summable sequences of real numbers. Let $\omega(F)$ denote the classes of all sequence spaces of fuzzy real numbers. A sequence $X = (X_k) \in \omega(F)$ is called analytic sequence of fuzzy numbers if $\sup_k |X_k|^{\frac{1}{k+1}} < \infty$. A sequence $X = (X_k) \in \omega(F)$ is called gai sequence of fuzzy numbers if $(k! \overline{\rho}(X_k, \bar{0}))^{\frac{1}{k+1}} \to 0$, as $k \to \infty$. Let $\mathbb{C}^{\mathcal{N}}$ denote the space of all sequences of complex numbers, where $\mathcal{N}$ is the set of non-negative integers. Tripathy et al. [31], defined and examined the forward and backward generalized difference sequence spaces: $U(\Delta_n^{(m)}) = \left\{(w_k) \in \mathbb{C}^{\mathcal{N}} : (\Delta_n^{(m)} w_k) \in U\right\}$ and $U(\Delta_n^m) = \left\{(w_k) \in \mathbb{C}^{\mathcal{N}} : (\Delta_n^m w_k) \in U\right\}$, where $m, n \in \mathcal{N}$, $U = \ell_\infty, c$ or c_0, with $\Delta_n^{(m)} w_k = \sum_{\nu=0}^{m} (-1)^\nu C_\nu^m \; w_{k+\nu n}$, and $\Delta_n^m w_k = \sum_{\nu=0}^{m} (-1)^\nu C_\nu^m \; w_{k-\nu n}$, respectively. If $n = 1$, the generalized difference sequence spaces reduced to $U(\Delta^{(m)})$ defined and investigated by Et and Çolak [32]. If $m = 1$, the generalized difference sequence spaces reduced to $U(\Delta_n)$ defined and investigated by Tripathy and Esi [33]. While, if $n = 1$ and $m = 1$, the generalized difference sequence spaces reduced to $U(\Delta)$ defined and studied by Kizmaz [34].

Definition 2.2. *[35] The backward generalized difference Δ_{n+1}^m is said to be an absolute non-decreasing, if $|x_i| \leq |y_i|$ for all $i \in \mathbb{N}$, then $\left|\Delta_{n+1}^m |x_i|\right| \leq \left|\Delta_{n+1}^m |y_i|\right|$.*

We indicate the space of all bounded, finite rank linear mappings from an infinite dimensional Banach space Δ into an infinite dimensional Banach space Λ by $\mathcal{L}(\Omega, \Lambda)$, and $\mathfrak{F}(\Omega, \Lambda)$ and when $\Delta = \Lambda$, we inscribe $\mathcal{L}(\Omega)$ and $\mathfrak{F}(\Omega)$. The space of approximable and compact bounded linear mappings from Ω into Λ will be denoted by $\Upsilon(\Omega, \Lambda)$ and $\mathcal{L}_c(\Omega, \Lambda)$, and if $\Omega = \Lambda$, we mark $\Upsilon(\Omega)$ and $\mathcal{L}_c(\Omega)$, respectively.

Definition 2.3. *[36] An s-number function is a mapping $s : \mathcal{L}(\Omega, \Lambda) \to \Re^{+\mathcal{N}}$ that gives all $V \in \mathcal{L}(\Omega, \Lambda)$ a $(s_d(V))_{d=0}^\infty$ holds the following conditions:*

(a): $\|V\| = s_0(V) \geq s_1(V) \geq s_2(V) \geq ... \geq 0$, *for every* $V \in \mathcal{L}(\Omega, \Lambda)$,

(b): $s_{l+d-1}(V_1 + V_2) \leq s_l(V_1) + s_d(V_2)$, *for every* $V_1, V_2 \in \mathcal{L}(\Omega, \Lambda)$ *and* $l, d \in \mathcal{N}$,

(c): $s_d(VYW) \leq \|V\| s_d(Y) \|W\|$, *for every* $W \in \mathcal{L}(\Delta_0, \Delta)$, $Y \in \mathcal{L}(\Omega, \Lambda)$ *and* $V \in \mathcal{L}(\Lambda, \Lambda_0)$, *where* Ω_0 *and* Λ_0 *are arbitrary Banach spaces,*

(d): *assume* $V \in \mathcal{L}(\Omega, \Lambda)$ *and* $\gamma \in \Re$, *then* $s_d(\gamma V) = |\gamma| s_d(V)$,

(e): *if* $rank(V) \leq d$, *then* $s_d(V) = 0$, *for all* $V \in \mathcal{L}(\Omega, \Lambda)$,

(f): $s_{l \geq a}(I_a) = 0$ *or* $s_{l<a}(I_a) = 1$, *where* I_a *indicates the unit mapping on the a-dimensional Hilbert space* ℓ_2^a.

We give here some examples of s-numbers:

(1): The q-th Kolmogorov number, denoted by $d_q(X)$, is marked by $d_q(X) = \inf_{\dim J \leq q} \sup_{\|f\| \leq 1} \inf_{g \in J} \|Xf - g\|$.

(2): The q-th approximation number, indicated by $\alpha_q(X)$, is marked by $\alpha_q(X) = \inf\{\|X - Y\| : Y \in \mathcal{L}(\Omega, \Lambda) \text{ and } \mathrm{rank}(Y) \leq q\}$.

Definition 2.4. *[3] Let* $\mathcal{L}$ *be the class of all bounded linear operators within any two arbitrary Banach spaces. A sub class* $\mathcal{U}$ *of* $\mathcal{L}$ *is said to be a mappings' ideal, if every* $\mathcal{U}(\Omega, \Lambda) = \mathcal{U} \cap \mathcal{L}(\Omega, \Lambda)$ *satisfies the following setups:*

(i): $I_\Gamma \in \mathcal{U}$, *where* Γ *indicates Banach space of one dimension.*

(ii): *The space* $\mathcal{U}(\Omega, \Lambda)$ *is linear over* $\Re$.

(iii): *If* $W \in \mathcal{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Omega, \Lambda)$ *and* $Y \in \mathcal{L}(\Lambda, \Lambda_0)$, *then* $YXW \in \mathcal{U}(\Delta_0, \Lambda_0)$.

Notations 2.5. *[29]*

$$\overline{\maltese}_{\mathbf{U}} := \left\{\overline{\maltese}_{\mathbf{U}}(\Omega, \Lambda)\right\}, \text{ where } \overline{\maltese}_{\mathbf{U}}(\Omega, \Lambda) := \left\{V \in \mathcal{L}(\Omega, \Lambda) : ((\overline{s_j(V)})_{j=0}^{\infty} \in \mathbf{U}\right\},$$

$$\overline{\maltese^{\alpha}}_{\mathbf{U}} := \left\{\overline{\maltese^{\alpha}}_{\mathbf{U}}(\Omega, \Lambda)\right\}, \text{ where } \overline{\maltese^{\alpha}}_{\mathbf{U}}(\Omega, \Lambda) := \left\{V \in \mathcal{L}(\Omega, \Lambda) : ((\overline{\alpha_j(V)})_{j=0}^{\infty} \in \mathbf{U}\right\},$$

$$\overline{\maltese^{d}}_{\mathbf{U}} := \left\{\overline{\maltese^{d}}_{\mathbf{U}}(\Omega, \Lambda)\right\}, \text{ where } \overline{\maltese^{d}}_{\mathbf{U}}(\Omega, \Lambda) := \left\{V \in \mathcal{L}(\Omega, \Lambda) : ((\overline{d_j(V)})_{j=0}^{\infty} \in \mathbf{U}\right\},$$

where

$$\overline{s_j(V)}(x) = \begin{cases} 1, & x = s_j(V) \\ 0, & x \neq s_j(V). \end{cases}$$

Definition 2.6. *[12] A function* $H \in [0, \infty)^{\mathcal{U}}$ *is said to be a pre-quasi norm on the ideal* $\mathcal{U}$ *if the following conditions hold:*

(1): *Assume* $V \in \mathcal{U}(\Omega, \Lambda)$, $H(V) \geq 0$ *and* $H(V) = 0$, *if and only if,* $V = 0$,

(2): *one has* $Q \geq 1$ *with* $H(\alpha V) \leq D|\alpha| H(V)$, *for all* $V \in \mathcal{U}(\Omega, \Lambda)$ *and* $\alpha \in \Re$,

(3): *there are* $P \geq 1$ *such that* $H(V_1 + V_2) \leq P[H(V_1) + H(V_2)]$, *for all* $V_1, V_2 \in \mathcal{U}(\Omega, \Lambda)$,

(4): *there are* $\sigma \geq 1$ *so that if* $V \in \mathcal{L}(\Delta_0, \Delta)$, $X \in \mathcal{U}(\Omega, \Lambda)$ *and* $Y \in \mathcal{L}(\Lambda, \Lambda_0)$ *then* $H(YXV) \leq \sigma \|Y\| H(X) \|V\|$.

Theorem 2.1. *[12] H is a pre-quasi norm on the ideal* $\mathcal{U}$, *whenever H is a quasi norm on the ideal* $\mathcal{U}$.

Lemma 2.1. *[37] If $\tau_a > 0$ and $v_a, t_a \in \Re$, for all $a \in \mathcal{N}$, then $|v_a + t_a|^{\tau_a} \leq 2^{K-1}(|v_a|^{\tau_a} + |t_a|^{\tau_a})$, where $K = \max\{1, \sup_a \tau_a\}$.*

3. Some characteristics of $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$

We have offered in this section sufficient conditions of the space of null variable exponent-weighted generalized difference gai sequences of fuzzy numbers, $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$, equipped with definite function h to be pre-quasi Banach (cssf). We have examined some algebraic and topological properties like completeness, solidness, symmetry, convergence-free etc. The Fatou property of various pre-quasi norms h on $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$ has been presented.

If $\tau = (\tau_a), \eta = (\eta_a) \in \Re^{+\mathcal{N}}$, where $\Re^{+\mathcal{N}}$ is the space of positive reals. The space of null variable exponent-weighted generalized difference gai sequences of fuzzy numbers is defined as:

$\chi_0^F(\Delta_{n+1}^m, \tau, \eta) = \left\{v = (v_a) \in \omega(F) : \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |\mu v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0 \text{ , for some } \mu > 0\right\}$, where $v_a = \overline{0}$ for $a < 0$, $\Delta_{n+1}^m |v_a| = \Delta_{n+1}^{m-1}|v_a| - \Delta_{n+1}^{m-1}|v_{a-1}|$ and $\Delta^0 v_a = v_a$, for all $a, n, m \in \mathcal{N}$.

Theorem 3.1. *If $(\tau_a) \in \ell_\infty$, then*

$$\chi_0^F(\Delta_{n+1}^m, \tau, \eta) = \left\{v = (v_a) \in \omega(F) : \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |\mu v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0 \text{ , for any } \mu > 0\right\}.$$

Proof.

$$\chi_0^F(\Delta_{n+1}^m, \tau, \eta) = \left\{v = (v_a) \in \omega(F) : \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |\mu v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0 \text{ , for some } \mu > 0\right\}$$

$$= \left\{v = (v_a) \in \omega(F) : \lim_{a\to\infty} |\mu|^{\frac{\tau_a}{a+1}} \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0 \text{ , for some } \mu > 0\right\}$$

$$= \left\{v = (v_a) \in \omega(F) : \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0\right\}$$

$$= \left\{v = (v_a) \in \omega(F) : \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |\mu v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0 \text{ , for any } \mu > 0\right\}.$$

□

It is clear to see that if $(\tau_a) \in \ell_\infty$, then

$$\lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(a! \left|\Delta_{n+1}^m |v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0 \Rightarrow \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left(\left|\Delta_{n+1}^m |v_a|\right|\right)^{\frac{1}{a+1}}, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0$$
$$\Rightarrow \lim_{a\to\infty} \left[\overline{\rho}\left(\eta_a \left|\Delta_{n+1}^m |v_a|\right|, \overline{0}\right)\right]^{\frac{\tau_a}{K}} = 0.$$

For $X = (X_k)$, a given sequence $S(X)$ denotes the set of all permutation of the elements of (X_k), that is $S(X) = \left\{(X_{\pi(k)})\right\}$.

Definition 3.1. **(1):** *A sequence space of fuzzy numbers $\mathbf{U}$ is said to be symmetric, if $S(X) \in \mathbf{U}$, for all $X \in \mathbf{U}$.*

(2): *A sequence space of fuzzy numbers* $\mathbf{U}$ *is said to be convergence free if* $(Y_k) \in \mathbf{U}$ *whenever* $(X_k) \in \mathbf{U}$ *and* $X_k = \bar{0}$ *implies* $Y_k = \bar{0}$.

Theorem 3.2. *If* $(\tau_a) \in \ell_\infty$*, then the space* $\left(\chi_0^F(\Delta, \tau, \eta)\right)_h$ *is symmetric.*

Proof. It is easy, so omitted. □

Theorem 3.3. *If* $(\tau_a) \in \ell_\infty$*, then the space* $\left(\chi_0^F(\Delta_{n+1}^{m+2}, \tau, \eta)\right)_h$ *is not symmetric.*

Proof. Consider the sequence $(X_k) = (\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \cdots)$. Then $(X_k) \in \left(\chi_0^F(\Delta^2, \tau, \eta)\right)_h$. Now if (Y_k) is the rearrangement of (X_k) defined by $(Y_k) = (\bar{0}, \bar{6}, \bar{2}, \bar{1}, \bar{4}, \bar{3}, \bar{5}, \cdots)$. Then $(Y_k) \notin \left(\chi_0^F(\Delta^2, \tau, \eta)\right)_h$. Therefore, the space $\left(\chi_0^F(\Delta^2, \tau, \eta)\right)_h$ is not symmetric. □

Theorem 3.4. *If* $(\tau_a) \in \ell_\infty$*, then the space* $\left(\chi_0^F(\Delta_{n+1}^{m+1}, \tau, \eta)\right)_h$ *is not convergence free.*

Proof. Consider the sequence $(X_k) = (\bar{1}, \bar{1}, \cdots)$. Then $(X_k) \in \left(\chi_0^F(\Delta^2, \tau, \eta)\right)_h$. Again if $(Y_k) = (\overline{k^2})$. Clearly, $(Y_k) \notin \left(\chi_0^F(\Delta^2, \tau, \eta)\right)_h$. Hence the space $\left(\chi_0^F(\Delta_{n+1}^{m+1}, \tau, \eta)\right)_h$ is not convergence free. □

Let us mark the space of all functions $h : \mathbf{U} \longrightarrow [0, \infty)$ by $[0, \infty)^{\mathbf{U}}$.

Definition 3.2. *[38] If* $\mathbf{U}$ *is a vector space. A function* $h \in [0, \infty]^{\mathbf{U}}$ *is said to be modular if the following conditions hold:*

(a): *Assume* $Y \in \mathbf{U}$, $Y = \overline{\vartheta} \Leftrightarrow h(Y) = 0$ *with* $h(Y) \geq 0$*, where* $\overline{\vartheta} = (\overline{0}, \overline{0}, \overline{0}, \ldots)$,
(b): $h(\eta Z) = h(Z)$ *verifies, for every* $Z \in \mathbf{U}$ *and* $|\eta| = 1$,
(c): *the inequality* $h(\alpha Y + (1-\alpha)Z) \leq h(Y) + h(Z)$ *holds, for every* $Y, Z \in \mathbf{U}$ *and* $\alpha \in [0, 1]$.

Definition 3.3. *[29] The linear space* $\mathbf{U}$ *is called a certain space of sequences of fuzzy numbers (cssf), when*

(1): $\{\overline{b}_q\}_{q \in \mathcal{N}} \subseteq \mathbf{U}$*, where* $\overline{b}_q = (\overline{0}, \overline{0}, ..., \overline{1}, \overline{0}, \overline{0}, \cdots)$*, while* $\overline{1}$ *displays at the* q^{th} *place,*
(2): $\mathbf{U}$ *is solid i.e., if* $Y = (Y_q) \in \omega(F)$, $Z = (Z_q) \in \mathbf{U}$ *and* $|Y_q| \leq |Z_q|$*, for every* $q \in \mathcal{N}$*, then* $Y \in \mathbf{U}$,
(3): $(Y_{[\frac{q}{2}]})_{q=0}^{\infty} \in \mathbf{U}$*, where* $[\frac{q}{2}]$ *denotes the integral part of* $\frac{q}{2}$*, if* $(Y_q)_{q=0}^{\infty} \in \mathbf{U}$.

Definition 3.4. *[29] A subclass* $\mathbf{U}_h$ *of* $\mathbf{U}$ *is said to be a pre-modular (cssf), if there is* $h \in [0, \infty)^{\mathbf{U}}$ *satisfies the following conditions:*

(i): *Assume* $Y \in \mathbf{U}$, $Y = \overline{\vartheta} \Leftrightarrow h(Y) = 0$ *with* $h(Y) \geq 0$*, where* $\overline{\vartheta} = (\overline{0}, \overline{0}, \overline{0}, \ldots)$,
(ii): *one has* $Q \geq 1$*, the inequality* $h(\alpha Y) \leq Q|\alpha| h(Y)$ *holds, for all* $Y \in \mathbf{U}$ *and* $\alpha \in \Re$,
(iii): *one has* $P \geq 1$*, the inequality* $h(Y + Z) \leq P(h(Y) + h(Z))$ *verifies, for all* $Y, Z \in \mathbf{U}$,
(iv): *suppose* $|Y_q| \leq |Z_q|$*, for all* $q \in \mathcal{N}$*, then* $h((Y_q)) \leq h((Z_q))$,
(v): *the inequality,* $h((Y_q)) \leq h((Y_{[\frac{q}{2}]})) \leq P_0 h((Y_q))$ *verifies, for some* $P_0 \geq 1$,
(vi): *if* E *is the space of finite sequences of fuzzy numbers, then the closure of* $E = \mathbf{U}_h$,
(vii): *one has* $\sigma > 0$ *with* $h(\overline{\alpha}, \overline{0}, \overline{0}, \overline{0}, ...) \geq \sigma |\alpha| h(\overline{1}, \overline{0}, \overline{0}, \overline{0}, ...)$*, where*

$$\overline{\alpha}(y) = \begin{cases} 1, & y = \alpha \\ 0, & y \neq \alpha. \end{cases}$$

Note that the notion of pre-modular vector spaces is more general than modular vector spaces. Some examples of pre-modular vector spaces but not modular vector spaces.

Example 3.5. *The function* $h(Z) = \sup_q \left[\overline{\rho}\left(\frac{1}{q+1}\left(\left|\Delta_2^3|Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{4q+4}{3q+4}}$ *is a pre-modular (not a modular) on the vector space* $\chi_0^F\left(\Delta_2^3, \left(\frac{q+1}{3q+4}\right), \left((q+1)^{-1}(q!)^{\frac{-1}{q+1}}\right)\right)$.
As for every $Z, Y \in \chi_0^F\left(\Delta_2^3, \left(\frac{q+1}{3q+4}\right), \left((q+1)^{-1}(q!)^{\frac{-1}{q+1}}\right)\right)$, *one has*

$$h\left(\frac{Z+Y}{2}\right) = \sup_q \left[\overline{\rho}\left(\frac{1}{q+1}\left(\left|\Delta_2^3\left|\frac{Z_q+Y_q}{2}\right|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{4q+4}{3q+4}} \leq 8(h(Z)+h(Y)).$$

Example 3.6. *The function* $h(Z) = \sup_q \left[\overline{\rho}\left(\frac{q+1}{q+2}\left(\left|\Delta|Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{2q+3}{q+4}}$ *is a pre-modular (not a modular) on the vector space* $\chi_0^F\left(\Delta, \left(\frac{2q+3}{q+4}\right), \left((q+1)(q+2)^{-1}(q!)^{\frac{-1}{q+1}}\right)\right)$.
As for every $Z, Y \in \chi_0^F\left(\Delta, \left(\frac{2q+3}{q+4}\right), \left((q+1)(q+2)^{-1}(q!)^{\frac{-1}{q+1}}\right)\right)$, *one has*

$$h\left(\frac{Z+Y}{2}\right) = \sup_q \left[\overline{\rho}\left(\frac{q+1}{q+2}\left(\left|\Delta\left|\frac{Z_q+Y_q}{2}\right|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{2q+3}{q+4}} \leq 2(h(Z)+h(Y)).$$

An example of pre-modular vector space and modular vector space.

Example 3.7. *The function* $h(Y) = \inf\left\{\alpha > 0 : \sup_q \left[\overline{\rho}\left(\frac{q+1}{q+2}\left(\left|\Delta|\frac{Y_q}{\alpha}|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{2q+3}{q+2}} \leq 1\right\}$ *is a pre-modular (modular) on the vector space* $\chi_0^F\left(\Delta, \left(\frac{2q+3}{q+2}\right), \left((q+1)(q+2)^{-1}(q!)^{\frac{-1}{q+1}}\right)\right)$.

Definition 3.8. *[29] If* $\mathbf{U}$ *is a (cssf). The function* $h \in [0,\infty)^{\mathbf{U}}$ *is said to be a pre-quasi norm on* $\mathbf{U}$, *if it verifies the following settings:*

- **(i):** *Suppose* $Y \in \mathbf{U}$, $Y = \overline{\vartheta} \Leftrightarrow h(Y) = 0$ *with* $h(Y) \geq 0$, *where* $\overline{\vartheta} = (\overline{0}, \overline{0}, \overline{0}, \ldots)$,
- **(ii):** *we have* $Q \geq 1$, *the inequality* $h(\alpha Y) \leq Q|\alpha|h(Y)$ *holds, for all* $Y \in \mathbf{U}$ *and* $\alpha \in \Re$,
- **(iii):** *one has* $P \geq 1$, *the inequality* $h(Y+Z) \leq P(h(Y)+h(Z))$ *verifies, for all* $Y, Z \in \mathbf{U}$.

Theorem 3.5. *[29] Suppose* $\mathbf{U}$ *is a pre-modular (cssf), then it is pre-quasi normed (cssf).*

Theorem 3.6. *[29]* $\mathbf{U}$ *is a pre-quasi normed (cssf), if it is quasi-normed (cssf).*

Definition 3.9. **(a):** *The function* h *on* $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$ *is named h-convex, if*

$$h(\alpha Y + (1-\alpha)Z) \leq \alpha h(Y) + (1-\alpha)h(Z),$$

for every $\alpha \in [0,1]$ *and* $Y, Z \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$.

(b): $\{Y_q\}_{q\in\mathcal{N}} \subseteq \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ *is h-convergent to* $Y \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, *if and only if,* $\lim_{q\to\infty} h(Y_q - Y) = 0$. *When the h-limit exists, then it is unique.*

(c): $\{Y_q\}_{q\in\mathcal{N}} \subseteq \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$ *is h-Cauchy, if* $\lim_{q,r\to\infty} h(Y_q - Y_r) = 0$.

(d): $\Gamma \subset \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$ *is h-closed, when for all h-converges* $\{Y_q\}_{a\in\mathcal{N}} \subset \Gamma$ *to* Y*, then* $Y \in \Gamma$.

(e): $\Gamma \subset \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$ *is h-bounded, if* $\delta_h(\Gamma) = \sup\left\{h(Y-Z):\ Y,Z\in\Gamma\right\} < \infty$.

(f): *The h-ball of radius* $\varepsilon \geq 0$ *and center* Y*, for every* $Y \in \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$*, is described as:*

$$\mathbf{B}_h(Y,\varepsilon) = \left\{Z \in \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h : h(Y-Z) \leq \varepsilon\right\}.$$

(g): *A pre-quasi norm h on* $\chi_0^F(\Delta_{n+1}^m,\tau,\eta)$ *holds the Fatou property, if for every sequence* $\{Z^q\} \subseteq \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$ *under* $\lim_{q\to\infty} h(Z^q - Z) = 0$ *and all* $Y \in \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$*, one has* $h(Y-Z) \leq \sup_r \inf_{q\geq r} h(Y - Z^q)$.

Recall that the Fatou property gives the h-closedness of the h-balls. We will denote the space of all increasing and decreasing sequences of real numbers by **I** and **D**, respectively.

Theorem 3.7. $\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$*, where* $h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}}$*, for every* $Y \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$*, is a pre-modular (cssf), if the following conditions are satisfied:*

a.: $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ *with* $\tau_0 > 0$*,*

b.: Δ_{n+1}^m *is an absolute non-decreasing,*

c.: $\left(\eta_q(q!)^{\frac{1}{q+1}}\right)_{q=0}^\infty \in \mathbf{D}$ *or,* $\left(\eta_q(q!)^{\frac{1}{q+1}}\right)_{q=0}^\infty \in \mathbf{I}\cap\ell_\infty$ *and there is* $C \geq 1$ *so that* $\eta_{2q+1}((2q+1)!)^{\frac{1}{2q+2}} \leq C\eta_q(q!)^{\frac{1}{q+1}}$.

Proof. (i) Clearly, $h(Y) \geq 0$ and $h(Y) = 0 \Leftrightarrow Y = \overline{\vartheta}$.
(1-i) Assume $Y,Z \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$. We have

$$\begin{aligned} h(Y+Z) &= \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q+Z_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\leq \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} + \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Z_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} = h(Y)+h(Z) < \infty, \end{aligned}$$

then $Y+Z \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$.
(iii) There are $P \geq 1$ with $h(Y+Z) \leq P(h(Y)+h(Z))$, for every $Y,Z \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$.
(1-ii) If $\alpha \in \Re$ and $Y \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$, one has

$$h(\alpha Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|\alpha Y_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} \leq \sup_q |\alpha|^{\frac{\tau_q}{(q+1)K}} \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} \leq Q|\alpha|h(v) < \infty.$$

Since $\alpha Y \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$, hence from parts (1-i) and (1-ii), we have $\chi_0^F(\Delta_{n+1}^m,\tau,\eta)$ is linear. Also $\overline{b}_p \in \chi_0^F(\Delta_{n+1}^m,\tau,\eta)$, for every $p \in \mathcal{N}$, as $h(\overline{b}_p) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|(\overline{b}_p)_q|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} = \sup_q \left(\eta_q(q!)^{\frac{1}{q+1}}\right)^{\frac{\tau_q}{K}} < \infty$.

(ii) One has $Q = \max\left\{1, \sup_q |\alpha|^{\frac{\tau_q}{(q+1)K}-1}\right\} \geq 1$ with $h(\alpha Y) \leq Q|\alpha|h(Y)$, for every $Y \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$ and $\alpha \in \Re$.

(2) If $|Y_q| \leq |Z_q|$, for every $q \in \mathcal{N}$ and $Z \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$. We obtain

$$h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \leq \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m |Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} = h(Z) < \infty,$$

then $Y \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$.

(iv) Evidently, from (2).

(3) Assume $(Y_q) \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$, $\left(\eta_q(q!)^{\frac{1}{q+1}}\right)_{q=0}^{\infty} \in \mathbf{I} \cap \ell_\infty$ and one has $C \geq 1$ such that $\eta_{2q+1}((2q+1)!)^{\frac{1}{2q+2}} \leq C\eta_q(q!)^{\frac{1}{q+1}}$, one can see

$$\begin{aligned} h\Big((Y_{[\frac{q}{2}]})\Big) &= \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m |Y_{[\frac{q}{2}]}|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\leq \sup_q \left[\overline{\rho}\left(\eta_{2q}\left(2q!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{2q+1}}, \overline{0}\right)\right]^{\frac{\tau_{2q}}{K}} + \sup_q \left[\overline{\rho}\left(\eta_{2q+1}\left((2q+1)!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{2q+2}}, \overline{0}\right)\right]^{\frac{\tau_{2q+1}}{K}} \\ &\leq \sup_q \left[\overline{\rho}\left(\eta_{2q}\left(2q!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{2q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} + \sup_q \left[\overline{\rho}\left(\eta_{2q+1}\left((2q+1)!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{2q+2}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \\ &\leq 2C^{\frac{\sup_q \tau_q}{K}} \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} = 2C^{\frac{\sup_q \tau_q}{K}} h\Big((Y_q)\Big), \end{aligned}$$

then $(Y_{[\frac{q}{2}]}) \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$. (v) From (3), one has $P_0 = 2C^{\frac{\sup_q \tau_q}{K}} \geq 1$.

(vi) Clearly, the closure of $E = \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$.

(vii) One gets $0 < \sigma \leq \sup_q |\alpha|^{\frac{\tau_q}{(q+1)K}-1}$, for $\alpha \neq 0$ or $\sigma > 0$, for $\alpha = 0$ with

$$h(\overline{\alpha}, \overline{0}, \overline{0}, \overline{0}, ...) \geq \sigma|\alpha|h(\overline{1}, \overline{0}, \overline{0}, \overline{0}, ...).$$

□

Theorem 3.8. *If the conditions of theorem 3.7 are satisfied, then* $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ *is a pre-quasi Banach (cssf), where* $h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m |Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}}$, *for all* $Y \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$.

Proof. According to Theorem 3.7 and Theorem 3.5, the space $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ is a pre-quasi normed (cssf). If $Y^l = (Y_q^l)_{q=0}^{\infty}$ is a Cauchy sequence in $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, hence for all $\varepsilon \in (0,1)$, then $l_0 \in \mathcal{N}$ such that for every $l, m \geq l_0$, we have

$$h(Y^l - Y^m) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m |Y_q^l - Y_q^m|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} < \varepsilon.$$

Therefore, $\overline{\rho}\left(\left(\left|\Delta_{n+1}^{m}|Y_q^l - Y_q^m|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right) < \varepsilon$. Since $(\Re[0,1], \overline{\rho})$ is a complete metric space. So (Y_q^m) is a Cauchy sequence in $\Re[0,1]$, for fixed $q \in \mathcal{N}$. This gives $\lim_{m\to\infty} Y_q^m = Y_q^0$, for fixed $q \in \mathcal{N}$. Then $h(Y^l - Y^0) < \varepsilon$, for all $l \geq l_0$. As $h(Y^0) = h(Y^0 - Y^l + Y^l) \leq h(Y^l - Y^0) + h(Y^l) < \infty$. Then $Y^0 \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$. □

Theorem 3.9. *The function* $h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}}$ *holds the Fatou property, when the conditions of theorem 3.7 are satisfied.*

Proof. Let $\{Z^r\} \subseteq \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ such that $\lim_{r\to\infty} h(Z^r - Z) = 0$. Since $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ is a pre-quasi closed space, we have $Z \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$. For every $Y \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, then

$$\begin{aligned}
h(Y-Z) &= \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q - Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \\
&\leq \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q - Z_q^r|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} + \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Z_q^r - Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \\
&\leq \sup_m \inf_{r\geq m} h(Y - Z^r).
\end{aligned}$$

□

Theorem 3.10. *The function* $h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\tau_q}$ *does not hold the Fatou property, for all* $Y \in \chi_0^F(\Delta_{n+1}^m, \tau, \eta)$*, if the conditions of theorem 3.7 are satisfied with* $\tau_0 > 1$.

Proof. Assume $\{Z^r\} \subseteq \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ such that $\lim_{r\to\infty} h(Z^r - Z) = 0$. As $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ is a pre-quasi closed space, we have $Z \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$. For all $Z \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, then

$$\begin{aligned}
h(Y-Z) &= \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q - Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\tau_q} \\
&\leq 2^{\sup_q \tau_q - 1}\left(\sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q - Z_q^r|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\tau_q} + \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Z_q^r - Z_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\tau_q}\right) \\
&\leq 2^{\sup_q \tau_q - 1} \sup_m \inf_{r\geq m} h(Y - Z^r).
\end{aligned}$$

□

Example 3.10. *For* $(\tau_q) \in [1,\infty)^{\mathcal{N}}$*, the function* $h(Y) = \inf\left\{\alpha > 0 : \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m\left|\frac{Y_q}{\alpha}\right|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\tau_q} \leq 1\right\}$ *is a norm on* $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$.

Example 3.11. *The function* $h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{3q+2}{3q+3}}$ *is a pre-quasi norm (not a norm) on* $\chi_0^F(\Delta_{n+1}^m, (\frac{3q+2}{q+1})_{q=0}^{\infty}), \eta)$.

Example 3.12. *The function* $h(Y) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}|Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{3q+2}{q+1}}$ *is a pre-quasi norm (not a quasi norm) on* $\chi^F_0(\Delta^m_{n+1}, (\frac{3q+2}{q+1})^\infty_{q=0}), \eta)$.

Example 3.13. *The function* $h(Y) = \sup_q \overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}|Y_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)$ *is a pre-quasi norm, quasi norm and not a norm on* $\chi^F_0(\Delta^m_{n+1}, (d), \eta)$*, for* $0 < d < 1$.

4. Structure of mappings' ideal

The structure of the mappings' ideal by $\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h$, where $h(g) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}|g_q|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}}$, for all $g \in \chi^F_0(\Delta^m_{n+1}, \tau, \eta)$, and extended $s-$fuzzy functions have been explained. We study enough setups on $\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h$ such that the class $\overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$ is complete. We investigate conditions setups (not necessary) on $\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h$ such that the closure of $\mathfrak{F} = \overline{\maltese}^{\alpha}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$. This gives a negative answer of Rhoades [39] open problem about the linearity of $s-$ type $\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h$ spaces. We explain enough setups on $\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h$ such that $\overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$ is strictly contained for different powers, weights and backward generalized differences, the class $\overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$ is simple, and the space of every bounded linear mappings which sequence of eigenvalues in $\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h$ equals $\overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$.

Theorem 4.1. *[29] If* $\mathbf{U}$ *is a (cssf), then* $\overline{\maltese}_{\mathbf{U}}$ *is a mappings' ideal.*

In view of Theorem 3.7 and Theorem 4.1, one has the following theorem.

Theorem 4.2. *If the conditions of theorem 3.7 are satisfied, then* $\overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$ *is a mappings' ideal.*

Theorem 4.3. *If the conditions of theorem 3.7 are satisfied, then the function* H *is a pre-quasi norm on* $\overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}$*, with* $H(Z) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}|\overline{s_q(Z)}|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}}$*, for every* $Z \in \overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}(\Omega, \Lambda)$.

Proof. **(1):** Suppose $X \in \overline{\maltese}_{\left(\chi^F_0(\Delta^m_{n+1}, \tau, \eta)\right)_h}(\Omega, \Lambda)$, $H(X) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}|\overline{s_q(X)}|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \geq 0$ and $H(X) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}|\overline{s_q(X)}|\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} = 0$, if and only if, $\overline{s_q(X)} = \overline{0}$, for all $q \in \mathcal{N}$, if and only if, $X = 0$,

(2): one has $Q \geq 1$ with $H(\alpha X) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}\overline{|s_q(\alpha X)|}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \leq Q|\alpha|H(X)$, for all $X \in \maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}(\Omega,\Lambda)$ and $\alpha \in \Re$,

(3): there are $PP_0 \geq 1$ so that for $X_1, X_2 \in \maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}(\Omega,\Lambda)$, we have

$$\begin{aligned} H(X_1+X_2) &= \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}\overline{|s_q(X_1+X_2)|}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \leq P\left(h(\overline{s_{[\frac{q}{2}]}(X_1)})^\infty_{q=0} + h(\overline{s_{[\frac{q}{2}]}(X_2)})^\infty_{q=0}\right) \\ &\leq PP_0\left(h(\overline{s_q(X_1)})^\infty_{q=0} + h(\overline{s_q(X_2)})^\infty_{q=0}\right), \end{aligned}$$

(4): there are $\varrho \geq 1$, if $X \in \mathcal{L}(\Delta_0,\Delta)$, $Y \in \maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}(\Omega,\Lambda)$ and $Z \in \mathcal{L}(\Lambda,\Lambda_0)$, then

$$H(ZYX) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}\overline{|s_q(ZYX)|}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \leq h(\ \|X\|\ \|Z\|\overline{s_q(Y)})^\infty_{q=0} \leq \varrho\ \|X\|H(Y)\ \|Z\|.$$

□

In the next theorems, we will use the notation $\left(\maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}, H\right)$, where $H(V) = h\left((\overline{s_q(V)})^\infty_{q=0}\right)$, for all $V \in \maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}$.

Theorem 4.4. *Assume the conditions of theorem 3.7 are satisfied, then* $\left(\maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}, H\right)$ *is a pre-quasi Banach mappings' ideal.*

Proof.
Let $(V_a)_{a\in\mathcal{N}}$ be a Cauchy sequence in $\maltese_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}(\Omega,\Lambda)$. Since $\mathcal{L}(\Omega,\Lambda) \supseteq S_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}(\Omega,\Lambda)$, then

$$H(V_r - V_a) = \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta^m_{n+1}\overline{|s_q(V_r-V_a)|}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \geq h\left(\overline{s_0(V_r-V_a)}, \overline{0}, \overline{0}, \overline{0}, ...\right) \geq \eta_0^{\frac{\tau_0}{K}}\ \|V_r - V_a\|^{\frac{\tau_0}{K}},$$

this implies $(V_a)_{a\in\mathcal{N}}$ is a Cauchy sequence in $\mathcal{L}(\Omega,\Lambda)$. Since $\mathcal{L}(\Omega,\Lambda)$ is a Banach space, one has $V \in \mathcal{L}(\Omega,\Lambda)$ such that $\lim_{a\to\infty} \|V_a - V\| = 0$ and as $(\overline{s_q(V_a)})^\infty_{q=0} \in \left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h$, for every $a \in \mathcal{N}$ and $\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h$ is a pre-modular (cssf). Then we have

$$\begin{aligned} H(V) = h\left((\overline{s_q(V)})^\infty_{q=0}\right) &\leq h\left((\overline{s_{[\frac{q}{2}]}(V-V_a)})^\infty_{q=0}\right) + h\left((\overline{s_{[\frac{q}{2}]}(V_a)}^\infty_{q=0})\right) \\ &\leq h\left((\ \|V_a - V\ \|\overline{1})^\infty_{q=0}\right) + 2C^{\frac{\sup_q \tau_q}{K}} h\left((\overline{s_q(V_a)})^\infty_{q=0}\right) < \varepsilon, \end{aligned}$$

for some $C \geq 1$, hence one has $(\overline{s_q(V)})_{q=0}^{\infty} \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, hence $V \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda)$. □

Definition 4.1. *A pre-quasi norm H on the ideal $\overline{\maltese}_{\mathbf{U}_h}$ holds the Fatou property if for all $\{T_q\}_{q\in\mathcal{N}} \subseteq \overline{\maltese}_{\mathbf{U}_h}(\Omega, \Lambda)$ such that $\lim_{q\to\infty} H(T_q - T) = 0$ and $M \in \overline{\maltese}_{\mathbf{U}_h}(\Omega, \Lambda)$, then*

$$H(M - T) \leq \sup_q \inf_{j\geq q} H(M - T_j).$$

Theorem 4.5. *If the conditions of theorem 3.7 are satisfied, then $\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}, H\right)$ does not hold the Fatou property.*

Proof. Let $\{T_q\}_{q\in\mathcal{N}} \subseteq \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda)$ with $\lim_{q\to\infty} H(T_q - T) = 0$. Since $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}$ is a pre-quasi closed ideal, then $T \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda)$, hence for all $M \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda)$, we have

$$\begin{aligned}
H(M - T) &= \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m \overline{s_q(M-T)}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \\
&\leq \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m \overline{s_{[\frac{q}{2}]}(M-T_i)}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} + \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m \overline{s_{[\frac{q}{2}]}(T_i-T)}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}} \\
&\leq 2C^{\frac{\sup_q \tau_q}{K}} \sup_r \inf_{i\geq r} \sup_q \left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m \overline{s_{[\frac{q}{2}]}(M-T_i)}\right|\right)^{\frac{1}{q+1}}, \overline{0}\right)\right]^{\frac{\tau_q}{K}}.
\end{aligned}$$

□

Theorem 4.6. $\overline{\maltese^{\alpha}}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda) =$ *the closure of $\mathfrak{F}(\Omega, \Lambda)$, if the conditions of theorem 3.7 are satisfied. But the converse is not necessarily true.*

Proof. As $\overline{b}_x \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, for all $x \in \mathcal{N}$ and $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ is a linear space. If $Z \in \mathfrak{F}(\Omega, \Lambda)$, one has $(\overline{\alpha_x(Z)})_{x=0}^{\infty} \in E$. Then the closure of $\mathfrak{F}(\Omega, \Lambda) \subseteq \overline{\maltese^{\alpha}}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda)$. Suppose $Z \in \overline{\maltese^{\alpha}}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega, \Lambda)$, one has $(\overline{\alpha_x(Z)})_{x=0}^{\infty} \in \left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$. Since $h(\overline{\alpha_x(Z)})_{x=0}^{\infty} < \infty$, if $\rho \in (0, 1)$, one has $x_0 \in \mathcal{N} - \{0\}$ so that $h((\overline{\alpha_x(Z)})_{x=x_0}^{\infty}) < \frac{\rho}{4}$. As $(\overline{\alpha_x(Z)})_{x=0}^{\infty}$ is decreasing, one gets

$$\begin{aligned}
&\sup_{x=x_0+1}^{2x_0} \left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta_{n+1}^m \overline{\alpha_{2x_0}(Z)}\right|\right)^{\frac{1}{x+1}}, \overline{0}\right)\right]^{\frac{\tau_x}{K}} \leq \sup_{x=x_0+1}^{2x_0} \left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta_{n+1}^m \overline{\alpha_x(Z)}\right|\right)^{\frac{1}{x+1}}, \overline{0}\right)\right]^{\frac{\tau_x}{K}} \\
&\leq \sup_{x=x_0}^{\infty} \left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta_{n+1}^m \overline{\alpha_x(Z)}\right|\right)^{\frac{1}{x+1}}, \overline{0}\right)\right]^{\frac{\tau_x}{K}} < \frac{\rho}{4}.
\end{aligned} \tag{4.1}$$

Then one has $Y \in \mathfrak{F}_{2x_0}(\Omega, \Lambda)$ such that $\text{rank}(Y) \leq 2x_0$ and

$$\sup_{x=2x_0+1}^{3x_0}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\|Z-Y\|}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} \leq \sup_{x=x_0+1}^{2x_0}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\|Z-Y\|}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} < \frac{\rho}{4}, \tag{4.2}$$

as $(\tau_q)_{q\in\mathcal{N}} \in \ell_\infty \cap \mathbf{I}$ with $\tau_0 > 0$, take

$$\sup_{x=0}^{x_0}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\|Z-Y\|}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} < \frac{\rho}{4}. \tag{4.3}$$

According to inequalities 4.1-4.3, then

$$\begin{aligned}
d(Z,Y) &= \sup_{x=0}^{\infty}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\alpha_x(Z-Y)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} \\
&\leq \sup_{x=0}^{3x_0-1}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\alpha_x(Z-Y)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=3x_0}^{\infty}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\alpha_x(Z-Y)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} \\
&\leq \sup_{x=0}^{3x_0}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\|Z-Y\|}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} + \\
&\sup_{x=x_0}^{\infty}\left[\overline{\rho}\left(\eta_{x+2x_0}\left((x+2x_0)!\left|\Delta^m_{n+1}\overline{\alpha_{x+2x_0}(Z-Y)}\right|\right)^{\frac{1}{x+2x_0+1}},\overline{0}\right)\right]^{\frac{\tau_{x+2x_0}}{K}} \\
&\leq \sup_{x=0}^{3x_0}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\|Z-Y\|}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=x_0}^{\infty}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\alpha_x(Z)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} \\
&\leq 3\sup_{x=0}^{x_0}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\|Z-Y\|}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} + \sup_{x=x_0}^{\infty}\left[\overline{\rho}\left(\eta_x\left(x!\left|\Delta^m_{n+1}\overline{\alpha_x(Z)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\frac{\tau_x}{K}} < \rho.
\end{aligned}$$

This implies $\overline{\maltese}^{\alpha}_{\left(\chi_0^F(\Delta^m_{n+1},\tau,\eta)\right)_h}(\Omega,\Lambda) \subseteq$ the closure of $\mathfrak{F}(\Omega,\Lambda)$. Contrarily, one has a counter example as $I_6 \in \overline{\maltese}^{\alpha}_{\left(\chi_0^F\left(\Delta,(0,0,1,1,\cdots),\left((x+1)^{-1}(x!)^{\frac{-1}{x+1}}\right)\right)\right)}(\Omega,\Lambda)$, but $\tau_0 > 0$ is not satisfied. □

Theorem 4.7. *Assume the conditions of theorem 3.7 are satisfied with $\tau_x^{(1)} < \tau_x^{(2)}$ and $\eta_x^{(2)} < \eta_x^{(1)}$, for every $x \in \mathcal{N}$, then*

$$\overline{\maltese}_{\left(\chi_0^F(\Delta^m_{n+2},(\tau_x^{(1)}),(\eta_x^{(1)})\right)_h}(\Omega,\Lambda) \subsetneq \overline{\maltese}_{\left(\chi_0^F(\Delta^{m+1}_{n+1},(\tau_x^{(2)}),(\eta_x^{(2)})\right)_h}(\Omega,\Lambda) \subsetneq \mathcal{L}(\Omega,\Lambda).$$

Proof. Suppose $Z \in \overline{\maltese}_{\left(\chi_0^F(\Delta^m_{n+2},(\tau_x^{(1)}),(\eta_x^{(1)})\right)_h}(\Omega,\Lambda)$, then $(\overline{s_x(Z)}) \in \left(\chi_0^F(\Delta^m_{n+2},(\tau_x^{(1)}),(\eta_x^{(1)})\right)_h$. We have

$$\sup_x\left[\overline{\rho}\left(\eta_x^{(2)}\left(x!\left|\Delta^{m+1}_{n+1}\overline{s_x(Z)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\tau_x^{(2)}} < \sup_x\left[\overline{\rho}\left(\eta_x^{(1)}\left(x!\left|\Delta^m_{n+2}\overline{s_x(Z)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\tau_x^{(1)}} < \infty,$$

then $Z \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda)$. Next, if we take $(\overline{s_x(Z)})_{x=0}^{\infty}$ with $\left(\Delta_{n+2}^{m}\overline{s_x(Z)}\right) = (\overline{1},\overline{1},\cdots)$, hence $\left(\Delta_{n+1}^{m+1}\overline{s_x(Z)}\right) = (\overline{1},\overline{0},\overline{0},\cdots)$, one has $Z \in \mathcal{L}(\Omega,\Lambda)$ so that

$$\lim_{x\to\infty}\left[\overline{\rho}\left(\eta_x^{(1)}\left(x!\left|\Delta_{n+2}^{m}\overline{s_x(Z)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\tau_x^{(1)}} \neq 0,$$

and

$$\lim_{x\to\infty}\left[\overline{\rho}\left(\eta_x^{(2)}\left(x!\left|\Delta_{n+1}^{m+1}\overline{s_x(Z)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\tau_x^{(2)}} = 0.$$

Therefore, $Z \notin \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)$ and $Z \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda)$.
Evidently, $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda) \subset \mathcal{L}(\Omega,\Lambda)$. After, if we choose $(\overline{s_x(Z)})_{x=0}^{\infty}$ so that $\left(\Delta_{n+1}^{m+1}\overline{s_x(Z)}\right) = (\overline{1},\overline{1},\cdots)$. One has $Z \in \mathcal{L}(\Omega,\Lambda)$ such that $Z \notin \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda)$. □

Lemma 4.1. *[3] Suppose $B \in \mathcal{L}(\Omega,\Lambda)$ and $B \notin \Upsilon(\Omega,\Lambda)$, then $D \in \mathcal{L}(\Omega)$ and $M \in \mathcal{L}(\Lambda)$ with $MBDe_b = e_b$, with $b \in \mathcal{N}$.*

Theorem 4.8. *[3] In general, one has*

$$\mathfrak{F}(\Omega) \subsetneq \Upsilon(\Omega) \subsetneq \mathcal{L}_c(\Omega) \subsetneq \mathcal{L}(\Omega).$$

Theorem 4.9. *If the conditions of theorem 3.7 are satisfied with $\tau_x^{(1)} < \tau_x^{(2)}$ and $\eta_x^{(2)} < \eta_x^{(1)}$, for all $x \in \mathcal{N}$, then*

$$\begin{aligned}&\mathcal{L}\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)\right)\\ &= \Upsilon\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)\right).\end{aligned}$$

Proof. Let $X \in \mathcal{L}\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)\right)$ and $X \notin \Upsilon\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)\right)$. In view of Lemma 4.1, one has $Y \in \mathcal{L}\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda)\right)$ and $Z \in \mathcal{L}\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)\right)$ so that $ZXYI_b = I_b$, then with $b \in \mathcal{N}$, we have

$$\begin{aligned}\|I_b\|_{\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)}))\right)_h}(\Omega,\Lambda)} &= \sup_x\left[\overline{\rho}\left(\eta_x^{(1)}\left(x!\left|\Delta_{n+2}^{m}\overline{s_x(I_b)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\tau_x^{(1)}}\\ &\leq \|ZXY\|\|I_b\|_{\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)}))\right)_h}(\Omega,\Lambda)}\\ &\leq \sup_x\left[\overline{\rho}\left(\eta_x^{(2)}\left(x!\left|\Delta_{n+1}^{m+1}\overline{s_x(I_b)}\right|\right)^{\frac{1}{x+1}},\overline{0}\right)\right]^{\tau_x^{(2)}}.\end{aligned}$$

This contradicts Theorem 4.7. As $X \in \Upsilon\Big(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)})\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)})\right)_h}(\Omega,\Lambda)\Big)$. □

Corollary 4.1. *Suppose the conditions of theorem 3.7 are satisfied with $\tau_x^{(1)} < \tau_x^{(2)}$ and $\eta_x^{(2)} < \eta_x^{(1)}$, for every $x \in \mathcal{N}$, then*

$$\mathcal{L}\Big(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)})\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)})\right)_h}(\Omega,\Lambda)\Big)$$
$$=\mathcal{L}_c\Big(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1},(\tau_x^{(2)}),(\eta_x^{(2)})\right)_h}(\Omega,\Lambda),\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+2}^{m},(\tau_x^{(1)}),(\eta_x^{(1)})\right)_h}(\Omega,\Lambda)\Big).$$

Proof. Obviously, since $\Upsilon \subset \mathcal{L}_c$. □

Definition 4.2. *[3] A Banach space Ω is said to be simple, if there is only one non-trivial closed ideal in $\mathcal{L}(\Omega)$.*

Theorem 4.10. *Assume the conditions of theorem 3.7 are verified, then $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}$ is simple.*

Proof. Let $X \in \mathcal{L}_c(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda))$ and $X \notin \Upsilon(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda))$. From Lemma 4.1, there exist $Y, Z \in \mathcal{L}(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda))$ with $ZXYI_b = I_b$.

This implies $I_{\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda)} \in \mathcal{L}_c(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda))$. Then $\mathcal{L}(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda)) = \mathcal{L}_c(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda))$, then $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}$ is simple Banach space. □

Notations 4.3.

$\left(\overline{\maltese}_{\mathbf{U}}\right)^{\lambda} := \left\{\left(\overline{\maltese}_{\mathbf{U}}\right)^{\lambda}(\Omega,\Lambda);\ \Delta \text{and } \Lambda \text{ are Banach Spaces}\right\}$, *where*

$\left(\overline{\maltese}_{\mathbf{U}}\right)^{\lambda}(\Omega,\Lambda) := \left\{X \in \mathcal{L}(\Omega,\Lambda) : ((\lambda_x(X))_{x=0}^{\infty} \in \mathbf{U} \text{ and } \|X - \overline{\rho}(\lambda_x(X),\overline{0})I\| \text{ is not invertible, with } x \in \mathcal{N}\right\}$.

Theorem 4.11. *If the conditions of theorem 3.7 are satisfied, then*

$$\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}\right)^{\lambda}(\Omega,\Lambda) = \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}(\Omega,\Lambda).$$

Proof.

Let $X \in \left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h}\right)^{\lambda}(\Omega,\Lambda)$, then $(\lambda_x(X))_{x=0}^{\infty} \in \left(\chi_0^F(\Delta_{n+1}^{m},\tau,\eta)\right)_h$ and $\|X - \overline{\rho}(\lambda_x(X),\overline{0})I\| = 0$, for all $x \in \mathcal{N}$. Therefore, $\lim_{q\to\infty}\left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^{m}|\lambda_q(X)\|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}} = 0$. One has $X = \overline{\rho}(\lambda_x(X),\overline{0})I$, for every $x \in \mathcal{N}$, so

$$\overline{\rho}(\overline{s_x(X)},\overline{0}) = \overline{\rho}(\overline{s_x(\overline{\rho}(\lambda_x(X),\overline{0})I)},\overline{0}) = \overline{\rho}(\lambda_x(X),\overline{0}),$$

for every $x \in \mathcal{N}$. Hence $(\overline{s_x(X)})_{x=0}^{\infty} \in \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$, then $X \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega,\Lambda)$. After, assume $X \in \overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega,\Lambda)$. Hence $(\overline{s_x(X)})_{x=0}^{\infty} \in \left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h$. We have

$$\lim_{q\to\infty}\left[\overline{\rho}\left(\eta_q\left(q!\left|\Delta_{n+1}^m|\overline{s_q(X)}|\right|\right)^{\frac{1}{q+1}},\overline{0}\right)\right]^{\frac{\tau_q}{K}}=0.$$

As Δ_{n+1}^m is continuous, then $\lim_{x\to\infty}\overline{\rho}(\overline{s_x(X)},\overline{0})=0$. Suppose $\|X-\overline{\rho}(\overline{s_x(X)},\overline{0})I\|^{-1}$ exists, with $x \in \mathcal{N}$. Hence $\|X-\overline{\rho}(\overline{s_x(X)},\overline{0})I\|^{-1}$ exists and bounded, for every $x \in \mathcal{N}$. As $\lim_{x\to\infty}\|X-\overline{\rho}(\overline{s_x(X)},\overline{0})I\|^{-1}=\|X\|^{-1}$ exists and bounded. As $\left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h},H\right)$ is a pre-quasi Mappings' ideal, one gets

$$I=XX^{-1}\in\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}(\Omega,\Lambda)\Rightarrow(\overline{s_x(I)})_{x=0}^{\infty}\in\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\Rightarrow\lim_{x\to\infty}\overline{\rho}(\overline{s_x(I)},\overline{0})=0.$$

We have a contradiction, since $\lim_{x\to\infty}\overline{\rho}(\overline{s_x(I)},\overline{0})=1$. Then $\|X-\overline{\rho}(\overline{s_x(X)},\overline{0})I\|=0$, with $x \in \mathcal{N}$. Which proves that $X \in \left(\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m,\tau,\eta)\right)_h}\right)^{\lambda}(\Omega,\Lambda)$. □

Theorem 4.12. *For $s-$ type $\mathbf{U}_h := \left\{f=(\overline{s_r(X)})\in\omega(F):\ X\in\mathcal{L}(\Omega,\Lambda) \text{ and } h(f)<\infty\right\}$. If $\overline{\maltese}_{\mathbf{U}_h}$ is a mappings' ideal, then the following conditions are verified:*

1. *$E \subset s-$ type $\mathbf{U}_h$.*
2. *Assume $\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$ and $\left(\overline{s_r(X_2)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$, then $\left(\overline{s_r(X_1+X_2)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$.*
3. *If $\lambda\in\Re$ and $\left(\overline{s_r(X)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$, then $|\lambda|\left(\overline{s_r(X)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$.*
4. *The sequence space $\mathbf{U}_h$ is solid. i.e., if $\left(\overline{s_r(Y)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$ and $\overline{s_r(X)}\le\overline{s_r(Y)}$, for all $r\in\mathcal{N}$ and $X,Y\in\mathcal{L}(\Omega,\Lambda)$, then $\left(\overline{s_r(X)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$.*

Proof. If $\overline{\maltese}_{\mathbf{U}_h}$ is a mappings' ideal.

(i): We have $\mathfrak{F}(\Omega,\Lambda)\subset\overline{\maltese}_{\mathbf{U}_h}(\Omega,\Lambda)$. Hence for all $X\in\mathfrak{F}(\Omega,\Lambda)$, we have $\left(\overline{s_r(X)}\right)_{r=0}^{\infty}\in E$. This gives $\left(\overline{s_r(X)}\right)_{r=0}^{\infty}\in s-$ type $\mathbf{U}_h$. Hence $E\subset s-$ type $\mathbf{U}_h$.

(ii): The space $\overline{\maltese}_{\mathbf{U}_h}(\Omega,\Lambda)$ is linear over $\Re$. Hence for each $\lambda\in\Re$ and $X_1,X_2\in\overline{\maltese}_{\mathbf{U}_h}(\Omega,\Lambda)$, we have $X_1+X_2\in\overline{\maltese}_{\mathbf{U}_h}(\Omega,\Lambda)$ and $\lambda X_1\in\overline{\maltese}_{\mathbf{U}_h}(\Omega,\Lambda)$. This implies

$$\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-\text{type }\mathbf{U}_h\text{ and }\left(\overline{s_r(X_2)}\right)_{r=0}^{\infty}\in s-\text{type }\mathbf{U}_h\ \Rightarrow\left(\overline{s_r(X_1+X_2)}\right)_{r=0}^{\infty}\in s-\text{type }\mathbf{U}_h$$

and

$$\lambda\in\Re\text{ and }\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-\text{type }\mathbf{U}_h\Rightarrow|\lambda|\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty}\in s-\text{type }\mathbf{U}_h.$$

(iii): If $A \in \mathcal{L}(\Omega_0, \Omega)$, $B \in \overline{\maltese}_{\mathbf{U}_h}(\Omega, \Lambda)$ and $D \in \mathcal{L}(\Lambda, \Lambda_0)$, then $DBA \in \overline{\maltese}_{\mathbf{U}_h}(\Omega_0, \Lambda_0)$, where Ω_0 and Λ_0 are arbitrary Banach spaces. Therefore, since $\left(\overline{s_r(B)}\right)_{r=0}^{\infty} \in s-$type $\mathbf{U}_h$, then $\left(\overline{s_r(DBA)}\right)_{r=0}^{\infty} \in s-$type $\mathbf{U}_h$. Since $\overline{s_r(DBA)} \leq \|D\|\overline{s_r(B)}\ \|A\|$. By using condition 3, if $\left(\|D\|\ \|A\|\overline{s_r(B)}\right)_{r=0}^{\infty} \in \mathbf{U}_h$, we have $\left(\overline{s_r(DBA)}\right)_{r=0}^{\infty} \in s-$type $\mathbf{U}_h$. This means $s-$type $\mathbf{U}_h$ is solid.

□

In view of Theorem 4.2 and Theorem 4.12, we conclude the following properties of the $s-$type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ space.

Theorem 4.13. *If $s-$type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h := \left\{f = (\overline{s_r(X)}) \in \omega(F) :\ X \in \mathcal{L}(\Omega, \Lambda) \text{ and } h(f) < \infty\right\}$, then the following conditions are verified:*

1. *$E \subset s-$ type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$.*
2. *Assume $\left(\overline{s_r(X_1)}\right)_{r=0}^{\infty} \in s-$ type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ and $\left(\overline{s_r(X_2)}\right)_{r=0}^{\infty} \in s-$ type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, then $\left(\overline{s_r(X_1 + X_2)}\right)_{r=0}^{\infty} \in s-$ type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$.*
3. *If $\lambda \in \Re$ and $\left(\overline{s_r(X)}\right)_{r=0}^{\infty} \in s-$type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$, then $|\lambda|\left(\overline{s_r(X)}\right)_{r=0}^{\infty} \in s-$type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$.*
4. *The sequence space $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ is solid. i.e., if $\left(\overline{s_r(Y)}\right)_{r=0}^{\infty} \in s-$ type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ and $\overline{s_r(X)} \leq \overline{s_r(Y)}$, for all $r \in \mathcal{N}$ and $X, Y \in \mathcal{L}(\Omega, \Lambda)$, then $\left(\overline{s_r(X)}\right)_{r=0}^{\infty} \in s-$ type $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$.*

Theorem 4.14. *The space $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^{m+1}, \tau, \eta)\right)_h}$ is not mappings' ideal, if the conditions (a) and (c) of theorem 3.7 are satisfied*

Proof. If we choose $m = 1$, $n = 1$, $w_k = \overline{1}$, $v_k = w_k$ for $k = 3s$ or $v_k = \overline{0}$, otherwise, for all $s, k \in \mathcal{N}$. We have $|v_k| \leq |w_k|$, for all $k \in \mathcal{N}$, $w \in \left(\chi_0^F(\Delta_2^2, \tau, \eta)\right)_h$ and $v \notin \left(\chi_0^F(\Delta_2^2, \tau, \eta)\right)_h$. Hence, the space $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ is not solid. □

5. Conclusion

In this paper, we have explained sufficient settings of the space $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$ equipped with definite function h to be pre-quasi Banach (cssf). The Fatou property of various pre-quasi norms h on $\chi_0^F(\Delta_{n+1}^m, \tau, \eta)$ has been investigated. The structure of the mappings' ideal by this space and extended $s-$fuzzy functions have been explained. We study enough setups on it such that the class $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h}$ is simple Banach and the closure of $\mathfrak{F} = \overline{\maltese}^{\alpha}_{\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h}$. We explain enough setups on it such that $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h}$ is strictly contained for different powers, weights and backward generalized differences and the space of every bounded linear mappings which sequence of eigenvalues in $\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h$ equals $\overline{\maltese}_{\left(\chi_0^F(\Delta_{n+1}^m, \tau, \eta)\right)_h}$. The existence results may be established under a wide range of flexible conditions. When it comes to the variable exponent in the above-mentioned space. Since many fixed point theorems in a particular space work by either expanding the self-mapping acting on it or expanding the space itself, as future work we can enlarge this space by $q-$analogue or discuss the fixed points of any contraction self-mapping acting on it and try to find the solutions for a class of non-linear summable and matrix equations of fuzzy functions in this space.

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