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Difference Cesáro Function Space on Rooted Tree Defined by Musielak-Orlicz Function

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Abstract. This paper aims to investigate the algebraic and topological properties of a newly constructed difference function space on a rooted tree defined by Musielak-Orlicz function.

1. Introduction

A function $\mathcal{M}$ from $[0, \infty)$ to itself which is continuous, non-decreasing and convex such that $\mathcal{M}(0) = 0, \mathcal{M}(\zeta) > 0$ for $\zeta > 0$ and $\mathcal{M}(\zeta) \to \infty$ as $\zeta \to \infty$ is known as Orlicz function.

In [11] Lindenstrauss and Tzafriri, defined the sequence space, denoted by $l_{\mathcal{M}}$, such that $\sum_{j=1}^{\infty} \mathcal{M}(\frac{\zeta_j}{\lambda}) < \infty$.

This space is called Orlicz sequence space and is Banach space equipped with the norm

$$||\zeta_j|| = \inf\{\lambda > 0 : \sum_{j=1}^{\infty} \mathcal{M}(\frac{\zeta_j}{\lambda}) \leq 1\}.$$

A sequence of Orlicz function is referred to as Musielak-Orlicz function (see [14, 15]).For further information on Cesàro sequence spaces and sequence spaces defined by Musielak-Orlicz functions, refer to ([10, 16–18, 26]) and references therein.

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For $1 < t < \infty$, the Cesáro sequence space Ces_t of real sequences (χ_j) is defined by,

$$Ces_t = \{\chi = (\chi_j) : \sum_{i=1}^{\infty} (\frac{1}{i} \sum_{j=1}^{i} |\chi_j|)^t < \infty\}$$

is a Banach space under the norm

$$||\chi_j|| = \left(\sum_{i=1}^{\infty} (\frac{1}{i} \sum_{j=1}^{i} |\chi_j|)^t \right)^{\frac{1}{t}}$$

This space is significant in the theory of matrix operators and was initially presented by Shiue [23]. Various authors have explored some geometric characteristics of this space. Kızmaz [8] introduced the concept of difference space, which was later generalized by Et. and Çolak [5] into the difference sequence space as follows:

$$Z(\Delta) = \{\chi = (\chi_j) \in \omega : (\Delta^{\nu} \chi_j) \in F\}$$

for $F = l_{\infty},\ c$ and c_0, where ν is non-negative integer and

$$\Delta^{\nu} \chi_j \Delta^{\nu-1} \chi_j - \Delta^{\nu-1} \chi_{j-1}, \Delta^0 \chi_j = \chi_j \ for\ all\ j \in \mathbb{N},$$

or equivalently,

$$\Delta^{\nu} \chi_j = \sum_{w=0}^{j} (-1)^{w} \binom{u}{w} \chi_{j+w}$$

Et. and Başasir [5] extended these spaces by considering $F = l_{\infty}(p),\ c(p)$ and $c_0(p)$.
Dutta [4] introduced the following difference sequence spaces using a new difference operator.

$$Z(\Delta_{\eta}) = \{\chi = (\chi_j) \in \omega : \Delta_n(\chi) \in F\}$$

for $F = l_{\infty},\ c$ and c_0, where $\Delta_{\eta}\chi = (\Delta_{\eta}\chi_j) = \{\chi_j - \chi_{j-n}\}$ for all $k, n \in \mathbb{N}$.
Başar and Atlay [1] introduced the generalized difference matrix $B = (b_{\eta j})$ for all $j, \eta \in \mathbb{N}$, which generalizes the $\Delta_{(1)}$-difference operator, by

$$b_{\eta j} = \begin{cases} \alpha & \text{when} \quad j = \eta \\ \beta & \text{when} \quad j = \eta - 1 \\ 0 & \text{when} \quad j > \eta \ or \ (0 \leq j < \eta - 1) \end{cases}$$

Başarir and Kayikçi [2] defined the matrix $B^{\nu} = (b^{\nu}_{\eta k})$ which simplifies the difference matrix Δ^{ν}_{1} for the case $\alpha = 1,\ \beta = 1$. The generalized B^{μ}-difference operator is equivalent to the following binomial representation:

$$B^{\nu}\chi = B^{\nu}(\chi_j) = \sum_{w}^{\nu} \binom{\nu}{0} r^{\nu-w} s^{w} \chi_{j-w}.$$

Recall that if any two vertex of graph is joined by a unique path then it is called tree, denoted by T. A tree with root o is called *rooted* tree and number of edges between root o and vertex χ is called order of χ which is denoted by $|\chi|$. Let c_i denotes the number of vertices whose order is i, for $i \in \mathbb{N}_0$. For more details, we refer to ([19,20,23]) and references therein.

Let $\mathcal{M} = (\mathfrak{I}_i)$ be Musielak-Orlicz function. For bounded sequence $t = (t_i)$ consisting of positive real numbers, we estalish the difference Cesáro function space on rooted tree T defined by Musielak-Orlicz function as follows:

$$Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t) = \{\mathfrak{f} : T \to \mathbb{C} : \sum_{i=0}^{\infty}\left[\mathfrak{I}_i\left(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\right)\right]^{t_i} < \infty \text{ for some } \phi > 0\}$$

We generalised this space as:

$$Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t) = \{\mathfrak{f} : T \to \mathbb{C} : \sum_{i=0}^{\infty}\left[\mathfrak{I}_i\left(\frac{1}{i+1}\sum_{j=1}^{i}\frac{1}{c_j}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\right)\right]^{t_i} < \infty \text{ for some } \phi > 0\}.$$

For any set S of function the space of multipliers of S, denoted by $M(S)$, is given by

$$M(S) = \{\mathfrak{f} : T \to \mathbb{C} : fg \in S \text{ for all } g \in S\}.$$

The following inequalities are used throughout the paper. Let $t = (t_i)$ be bounded sequence of strictly positive real numbers. If $H = \sup_i(t_i)$ $C = \max(1, 2^{H-1})$, then for any complex numbers a_i, b_i,

$$|a_i + b_i|^{t_i} \le C(|a_i|^{t_i} + |b_i|^{t_i}) \tag{1.1}$$

Also for any complex number a,

$$|a|^{t_i} \le max(1, |a|^H) \tag{1.2}$$

2. Main Results

Theorem 1. *Let $t = (t_i)$ be bounded sequence of positive real numbers and T be a rooted tree then for any Musielak-Orlicz $\mathcal{M} = (\mathfrak{I}_i)$, the spaces $Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ and $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ are linear over $\mathbb{C}$.*

Proof. Let $\mathfrak{f}$, $\mathfrak{g} \in Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ then $\exists\ \phi_1 > 0$ and $\phi_2 > 0$ such that

$$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi_1}\Big)\Big]^{t_i} < \infty$$

and

$$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{g}(\xi)]|}{\phi_2}\Big)\Big]^{t_i} < \infty$$

Let α, $\beta \in \mathbb{C}$ and define $\phi_3 = \max(2|\alpha|\phi_1, 2|\beta|\phi_2)$. Since $\mathfrak{I}_i$ is non-decreasing and convex,

$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|\alpha B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)] + \beta B^{\nu}_{\Lambda}[\mathfrak{g}(\xi)]|}{\phi_3}\Big)\Big]^{t_i}$

$$\le \sum_{i=0}^{\infty}\frac{1}{2^{t_i}}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi_1}\Big) + \mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{g}(\xi)]|}{\phi_2}\Big)\Big]^{t_i}$$

$$\begin{aligned} &\leq \max(1, 2^{C-1})\Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi_1}\Big)\Big]^{t_1} + \sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\mathfrak{g}(\xi)]|}{\phi_2}\Big)\Big]^{t_i}\Big) \\ &< \infty. \end{aligned}$$

Hence, the required result. Likewise, we show that $Ces(\mathcal{M}, B_\Lambda^\nu, T, t)$ is linear space over $\mathbb{C}$. □

Theorem 2. *Let T be a rooted tree and $t = (t_i)$ be bounded sequence of positive real numbers, then for any Musielak-Orlicz function $\mathcal{M} = (\mathfrak{I}_i)$,*

(1) *$Ces^c(\mathcal{M}, B_\Lambda^\nu, T, t)$ is paranormed space over $\mathbb{C}$, paranormed by*

$$\delta(f(\xi)) = \Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi_1}\Big)\Big]^{t_i}\Big)^{\frac{1}{H}}, \tag{2.1}$$

where $H = \sup_i(t_i)$.

(2) *$Ces_p(\mathcal{M}, B_\Lambda^\nu, T, t)$ is paranormed linear space over $\mathbb{C}$, paranormed by*

$$\gamma(f(\xi)) = \Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{i+1}\sum_{j=1}^{i}\frac{1}{c_j}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{H}}, \tag{2.2}$$

where $H = sup_i(t_i)$.

Proof. To finalize the result, we just need to show that δ is sub-additive and multiplication is continuous. For this, let $\mathfrak{f}$, $\mathfrak{g} \in Ces^c(\mathfrak{I}, B_\Lambda^\nu, T, t)$ and by using the Minkowski's inequality, we have

$$\begin{aligned} &\Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\tfrac{1}{c_i}\sum_{|\xi|=i}\tfrac{|B_\Lambda^\nu[(\mathfrak{f}+\mathfrak{g})(\xi)]|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{H}} \\ &\leq \Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\Big(\frac{|B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi} + \frac{|B_\Lambda^\nu[\mathfrak{g}(\xi)]|}{\phi}\Big)\Big)\Big]^{t_i}\Big)^{\frac{1}{H}} \\ &\leq \Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{H}} + \Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\mathfrak{g}(\xi)]|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{H}} \end{aligned}$$

This shows that δ is sub-additive. Next, let $\lambda \in \mathbb{C}$. By definition, we have

$$\delta(f(\xi)) = \Big(\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B_\Lambda^\nu[\lambda\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{H}} \leq L_\lambda^{\frac{C}{H}}\delta(\mathfrak{f}(\xi))$$

where $L_\lambda \in \mathbb{N}_0$ such that $|\lambda| \leq L_\lambda$. Let $\lambda \to 0$ and for fixed ξ, $\delta(\mathfrak{f}(\xi)) = 0$. By definition for $|\lambda| < 1$, we have

$$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|\lambda B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i} < \epsilon,\ i > i_0(\epsilon) \tag{2.3}$$

Also, for $1 \leq i \leq i_0$, for sufficiently small λ. Since $\mathcal{M} = (\mathfrak{I}_i)$ is continous, we have

$$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|\lambda B_\Lambda^\nu[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i} < \epsilon. \tag{2.4}$$

By above equations, it imply that $\delta(\mathfrak{f}(\xi)) \to 0$ as $\lambda \to 0$ and hence, the result. Similarly, $Ces(\mathcal{M}, B_\Lambda^\nu, T, t)$ is paranormed space paranormed by (2.2). □

Theorem 3. *Let $t = (t_i)$ be a bounded sequence of positive real numbers and T be a rooted tree. Then for any Musielak-Orlicz function $\mathcal{M} = (\mathfrak{I}_i)$,*

(1) *$Ces^c(\mathcal{M}, B^\nu_\Lambda, T, t)$ is a complete paranormed space paranormed defined by (2.1).*
(2) *$Ces(\mathcal{M}, B^\nu_\Lambda, T, t)$ is a complete paranormed space paranormed defined by (2.2).*

Proof. To establish this, it is sufficient to prove the completeness property $Ces(\mathcal{M}, B^\nu_\Lambda, T, t)$. Let $(\mathfrak{f}^{(m)})$ be a Cauchy sequence in $Ces(\mathcal{M}, B^\nu_\Lambda, T, t)$. Let $r\xi_0$ be fixed. Then for each $\frac{\epsilon}{r\xi_0} > 0$, $\exists\, N \in \mathbb{N}_0$ such that

$$\delta(B^\mu_\Lambda \mathfrak{f}^{(m)} - B^\nu_\Lambda \mathfrak{f}^{(n)}) < \frac{\epsilon}{r\xi_0},$$

for all $m,\ n \geq N$,
Therefore,

$$\Big(\sum_{i=1}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\frac{\sum_{j=1}^{i}|B^\nu_\Lambda[\mathfrak{f}^{(m)}(\xi_j)] - B^\nu_\Lambda[\mathfrak{f}(\xi_j)^{(n)}]|}{\delta(B^\nu_\Lambda[\mathfrak{f}^{(m)}(\xi)] - B^\nu_\Lambda[\mathfrak{f}^{(n)}(\xi)])}\Big)\Big]^{t_i}\Big)^{\frac{1}{K}} \leq 1.$$

implies

$$\sum_{i=1}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\frac{\sum_{j=1}^{i}|B^\nu_\Lambda[\mathfrak{f}^{(m)}(\xi_j)] - B^\nu_\Lambda[\mathfrak{f}^{(n)}(\xi_j)]|}{\delta(B^\nu_\Lambda[\mathfrak{f}^{(m)}(\xi)] - B^\nu_\Lambda[\mathfrak{f}^{(n)}(\xi)])}\Big)\Big]^{t_i} \leq 1.$$

Since $1 \leq t_i < \infty$, it follows that $\mathfrak{I}_i\Big(\frac{1}{c_i}\frac{\sum_{j=1}^{i}|B^\nu_\Lambda[\mathfrak{f}^{(m)}(\xi_j)] - B^\nu_\Lambda[\mathfrak{f}^{(n)}(\xi_j)]|}{\delta(B^\nu_\Lambda[\mathfrak{f}^{(m)}(\xi)] - B^\nu_\Lambda[\mathfrak{f}^{(n)}](\xi)}\Big) \leq 1$, for each $i \geq 1$.
We choose $r > 0$ such that $(\frac{\xi_0}{2})rt(\frac{\xi_0}{2}) \geq 1$, where t is the kernel associated with $\mathfrak{I}_i$. Hence,

$$\mathfrak{I}_i\Big(\frac{1}{c_i}\frac{\sum_{j=1}^{i}|B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi_j) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi_j)|}{\gamma(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi))}\Big) \leq (\frac{\xi_0}{2})rt(\frac{\xi_0}{2})$$

for each $i \in \mathbb{N}$. Using the integral representation of Orlicz function, we get
$\frac{1}{c_i}\sum_{j=1}^{i}|B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi_j) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi_j)| \leq \frac{r\xi_0}{2}\delta(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi)) < \frac{\epsilon}{2}$, for all $m, n \geq N$.
Hence for each fixed j, $(B^\nu_\Lambda f^{(m)}(\xi_j))$ is Cauchy sequence in $\mathbb{C}$. Since $\mathbb{C}$ is complete, $(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi_j)) \to (B^\nu_\Lambda \mathfrak{f}(\xi_j))$ as $m \to \infty$. For given $\epsilon > 0$, choose an integer $i_0 > 1$ such that $\delta(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi)) < \epsilon$ for all $m, n \geq i_0$, such that $\delta(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi)) < \rho < \epsilon$.
Since

$$\Big(\sum_{i=1}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\frac{\sum_{j=1}^{i}|B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi_j) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi_j)|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{K}} \leq 1$$

for all $m, n \geq i_0$.
Now, by continuity of $\mathfrak{I}_i$ and taking $n \to \infty$ in above equality, we have

$$\Big(\sum_{i=1}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\frac{\sum_{j=1}^{i}|B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi_j) - B^\nu_\Lambda \mathfrak{f}(\xi_j)|}{\phi}\Big)\Big]^{t_i}\Big)^{\frac{1}{K}} \leq 1$$

for all $m \geq i_0$.
Letting $m \to \infty$, we get $\delta(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi) - B^\nu_\Lambda \mathfrak{f}(\xi)) < \epsilon$ for all $m, n \geq i_0$, such that $\delta(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi) - B^\nu_\Lambda \mathfrak{f}^{(n)}(\xi)) < \rho < \epsilon$ for all $m \geq i_0$. Thus $(B^\nu_\Lambda \mathfrak{f}^{(m)}(\xi))$ converges to $(B^\nu_\Lambda \mathfrak{f}(\xi))$ in paranorm of

$Ces(\mathcal{M}, B^{\nu}_{\Lambda}, t)$. Since $(B^{\nu}_{\Lambda}\mathfrak{f}^{(m)}(\xi)) \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ and $\mathfrak{I}_i$ is continuous, it follows that $(B^{\nu}_{\Lambda}\mathfrak{f}(\xi)) \in Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ and hence, the result. Likewise, $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ is complete paranormed space paranormed defined by (2.2). □

Theorem 4. *If $s = (s_i)$ and $t = (t_i)$ are bounded sequence of positive real numbers such that $0 < s_i \le t_i < \infty$ for each i and $\mathcal{M} = (\mathfrak{I}_i)$ be Musielak-Orlicz function, then $Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, s) \subset Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ and $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, s) \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$.*

Proof. Let $\mathfrak{f} \in Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, s)$. Then $\exists\, \phi > 0$ such that

$$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{s_i} < \infty$$

This implies that $\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big) \le 1$ for sufficiently large value of i, say $i \ge i_0$ for fixed $i_0 \in \mathbb{N}_0$. Since $\mathfrak{I}_i$ is increasing and $s_i \le t_i$,

$$\sum_{i\ge i_0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{s_i} \le \sum_{i\ge i_0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i} < \infty.$$

Therefore, $\mathfrak{f} \in Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$.
Likewise, we show that $Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, s) \subset Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$ and hence, the result. □

Theorem 5. *Let $t = (t_i)$ be bounded sequence of positive real numbers and $\mathcal{M} = (\mathfrak{I}_i)$ be Musielak-Orlicz function, then $L_\infty(T) \subset S(Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t))$ and $L_\infty(T) \subset S(Ces(\mathcal{M}, B^{\nu}_{\Lambda}, T, t))$.*

Proof. Let $\mathfrak{g} \in L_\infty(T)$, $K = \sup_{|\xi|}|\mathfrak{g}(\xi)|$ and $\mathfrak{f} \in Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$. Then

$$\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i} < \infty, \text{ for some } \phi > 0.$$

Since $\mathfrak{I}_i$ satisfies Δ_2-condition, there exists a constant N such that

$$\begin{aligned}\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{g}\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i} &\le \sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|\mathfrak{g}(\xi)||B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i}\\ &\le \sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(1+[M]\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i}\\ &\le (N(1+[K]))^H\sum_{i=0}^{\infty}\Big[\mathfrak{I}_i\Big(\frac{1}{c_i}\sum_{|\xi|=i}\frac{|B^{\nu}_{\Lambda}[\mathfrak{f}(\xi)]|}{\phi}\Big)\Big]^{t_i}\\ &< \infty,\end{aligned}$$

where $[K]$ denotes the integral part of K and hence $\mathfrak{g} \in Ces^c(\mathcal{M}, B^{\nu}_{\Lambda}, T, t)$. Similarly, we show the other inequality. □

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