Int. J. Anal. Appl. (2024), 22:229

International Journal of Analysis and Applications

# Distributed State-Dependent with Conjugate Feedback Control

**Sh. A. Abd El-Salam[1], A. M. A. El-Sayed[2], M. E. I. El-Gendy[3,*]**

[1]*Department of Mathematics and Computer Science, Faculty of Sciences, Damanhour University, Damanhour, Egypt*

[2]*Department of Mathematics and Computer Science, Faculty of Sciences, Alexandria University, Alexandria, Egypt*

[3]*Corresponding author: Department of Mathematics, College of Science, Qassim University, P.O. Box 6644, Burayda 51452, Saudi Arabia*

[*]*Corresponding author:* m.elgendy@qu.edu.sa

**Abstract.** Our goal in this work is to study the existence of solution of the Distributed state-dependent integral equation

$$x(\ell) = a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(s, y(s))ds,$$

with conjugate feedback control

$$y(\ell) = a_2(\ell) + \int_0^{\varphi_2(y(\ell))} f_2(s, x(s))ds.$$

Then some properties of this solution will be studied like uniqueness, continuous dependence and Hyers-Ulam stability.

## 1. Introduction

Self-reference is a concept that involves referring to oneself or one's own attributes, characteristics, or actions. It can occur in language, logic, mathematics, philosophy, and other fields.

In natural or formal languages, self-reference occurs when a sentence, idea or formula refers to itself. The reference may be expressed either directly through some intermediate sentence or formula or by means of some encoding.

In philosophy, self-reference also refers to the ability of a subject to speak of or refer to itself, that is, to have the kind of thought expressed by the first person nominative singular pronoun "I" in English.

Received: Oct. 17, 2024.

2020 *Mathematics Subject Classification.* 46B45, 46B50, 46E15, 34A12, 45M10.

*Key words and phrases.* Self-reference; continuous dependence; Hyers-Ulam stability; nonlinear equations.

https://doi.org/10.28924/2291-8639-22-2024-229

ISSN: 2291-8639

In mathematics and computability theory, self-reference (also known as impredicative) is the key concept in proving limitations of many systems. Gödel's theorem uses it to show that no formal consistent system of mathematics can ever contain all possible mathematical truths, because it cannot prove some truths about its own structure. The halting problem equivalent, in computation theory, shows that there is always some task that a computer cannot perform, namely reasoning about itself. These proofs relate to a long tradition of mathematical paradoxes such as Russell's paradox and Berry's paradox, and ultimately to classical philosophical paradoxes.

Self-reference (state-dependent) was studied in many papers foe examples ( [5] and [7]- [15]).

Also, studying the stability of the solution is considered one of the important studies in our lives. In order to model a physical process, an equation or problem can be used if a small alteration to it results in a corresponding small alteration in the outcome. When this occurs, the equation or problem is said to be stable.

Continuous Dependence, another important concept in stability theory, addresses the behavior of solutions in mathematical problems under varying conditions. It ensures that small changes in the initial conditions or parameters of a problem result in correspondingly small changes in the solution.

Hyers-Ulam stability when applied to the problem specifically, evaluates the model's robustness to disturbances, while Continuous dependence is applied to the unique solution of a problem to examine how the solutions are affected when its parameters are changed slightly.

Hyers-Ulam stability was studied im many papers for example [18]- [19].

A system of equations is said to be coupled if knowledge of one variable depends upon knowing the value of another variable. This kind of problem was studied in many paper for examples ( [1]- [3], [6], and [16]- [17]).

Here we study the existence of solution of the Distributed state-dependent integral equation

$$x(\ell) = a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(s, y(s))ds, \tag{1.1}$$

with conjugate feedback control

$$y(\ell) = a_2(\ell) + \int_0^{\varphi_2(y(\ell))} f_2(s, x(s))ds. \tag{1.2}$$

Our paper organized as follows: Section 2 contains main results for our problem (1.1)-(1.2). In Section 3, the uniqueness of this solution will be proved. In Section 4, continuous dependence of the solution on $a_i(.)$, $\varphi_i(.)$ and $f_i(.,.)$ will be studied. In section 5, Hyers-Ulam stability will be studied. Finally, some examples are given in Section 6.

The following theorem will be needed.

**Theorem 1.1.** ***(Schauder fixed point Theorem)*** *[4]*
*Let $\mathfrak{U}$ be a convex subset of a Banach space $X$, and $\mathfrak{T} : \mathfrak{U} \to \mathfrak{U}$ is compact, continuous map. Then $\mathfrak{T}$ has at least one fixed point in $\mathfrak{U}$.*

## 2. Main Results

Let $\mathbb{C} = C[0,T]$ be the space of continuous functions on $\mathbb{I} = [0,T]$, with supremum norm $\|u\|_{\mathbb{C}} = \sup_{\ell\in\mathbb{I}} |u(\ell)|$ for any $u \in \mathbb{C}$.

Define

$$\mathbb{E} = \mathbb{C}\times\mathbb{C} = \left\{(x(\ell), y(\ell)) : (x(\ell), y(\ell)) \in \mathbb{E} \text{ and } \|(x,y)\|_{\mathbb{E}} = \|x\|_{\mathbb{C}} + \|y\|_{\mathbb{C}} \le r\right\},$$

and define the operator $\mathbb{T}$ by

$$\mathbb{T}(x,y)(\ell) = (\mathbb{T}_1 y(\ell), \mathbb{T}_2 x(\ell)),$$

where

$$\mathbb{T}_1 y(\ell) = a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(s, y(s))ds,$$
$$\mathbb{T}_2 x(\ell) = a_2(\ell) + \int_0^{\varphi_2(y(\ell))} f_2(s, x(s))ds.$$

We will prove that the operator $\mathbb{T}$ has a fixed point, hence the solution of problem (1.1)-(1.2) exists.

Suppose the following assumptions:

(i) $f_i : \mathbb{I}\times\mathbb{R} \to \mathbb{R}, i = 1,2$ such that
   (1) for each $\ell \in \mathbb{I}$, $f_i(\ell, .)$ are continuous,
   (2) for each $u \in \mathbb{R}$, $f_i(., u)$ are measurable,
   (3)
$$|f_i(\ell,u)| \le b(\ell) + c\,|u(\ell)|, \text{ for all } (\ell,u) \in \mathbb{I}\times\mathbb{R},$$
   where $b(.)$ is bounded, measurable and $b = \sup_{\ell\in\mathbb{I}} b(\ell)$ and $c \ge 0$ is a constant.

(ii) $a_i : \mathbb{I} \to \mathbb{R}$ are continuous functions and $a = \sup_{\ell\in\mathbb{I}} |a_i(\ell)|,\ i = 1,2$.

(iii) $\varphi_i : \mathbb{I} \to \mathbb{I}$ such that
$$|\varphi_i(\ell) - \varphi_i(s)| \le |\ell - s|,\ \varphi_i(0) = 0,\ i = 1,2.$$

(iv) there exists $r$ satisfies the quadratic equation
$$a + (b - \frac{1}{2})r + cr^2 = 0.$$

**Theorem 2.1.** *Assume that the assumptions (i-iv) are satisfied. Then problem (1.1)-(1.2) has at least one solution $(x,y) \in \mathbb{E}$.*

**Proof.** Define the subset $\mathfrak{S}_r$ by

$$\mathfrak{S}_r = \{(x(\ell), y(\ell)) \in \mathbb{E} : \|(x,y)\|_{\mathbb{E}} \le r\}.$$

The set $\mathfrak{S}_r$ is nonempty, closed and convex.
For $(x, y) \in \mathfrak{S}_r$, we have

$$\begin{aligned}
|\mathbb{T}_2 x(\ell)| &= \left| a_2(\ell) + \int_0^{\varphi_2(y(\ell))} f_2(s, x(s)) ds \right| \\
&\leq |a_2(\ell)| + \int_0^{\varphi_2(y(\ell))} |f_2(s, x(s))| ds \\
&\leq a + \int_0^{\varphi_2(y(\ell))} \Big( b(s) + c|x(s)| \Big) ds \\
&\leq a + b\, \varphi_2(y(\ell)) + c\, r\, \varphi_2(y(\ell)), \\
\|\mathbb{T}_2 x\|_C &\leq a + b\, r + c\, r^2,
\end{aligned}$$

then the operator $\mathbb{T}_2$ is uniformly bounded on $\mathfrak{S}_r$.
Similarly; we get

$$\|\mathbb{T}_1 y\|_C \leq a + b\, r + c\, r^2,$$

then the operator $\mathbb{T}_1$ is uniformly bounded on $\mathfrak{S}_r$.
Now,

$$\begin{aligned}
\|\mathbb{T}(x, y)\|_{\mathbb{E}} &= \|\mathbb{T}_1 y\|_C + \|\mathbb{T}_2 x\|_C \\
&\leq 2(a + b\, r + c\, r^2) = r.
\end{aligned}$$

Therefore, $\mathbb{T}$ is uniformly bounded on $\mathfrak{S}_r$.
Now, we show that $\mathbb{T}$ is a completely continuous operator. Indeed, let $\ell_1, \ell_2 \in \mathbb{I}, \ell_1 < \ell_2$ such that $|\ell_2 - \ell_1| < \delta$, we have

$$\begin{aligned}
|\mathbb{T}_2 x(\ell_2) - \mathbb{T}_2 x(\ell_1)| &= \left| a_2(\ell_2) + \int_0^{\varphi_2(y(\ell_2))} f_2(s, x(s)) ds \right. \\
&\quad - \left. a_2(\ell_1) + \int_0^{\varphi_2(y(\ell_1))} f_2(s, x(s)) ds \right| \\
&\leq |a_2(\ell_2) - a_2(\ell_1)| + \left| \int_0^{\varphi_2(y(\ell_2))} f_2(s, x(s)) ds - \int_0^{\varphi_2(y(\ell_1))} f_2(s, x(s)) ds \right| \\
&= |a_2(\ell_2) - a_2(\ell_1)| + \left| \int_{\varphi_2(y(\ell_1))}^{\varphi_2(y(\ell_2))} f_2(s, x(s)) ds \right| \\
&\leq |a_2(\ell_2) - a_2(\ell_1)| + \int_{\varphi_2(y(\ell_1))}^{\varphi_2(y(\ell_2))} \Big( b(s) + c|x(s)| \Big) ds \\
&\leq |a_2(\ell_2) - a_2(\ell_1)| + (b + c\, r)|\varphi_2(y(\ell_2)) - \varphi_2(y(\ell_1))| \\
&\leq |a_2(\ell_2) - a_2(\ell_1)| + (b + c\, r)|y(\ell_2) - y(\ell_1)|.
\end{aligned}$$

Similarly;

$$|\mathbb{T}_1 y(\ell_2) - \mathbb{T}_1 y(\ell_1)| \leq |a_1(\ell_2) - a_1(\ell_1)| + (b + c\, r)|x(\ell_2) - x(\ell_1)|.$$

Then, the operator $\mathbb{T}(x, y)$ is an equi-continuous operator. Therefore from Arzela-Ascoli Theorem we deduce that the operator $\{\mathbb{T}(x, y)\}$ is relatively compact.

Let $\mathbb{T} : \mathfrak{S}_r \to \mathfrak{S}_r$, for $(x, y) \in \mathfrak{S}_r \Rightarrow \mathbb{T}$ is a continuous operator:
indeed, let $\{x_n(\ell), y_n(\ell)\}$ is a sequence in $\mathfrak{S}_r$ converges to $(x_0(\ell), y_0(\ell))$ for every $\ell \in \mathbb{I}$. Then

$$\begin{aligned}
|\mathbb{T}_2 x_n(\ell) - \mathbb{T}_2 x_0(\ell)| &= \left| \int_0^{\varphi_2(y_n(\ell))} f_2(s, x_n(s))ds - \int_0^{\varphi_2(y_0(\ell))} f_2(s, x_0(s))ds \right| \\
&= \left| \int_0^{\varphi_2(y_n(\ell))} f_2(s, x_n(s))ds - \int_0^{\varphi_2(y_0(\ell))} f_2(s, x_n(s))ds \right. \\
&+ \left. \int_0^{\varphi_2(y_0(\ell))} f_2(s, x_n(s))ds - \int_0^{\varphi_2(y_0(\ell))} f_2(s, x_0(s))ds \right| \\
&\leq \int_{\varphi_2(y_0(\ell))}^{\varphi_2(y_n(\ell))} |f_2(s, x_n(s))|ds \\
&+ \int_0^{\varphi_2(y_0(\ell))} \left| f_2(s, x_n(s)) - f_2(s, x_0(s)) \right| ds \\
&\leq \int_{\varphi_2(y_0(\ell))}^{\varphi_2(y_n(\ell))} \Big( b(s) \ + \ c|x_n(s)| \Big) ds + \varepsilon \, |\varphi_2(y_0(\ell))| \\
&\leq (b + c\, r)|\varphi_2(y_n(\ell)) - \varphi_2(y_0(\ell))| + \varepsilon \, |y_0(\ell)| \\
&\leq (b + c\, r)|y_n(\ell) - y_0(\ell)| + \varepsilon \, r.
\end{aligned}$$

Similarly;

$$|\mathbb{T}_1 y_n(\ell) - \mathbb{T}_1 y_0(\ell)| \ \leq \ (b + c\, r)|x_n(\ell) - x_0(\ell)| + \varepsilon \, r.$$

Then,

$$\mathbb{T}(x, y) \ = \ (\mathbb{T}_1 y, \mathbb{T}_2 x)$$

is a continuous operator from $\mathbb{E}$ to $\mathbb{E}$.

Therefore, the conditions of the Schauder fixed point Theorem hold, which implies that $\mathbb{T}$ has a fixed point in $\mathfrak{S}_r$. Then problem (1.1)-(1.2) has a solution $(x, y) \in \mathbb{E}$.

**Corollary 2.1.** *Let the assumptions of Theorem (2.1) be satisfied, if* $x(\ell) = y(\ell), a_1(\ell) = a_2(\ell), \varphi_1 = \varphi_2$ *and* $f_1 = f_2$ *in the Distributed state-dependent integral equation (1.1) with conjugate feedback control (1.2),* *then we deduce that the following equation*

$$x(\ell) = a(\ell) + \int_0^{\varphi(x(\ell))} f(s,x(s))ds$$

*has a least one solution* $x \in \mathbb{C}(\mathbb{I})$.

## 3. Uniqueness of the Solution

**Theorem 3.1.** *Suppose that the conditions (i-2) and (ii) - (iv) of Theorem 2.1 are satisfied in addition to the following assumptions:*

$$|f_i(\ell,u) - f_i(\ell,v)| \leq k\,|u - v|,\ i = 1,2. \tag{3.1}$$

*Then problem (1.1)-(1.2) has a unique solution.*

**Proof.** From assumption (3.1), we get

$$|f_i(\ell,u) - f_i(\ell,0)| \leq k\,|u|,$$

but since

$$|f_i(\ell,u)| - |f_i(\ell,0)| \leq |f_i(\ell,u) - f_i(\ell,0)| \leq k\,|u|,$$

therefore

$$|f_i(\ell,u)| \leq |f_i(\ell,0)| + k\,|u|,$$

i.e. assumptions $(i-1)$ and $(i-3)$ of theorem 2.1 are satisfied. Then from theorem 2.1 the solution exists.

Now we prove the uniqueness of this solution:

Let $(x_1, y_1)$ be a solution of problem (1.1)-(1.2), then

$$\begin{aligned}
|x(\ell) - x_1(\ell)| &= \Big|a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds \\
&- a_1(\ell) - \int_0^{\varphi_1(x_1(\ell))} f_1(s,y_1(s))ds\Big| \\
&= \Big|\int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(x_1(\ell))} f_1(s,y(s))ds \\
&+ \int_0^{\varphi_1(x_1(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(x_1(\ell))} f_1(s,y_1(s))ds\Big| \\
&\leq \Big|\int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(x_1(\ell))} f_1(s,y(s))ds\Big| \\
&+ \Big|\int_0^{\varphi_1(x_1(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(x_1(\ell))} f_1(s,y_1(s))ds\Big| \\
&\leq \int_{\varphi_1(x_1(\ell))}^{\varphi_1(x(\ell))} \Big|f_1(s,y(s))\Big|ds \\
&+ \int_0^{\varphi_1(x_1(\ell))} \Big|f_1(s,y(s)) - f_1(s,y_1(s))\Big|ds
\end{aligned}$$

$$\begin{aligned}
&\leq \left(|f_1(\ell,0)|+k|y(\ell)|\right)\Big|\varphi_1(x(\ell))-\varphi_1(x_1(\ell))\Big| \\
&+ k|y(\ell)-y_1(\ell)|\,|\varphi_1(x_1(\ell))| \\
&\leq \left(|f_1(\ell,0)|+k|y(\ell)|\right)\Big|x(\ell)-x_1(\ell)\Big| \\
&+ k\,|x_1(\ell)|\,|y(\ell)-y_1(\ell)|.
\end{aligned}$$

Then,

$$\|x-x_1\|_C \leq (d+k\,r)\|x-x_1\|_C + k\,r\,\|y-y_1\|_C,$$

where $d=\sup_{\ell\in\mathbb{I}}|f_i(\ell,0)|, i=1,2$.
Similarly,

$$\|y-y_1\|_C \leq (d+k\,r)\|y-y_1\|_C + k\,r\,\|x-x_1\|_C.$$

Then

$$\begin{aligned}
\|(x,y)-(x_1,y_1)\|_{\mathbb{E}} &= \|x-x_1\|_C + \|y-y_1\|_C \\
&\leq (d+k\,r)\|x-x_1\|_C + k\,r\,\|y-y_1\|_C \\
&+ (d+k\,r)\|y-y_1\|_C + k\,r\,\|x-x_1\|_C \\
&\leq (d+2k\,r)\|x-x_1\|_C + (d+2k\,r)\|y-y_1\|_C \\
&\leq (d+2kr)\left(\|x-x_1\|_C + \|y-y_1\|_C\right) \\
&= \alpha\,\|(x,y)-(x_1,y_1)\|_{\mathbb{E}},
\end{aligned}$$

where

$$\alpha = d + 2\,k\,r.$$

Which implies

$$\|(x,y)-(x_1,y_1)\|_{\mathbb{E}} = 0 \text{ then } (x,y) = (x_1,y_1).$$

**Corollary 3.1.** *Let the assumptions of Theorem (3.1) be satisfied, if $x(\ell)=y(\ell), a_1(\ell)=a_2(\ell), \varphi_1=\varphi_2$ and $f_1=f_2$ in the Distributed state-dependent integral equation (1.1) with conjugate feedback control (1.2), then we deduce that there exists a unique continuous solution of the following equation*

$$x(\ell) = a(\ell) + \int_0^{\varphi(x(\ell))} f(s,x(s))ds.$$

## 4. Continuous Dependence of the Solution

**Firstly**, we study the continuous dependence of the solution of problem (1.1)-(1.2) on $a_i(\ell), i = 1,2$.

**Theorem 4.1.** *Let the assumptions of Theorem 3.1 be satisfied. Then the solution of problem (1.1)-(1.2) depends continuously on $a_i(\ell), i=1,2$,*

**Proof.** Let $(\widetilde{x}(\ell),\widetilde{y}(\ell))$ be a solution of:

$$\begin{cases} \widetilde{x}(\ell) \ = \ \widetilde{a}_1(\ell) \ + \ \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds, \\ \\ \widetilde{y}(\ell) \ = \ \widetilde{a}_2(\ell) \ + \ \int_0^{\varphi_2(\widetilde{y}(\ell))} f_2(s,\widetilde{x}(s))ds. \end{cases} \tag{4.1}$$

$$\begin{aligned}
|x(\ell)-\widetilde{x}(\ell)| &= \Bigg| a_1(\ell) \ + \ \int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds \\
&- \ \widetilde{a}_1(\ell) \ - \ \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds \Bigg| \\
&\le |a_1(\ell)-\widetilde{a}_1(\ell)| + \Bigg| \int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds \\
&+ \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds \Bigg| \\
&\le |a_1(\ell)-\widetilde{a}_1(\ell)| + \Bigg| \int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds \Bigg| \\
&+ \Bigg| \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds \Bigg| \\
&\le |a_1(\ell)-\widetilde{a}_1(\ell)| + \int_{\varphi_1(\widetilde{x}(\ell))}^{\varphi_1(x(\ell))} \Big| f_1(s,y(s)) \Big| ds \\
&+ \int_0^{\varphi_1(\widetilde{x}(\ell))} \Big| f_1(s,y(s)) \ - \ f_1(s,\widetilde{y}(s)) \Big| ds \\
&\le |a_1(\ell)-\widetilde{a}_1(\ell)| + \Big( |f_1(\ell,0)| + k|y(\ell)| \Big) \Big| \varphi_1(x(\ell)) - \varphi_1(\widetilde{x}(\ell)) \Big| \\
&+ \ k|y(\ell)-\widetilde{y}(\ell)| \, |\varphi_1(\widetilde{x}(\ell))| \\
&\le |a_1(\ell)-\widetilde{a}_1(\ell)| + \Big( |f_1(\ell,0)| + k|y(\ell)| \Big) |x(\ell)-\widetilde{x}(\ell)| \\
&+ \ k\,|\widetilde{x}(\ell)| \, |y(\ell)-\widetilde{y}(\ell)|.
\end{aligned}$$

Then,

$$\begin{aligned}
\|x-\widetilde{x}\|_{\mathbb{C}} &\le \|a_1 \ - \ \widetilde{a}_1\|_{\mathbb{C}} + (d+k\,r)\|x-\widetilde{x}\|_{\mathbb{C}} + k\,r\,\|y-\widetilde{y}\|_{\mathbb{C}}, \\
\|x-\widetilde{x}\|_{\mathbb{C}} &\le \frac{1}{1-(d+k\,r)} \Big( \|a_1 \ - \ \widetilde{a}_1\|_{\mathbb{C}} + k\,r\,\|y-\widetilde{y}\|_{\mathbb{C}} \Big).
\end{aligned}$$

Similarly;

$$\|y-\widetilde{y}\|_{\mathbb{C}} \le \frac{1}{1-(d+k\,r)} \Big( \|a_2 \ - \ \widetilde{a}_2\|_{\mathbb{C}} + k\,r\,\|x-\widetilde{x}\|_{\mathbb{C}} \Big).$$

Then,

$$\|(x,y) \ - \ (\widetilde{x},\widetilde{y})\|_{\mathbb{E}} \ = \ \|x \ - \ \widetilde{x}\|_{\mathbb{C}} \ + \ \|y \ - \ \widetilde{y}\|_{\mathbb{C}}$$

$$\begin{aligned}
&\leq \frac{1}{1-(d+k\,r)}\Big(\|a_1 - \widetilde{a}_1\|_{\mathbb{C}} + k\,r\,\|y-\widetilde{y}\|_{\mathbb{C}}\Big)\\
&+ \frac{1}{1-(d+k\,r)}\Big(\|a_2 - \widetilde{a}_2\|_{\mathbb{C}} + k\,r\,\|x-\widetilde{x}\|_{\mathbb{C}}\Big)\\
&= \beta\Big(\|a_1 - \widetilde{a}_1\|_{\mathbb{C}} + \|a_2 - \widetilde{a}_2\|_{\mathbb{C}}\Big) + \eta\Big(\|x - \widetilde{x}\|_{\mathbb{C}} + \|y - \widetilde{y}\|_{\mathbb{C}}\Big)\\
&= \beta\,\|(a_1,a_2) - (\widetilde{a}_1,\widetilde{a}_2)\|_{\mathbb{E}} + \eta\,\|(x,y) - (\widetilde{x},\widetilde{y})\|_{\mathbb{E}},
\end{aligned}$$

where

$$\beta = \frac{1}{1-(d+k\,r)} \text{ and } \eta = \frac{k\,r}{1-(d+k\,r)}.$$

Therefore, if $\|(a_1,a_2) - (\widetilde{a}_1,\widetilde{a}_2)\| < \delta_1 \Rightarrow \|(x,y) - (\widetilde{x},\widetilde{y})\| < \varepsilon_1 = \frac{\beta}{1-\eta}\delta_1$.

**Secondly**, we show that the solution of problem (1.1)-(1.2) depends continuously on $\varphi_i(\ell), i = 1,2$.

**Theorem 4.2.** *Let the assumptions of Theorem 3.1 be satisfied. Then the solution of problem (1.1)-(1.2) depends continuously on $\varphi_i(\ell), i = 1,2$,*

**Proof.** Let $(\widetilde{x}(\ell),\widetilde{y}(\ell))$ be a solution of:

$$\begin{cases} \widetilde{x}(\ell) = a_1(\ell) + \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds,\\[2ex] \widetilde{y}(\ell) = a_2(\ell) + \int_0^{\widetilde{\varphi}_2(\widetilde{y}(\ell))} f_2(s,\widetilde{x}(s))ds. \end{cases} \tag{4.2}$$

$$\begin{aligned}
|x(\ell)-\widetilde{x}(\ell)| &= \Big|a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - a_1(\ell) - \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds\Big|\\
&= \Big|\int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,y(s))ds\\
&+ \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,y(s))ds - \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds\Big|\\
&\leq \Big|\int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,y(s))ds\Big|\\
&+ \Big|\int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,y(s))ds - \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} f_1(s,\widetilde{y}(s))ds\Big|\\
&\leq \int_{\widetilde{\varphi}_1(\widetilde{x}(\ell))}^{\varphi_1(x(\ell))} \Big|f_1(s,y(s))\Big|ds + \int_0^{\widetilde{\varphi}_1(\widetilde{x}(\ell))} \Big|f_1(s,y(s)) - f_1(s,\widetilde{y}(s))\Big|ds\\
&\leq \Big(|f_1(\ell,0)| + k|y(\ell)|\Big)\Big|\varphi_1(x(\ell)) - \widetilde{\varphi}_1(\widetilde{x}(\ell))\Big|\\
&+ k|y(\ell)-\widetilde{y}(\ell)|\,|\widetilde{\varphi}_1(\widetilde{x}(\ell))|\\
&\leq \Big(|f_1(\ell,0)| + k|y(\ell)|\Big)\Big|\varphi_1(x(\ell)) - \varphi_1(\widetilde{x}(\ell)) + \varphi_1(\widetilde{x}(\ell)) - \widetilde{\varphi}_1(\widetilde{x}(\ell))\Big|
\end{aligned}$$

$$\begin{aligned}
&+ \quad k\,|\widetilde{x}(\ell)|\,|y(\ell)-\widetilde{y}(\ell)| \\
&\leq \quad (d+k\,r)\Big(|\varphi_1(x(\ell))-\varphi_1(\widetilde{x}(\ell))|+|(\varphi_1-\widetilde{\varphi}_1)(\widetilde{x}(\ell))|\Big)+k\,r|y(\ell)-\widetilde{y}(\ell)| \\
&\leq \quad (d+k\,r)\Big(|x(\ell)-\widetilde{x}(\ell)|+|(\varphi_1-\widetilde{\varphi}_1)(\widetilde{x}(\ell))|\Big)+k\,r\,|y(\ell)-\widetilde{y}(\ell)|.
\end{aligned}$$

Then,

$$\|x-\widetilde{x}\|_C \leq \frac{d+k\,r}{1-(d+k\,r)}\|\varphi_1-\widetilde{\varphi}_1\|_C+\frac{k\,r}{1-(d+k\,r)}\,\|y-\widetilde{y}\|_C.$$

Similarly;

$$\|y-\widetilde{y}\|_C \leq \frac{d+k\,r}{1-(d+k\,r)}\|\varphi_2-\widetilde{\varphi}_2\|_C+\frac{k\,r}{1-(d+k\,r)}\,\|x-\widetilde{x}\|_C.$$

Then,

$$\begin{aligned}
\|(x,y)-(\widetilde{x},\widetilde{y})\|_{\mathbb{E}} &= \|x-\widetilde{x}\|_C+\|y-\widetilde{y}\|_C \\
&\leq \frac{d+k\,r}{1-(d+k\,r)}\|\varphi_1-\widetilde{\varphi}_1\|_C+\frac{k\,r}{1-(d+k\,r)}\,\|y-\widetilde{y}\|_C \\
&+ \frac{d+k\,r}{1-(d+k\,r)}\|\varphi_2-\widetilde{\varphi}_2\|_C+\frac{k\,r}{1-(d+k\,r)}\,\|x-\widetilde{x}\|_C \\
&= \xi\Big(\|\varphi_1-\widetilde{\varphi}_1\|_C+\|\varphi_2-\widetilde{\varphi}_2\|_C\Big)+\eta\Big(\|x-\widetilde{x}\|_C+\|y-\widetilde{y}\|_C\Big) \\
&= \xi\,\|(\varphi_1,\varphi_2)-(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\|_{\mathbb{E}}+\eta\,\|(x,y)-(\widetilde{x},\widetilde{y})\|_{\mathbb{E}},
\end{aligned}$$

where

$$\xi=\frac{d+k\,r}{1-(d+k\,r)}.$$

Therefore, if $\|(\varphi_1,\varphi_2)-(\widetilde{\varphi}_1,\widetilde{\varphi}_2)\|_{\mathbb{E}}<\delta_2 \Rightarrow \|(x,y)-(\widetilde{x},\widetilde{y})\|_{\mathbb{E}}<\varepsilon_2=\frac{\xi}{1-\eta}\delta_2$.

**Finally**, we show that the solution of problem (1.1)-(1.2) depends continuously on $f_i(\ell), i=1,2$.

**Theorem 4.3.** *Let the assumptions of Theorem 3.1 be satisfied. Then the solution of problem (1.1)-(1.2) depends continuously on $f_i(\ell), i=1,2$,*

**Proof.** Let $(\widetilde{x}(\ell),\widetilde{y}(\ell))$ be a solution of:

$$\begin{cases} \widetilde{x}(\ell) = a_1(\ell) + \int_0^{\varphi_1(\widetilde{x}(\ell))}\widetilde{f}_1(s,\widetilde{y}(s))ds, \\ \\ \widetilde{y}(\ell) = a_2(\ell) + \int_0^{\varphi_2(\widetilde{y}(\ell))}\widetilde{f}_2(s,\widetilde{x}(s))ds. \end{cases} \tag{4.3}$$

$$\begin{aligned}
|x(\ell)-\widetilde{x}(\ell)| &= \Big|a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds \\
&\quad - a_1(\ell) - \int_0^{\varphi_1(\widetilde{x}(\ell))}\widetilde{f}_1(s,\widetilde{y}(s))ds\Big|
\end{aligned}$$

$$
\begin{aligned}
&= \Big|\int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds \\
&+ \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} \widetilde{f_1}(s,\widetilde{y}(s))ds\Big| \\
&\leq \Big|\int_0^{\varphi_1(x(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds\Big| \\
&+ \Big|\int_0^{\varphi_1(\widetilde{x}(\ell))} f_1(s,y(s))ds - \int_0^{\varphi_1(\widetilde{x}(\ell))} \widetilde{f_1}(s,\widetilde{y}(s))ds\Big| \\
&\leq \int_{\varphi_1(\widetilde{x}(\ell))}^{\varphi_1(x(\ell))} \Big|f_1(s,y(s))\Big|ds \\
&+ \int_0^{\varphi_1(\widetilde{x}(\ell))} \Big|f_1(s,y(s)) \; - \; \widetilde{f_1}(s,\widetilde{y}(s))\Big|ds \\
&\leq \Big(|f_1(\ell,0)| + k|y(\ell)|\Big)\Big|\varphi_1(x(\ell)) - \varphi_1(\widetilde{x}(\ell))\Big| \\
&+ \int_0^{\varphi_1(\widetilde{x}(\ell))} \Big|f_1(s,y(s)) - f_1(s,\widetilde{y}(s)) + f_1(s,\widetilde{y}(s)) - \widetilde{f_1}(s,\widetilde{y}(s))\Big|ds \\
&\leq \Big(|f_1(\ell,0)| + k|y(\ell)|\Big)|x(\ell) - \widetilde{x}(\ell)| \\
&+ \int_0^{\varphi_1(\widetilde{x}(\ell))} \Big(k|y(s) - \widetilde{y}(s)| + |f_1(s,\widetilde{y}(s)) - \widetilde{f_1}(s,\widetilde{y}(s))|\Big)ds \\
&\leq (d + k\,r)\|x - \widetilde{x}\|_{\mathbb{C}} + |\varphi_1(\widetilde{x}(\ell))|\Big(k\|y - \widetilde{y}\|_{\mathbb{C}} + \|f_1 - \widetilde{f_1}\|_{\mathbb{C}}\Big) \\
&\leq (d + k\,r)\|x - \widetilde{x}\|_{\mathbb{C}} + r\Big(k\|y - \widetilde{y}\|_{\mathbb{C}} + \|f_1 - \widetilde{f_1}\|_{\mathbb{C}}\Big).
\end{aligned}
$$

Then,

$$
\|x - \widetilde{x}\|_{\mathbb{C}} \;\leq\; \frac{r}{1 \,-\, (d + k\,r)}\|f_1 - \widetilde{f_1}\|_{\mathbb{C}} + \frac{k\,r}{1 \,-\, (d + k\,r)}\,\|y - \widetilde{y}\|_{\mathbb{C}}.
$$

Similarly;

$$
\|y - \widetilde{y}\|_{\mathbb{C}} \;\leq\; \frac{r}{1 \,-\, (d + k\,r)}\|f_2 - \widetilde{f_2}\|_{\mathbb{C}} + \frac{k\,r}{1 \,-\, (d + k\,r)}\,\|x - \widetilde{x}\|_{\mathbb{C}}.
$$

Then,

$$
\begin{aligned}
\|(x,y) \,-\, (\widetilde{x},\widetilde{y})\|_{\mathbb{E}} &= \|x \,-\, \widetilde{x}\|_{\mathbb{C}} \,+\, \|y \,-\, \widetilde{y}\|_{\mathbb{C}} \\
&\leq \frac{r}{1 \,-\, (d + k\,r)}\|f_1 - \widetilde{f_1}\|_{\mathbb{C}} + \frac{k\,r}{1 \,-\, (d + k\,r)}\,\|y - \widetilde{y}\|_{\mathbb{C}} \\
&+ \frac{r}{1 \,-\, (d + k\,r)}\|f_2 - \widetilde{f_2}\|_{\mathbb{C}} + \frac{k\,r}{1 \,-\, (d + k\,r)}\,\|x - \widetilde{x}\|_{\mathbb{C}} \\
&= \lambda\Big(\|f_1 \,-\, \widetilde{f_1}\|_{\mathbb{C}} + \|f_2 - \widetilde{f_2}\|_{\mathbb{C}}\Big) + \eta\Big(\|x \,-\, \widetilde{x}\|_{\mathbb{C}} + \|y \,-\, \widetilde{y}\|_{\mathbb{C}}\Big)
\end{aligned}
$$

$$= \lambda \,\|(f_1, f_2) - (\widetilde{f_1}, \widetilde{f_2})\|_{\mathbb{E}} + \eta \,\|(x, y) - (\widetilde{x}, \widetilde{y})\|_{\mathbb{E}},$$

where

$$\lambda = \frac{r}{1 - (d + k\, r)}.$$

Therefore, if $\|(f_1, f_2) - (\widetilde{f_1}, \widetilde{f_2})\|_{\mathbb{E}} < \delta_3 \Rightarrow \|(x, y) - (\widetilde{x}, \widetilde{y})\|_{\mathbb{E}} < \varepsilon_3 = \frac{\lambda}{1-\eta}\delta_3$.

**Corollary 4.1.** *Let the assumptions of Theorem (3.1) be satisfied, if $x(\ell) = y(\ell), a_1(\ell) = a_2(\ell), \varphi_1 = \varphi_2$ and $f_1 = f_2$ in the Distributed state-dependent integral equation (1.1) with conjugate feedback control (1.2), then we deduce that the solution of the following equation*

$$x(\ell) = a(\ell) + \int_0^{\varphi(x(\ell))} f(s, x(s))ds.$$

*exists and depends continuously on each of $a_i(.)$, $f_i(.,.)$ and $\phi_i(.)$.*

## 5. Hyers-Ulam Stability

**Theorem 5.1.** *Consider the Distributed state-dependent integral equation*

$$x(\ell) = a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(\theta, y(\theta))d\theta, \ \ell \in \mathbb{I} \tag{5.1}$$

*with conjugate feedback control*

$$y(\ell) = a_2(\ell) + \int_0^{\varphi_2(y(\ell))} f_2(\theta, x(\theta))d\theta. \tag{5.2}$$

*Let the solution of (5.1) with (5.2) exists, then this solution is Hyers-Ulam stable if $\forall \epsilon > 0, \exists\, \delta^*(\epsilon) > 0$ such that any solution $(x_s(\ell), y_s(\ell))$ of (5.1) with (5.2) satisfies*

$$\left| x_s(\ell) - a_1(\ell) - \int_0^{\varphi_1(x_s(\ell))} f_1(\theta, y_s(\theta))d\theta \right| < \delta^*,$$

$$\left| y_s(\ell) - a_2(\ell) - \int_0^{\varphi_2(y_s(\ell))} f_2(\theta, x_s(\theta))d\theta \right| < \delta^*,$$

*then*

$$\|(x, y) - (x_s, y_s)\|_{\mathbb{E}} < \epsilon.$$

**Theorem 5.2.** *Let the assumptions of Theorem 3.1 be satisfied. Then the solution of problem (1.1)-(1.2) is Hyers-Ulam stable.*

**Proof.** Let $(x_s(\ell), y_s(\ell))$ be a solution of:

$$\begin{cases} x_s(\ell) = a_1(\ell) + \int_0^{\varphi_1(x_s(\ell))} f_1(\theta, y_s(\theta))d\theta, \\ \\ y_s(\ell) = a_2(\ell) + \int_0^{\varphi_2(y_s(\ell))} f_2(\theta, x_s(\theta))d\theta. \end{cases} \tag{5.3}$$

$$\begin{aligned}
|x(\ell)-x_s(\ell)| &= \left|a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(\theta,y(\theta))d\theta - x_s(\ell)\right| \\
&= \left|a_1(\ell) + \int_0^{\varphi_1(x(\ell))} f_1(\theta,y(\theta))d\theta - \int_0^{\varphi_1(x_s(\ell))} f_1(\theta,y(\theta))d\theta\right. \\
&+ \int_0^{\varphi_1(x_s(\ell))} f_1(\theta,y(\theta))d\theta - \int_0^{\varphi_1(x_s(\ell))} f_1(\theta,y_s(\theta))d\theta \\
&+ \left.\int_0^{\varphi_1(x_s(\ell))} f_1(\theta,y_s(\theta))d\theta - x_s(\ell)\right| \\
&\le \int_{\varphi_1(x_s(\ell))}^{\varphi_1(x(\ell))} |f_1(\theta,y(\theta))|d\theta + \int_0^{\varphi_1(x_s(\ell))} \left|f_1(\theta,y(\theta)) - f_1(\theta,y_s(\theta))\right|d\theta \\
&+ \left|a_1(\ell) + \int_0^{\varphi_1(x_s(\ell))} f_1(\theta,y_s(\theta))d\theta - x_s(\ell)\right| \\
&\le \Big(|f_1(\ell,0)| + k|y(\ell)|\Big)\left|\varphi_1(x(\ell)) - \varphi_1(x_s(\ell))\right| \\
&+ k\,|y(\ell)-y_s(\ell)|\,|\varphi_1(x_s(\ell))| + \delta^* \\
&\le (d+k\,r)|x(\ell)-x_s(\ell)| + k\,r\,|y(\ell)-y_s(\ell)| + \delta^*, \\
\|x-x_s\|_C &\le (d+k\,r)\|x-x_s\|_C + k\,r\,\|y-y_s\|_C + \delta^*.
\end{aligned}$$

Similarly;

$$\|y-y_s\|_C \le (d+k\,r)\|y-y_s\|_C + k\,r\,\|x-x_s\|_C + \delta^*.$$

Now

$$\begin{aligned}
\|(x,y) - (x_s,y_s)\|_{\mathbb{E}} &= \|x - x_s\|_C + \|y - y_s\|_C \\
&\le (d+k\,r)\|x-x_s\|_C + k\,r\,\|y-y_s\|_C + \delta^* \\
&+ (d+k\,r)\|y-y_s\|_C + k\,r\,\|x-x_s\|_C + \delta^* \\
&= (d+2\,k\,r)\|x-x_s\|_C + (d+2\,k\,r)\|y-y_s\|_C + 2\,\delta^* \\
&= \alpha\Big(\|x - x_s\|_C + \|y - y_s\|_C\Big) + 2\,\delta^* \\
&= \alpha\,\|(x,y) - (x_s,y_s)\|_{\mathbb{E}} + 2\,\delta^*,
\end{aligned}$$

which implies

$$\|(x,y) - (x_s,y_s)\|_{\mathbb{E}} \le \frac{2\,\delta^*}{1-\alpha} = \epsilon.$$

**Corollary 5.1.** *Let the assumptions of Theorem (3.1) be satisfied, if $x(\ell) = y(\ell), a_1(\ell) = a_2(\ell), \varphi_1 = \varphi_2$ and $f_1 = f_2$ in the Distributed state-dependent integral equation (1.1) with conjugate feedback control (1.2), then we deduce that the solution of the following equation*

$$x(\ell) = a(\ell) + \int_0^{\varphi(x(\ell))} f(s,x(s))ds.$$

*exists and this solution is Hyers-Ulam stable.*

## 6. Some Particular Cases

Letting $\phi_i(u) = \eta_i u, u \in \mathbb{C}$ and $\eta_i \in (0,1), i = 1,2$, then we have the Distributed state-dependent integral equation

$$x(\ell) \ = \ a_1(\ell) \ + \ \int_0^{\eta_1 x(\ell)} f_1(s, y(s))ds,$$

with conjugate feedback control

$$y(\ell) \ = \ a_2(\ell) \ + \ \int_0^{\eta_2 y(\ell)} f_2(s, x(s))ds.$$

Note that $\phi_i(u) = \eta_i u$ satisfy condition $(iii)$. Then from Theorem (2.1), we obtain that the solution of this problem exists, from Theorem (3.1), we get that this solution is unique and from section 4, we get that this solution depends continuously on each of $a_i(.)$, $\eta_i$ and $f_i(.,.)$. Also, this solution is Hyers-Ulam stable.

Now, if

$$f_1(\ell, y(\ell)) \ = \ f_2(\ell, x(\ell)) \ = \ (1+2\ell)^2$$

(note that each of $f_1$ and $f_2$ satisfies condition $(i)$) and if $x(\ell) = y(\ell)$, $a_1(\ell) = a_2(\ell)$ and $\eta_1 = \eta_2$. Then, we obtain the problem

$$x(\ell) \ = \ a(\ell) + \int_0^{\eta x(\ell)} (1+2s)^2 ds,$$

Now, by simple calculations, we get that the problem:

$$\begin{aligned} x(\ell) &= a(\ell) + \int_0^{\eta x(\ell)} (1+2s)^2 ds \\ &= a(\ell) + \frac{1}{2}\frac{(1+2s)^3}{3}\bigg|_0^{\eta x(\ell)} \\ &= a(\ell) + \frac{1}{2}\bigg(\frac{[1+2\eta\, x(\ell)]^3}{3} - \frac{1}{3}\bigg) \end{aligned}$$

has a unique continuous solution which depends continuously on each of $a(.)$ and $\eta$. Also, this solution is Hyers-Ulam stable.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References

[1] B. Ahmad, A. Alsaedi, Existence and Uniqueness of Solutions for Coupled Systems of Higher-Order Nonlinear Fractional Differential Equations, Fixed Point Theory Appl. 2010 (2010), 364560. https://doi.org/10.1155/2010/364560.

[2] B. Ahmad, S.K. Ntouyas, A. Alsaedi, On a Coupled System of Fractional Differential Equations with Coupled Nonlocal and Integral Boundary Conditions, Chaos Solitons Fractals 83 (2016), 234–241. https://doi.org/10.1016/j.chaos.2015.12.014.

[3] K.L. Baluja, P.G. Burke, L.A. Morgan, R-Matrix Propagation Program for Solving Coupled Second-Order Differential Equations, Comput. Phys. Commun. 27 (1982), 299-307.

[4] K. Deimling, Nonlinear Functional Analysis, Dover publications, Mineola, 2010.

[5] E. Eder, The Functional Differential Equation $x'(t) = x(x(t))$, J. Differ. Equ. 54 (1984), 390–400. https://doi.org/10.1016/0022-0396(84)90150-5.

[6] A.M.A. El-Sayed, S.A. Abd El-Salam, Coupled System of a Fractional Order Differential Equations with Weighted Initial Conditions, Open Math. 17 (2019), 1737–1749. https://doi.org/10.1515/math-2019-0120.

[7] A.M.A. El-Sayed, H. El-Owaidy, R. Gamal Ahmed, Solvability of a Boundary Value Problem of Self-Reference Functional Differential Equation with Infinite Point and Integral Conditions, J. Math. Comput. Sci. 21 (2020), 296–308. https://doi.org/10.22436/jmcs.021.04.03.

[8] A.M.A. El-Sayed, R. Gamal Aahmed, Solvability of the Functional Integro-Differential Equation with Self-Reference and State-Dependence, J. Nonlinear Sci. Appl. 13 (2020), 1–8. https://doi.org/10.22436/jnsa.013.01.01.

[9] H.R. Ebead, Self-Reference (State-Dependence) Quadratic Integral Equation of Fractional Order, J. Fract. Calc. Appl. 15 (2024), 1-15.

[10] F. Hartung, T. Krisztin, H.-O. Walther, J. Wu, Chapter 5 Functional Differential Equations with State-Dependent Delays: Theory and Applications, in: Handbook of Differential Equations: Ordinary Differential Equations, Elsevier, 2006: pp. 435–545. https://doi.org/10.1016/S1874-5725(06)80009-X.

[11] H.H.G. Hashem, Continuous Dependence of Solutions of Coupled Systems of State Dependent Functional Equations, Adv. Differ. Equ. Control Process. 22 (2020), 121–135. https://doi.org/10.17654/DE022020121.

[12] U.V. Lê, E. Pascali, An Existence Theorem for Self-Referred and Hereditary Differential Equations, Adv. Differ. Equ. Control Process. 1 (2008), 25-32.

[13] N.T.T. Lan, P. Eduardo, A Two-Point Boundary Value Problem for a Differential Equation With Self-Reference, Electron. J. Math. Anal. Appl. 6 (2018), 25–30.

[14] M. Feckan, On a Certain Type of Functional Differential Equations, Math. Slovaca 43 (1993), 39–43. http://dml.cz/dmlcz/130391.

[15] M. Miranda Jr., E. Pascali, On a Type of Evolution of Self-Referred and Hereditary Phenomena, Aequat. Math. 71 (2006), 253–268. https://doi.org/10.1007/s00010-005-2821-7.

[16] I. Talib, N.A. Asif. C. Tunc, Existence of Solutions to Second-Order Nonlinear Coupled Systems With Nonlinear Coupled Boundary Conditions, Electron. J. Differ. Equ. 2015 (2015), 313.

[17] R. Tian, Z. Zhang, Existence and Bifurcation of Solutions for a Double Coupled System of Schrödinger Equations, Sci. China Math. 58 (2015), 1607–1620. https://doi.org/10.1007/s11425-015-5028-y.

[18] S.M. Ulam, A Collection of the Mathematical Problems, Interscience Publishers, New York, 1960.

[19] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964.