Int. J. Anal. Appl. (2025), 23:21

International Journal of Analysis and Applications

On Functional Equation Mixed q-Fractional Integral and Fractional Derivative with q-Integral Condition

Reda Gamal Ahmed[1,*], Ahmed M. A. El-Sayed[2], Muhanna Ali H. Alrashdi[3]

[1]*Faculty of Science, Al-Azhar University, 11884 Nasr City, Cairo, Egypt*
[2]*Faculty of Science, Alexandria University, Alexandria 21544, Egypt*
[3]*College of Science, Department of Mathematics, University of Hail, Saudi Arabia*

[*]*Corresponding author:* redagamal@azhar.edu.eg

Abstract. This paper examines the existence and uniqueness of the solution to the mixed q-fractional integral and fractional derivative with q-integral condition nonlocal problem of the functional integro-differential equation. Moreover, we establish the problem's Hyers-Ulam stability. The solution's continuous dependence will be examined. We'll provide an example to illustrate the findings.

1. Introduction

Both fractional and quantum calculus (q-calculus) have a lengthy history. Quantum calculus, or q-calculus, is a well-established discipline with several applications in complex analysis, hypergeometric series, quantum mechanics, and particle physics. Jackson developed the q-calculus first [1]. Books [2, 3] contain basic definitions and properties of q-calculus. q-calculus, This area of research has several applications, see [4–6] and references therein. There are several developments and applications of the q-calculus in mathematical physics, the theory of relativity and special functions [7,8]. In several papers [9–12,15], integro–differential equation with Infinite-Point Boundary Conditions have been studied. Reda et al. [5] used the fixed point theory to investigate the existence and uniqueness of some differential equations, including the fractional q-integral with the nonlocal q-integral condition. In this paper, we are concerned with q-fractional functional

Received: Dec. 12, 2024.

2020 *Mathematics Subject Classification.* 05A30, 26A15, 26A33, 26E70.

Key words and phrases. functional equations; existence of solutions; continuous dependence; continuous solutions; nonlocal condition; q-calculus.

https://doi.org/10.28924/2291-8639-23-2025-21
ISSN: 2291-8639

integro-differential equation (nonlocal problem (n-p))

$$\begin{cases} D^{\beta+1}\zeta(t) = \varphi\Big(t, \zeta(t), \lambda I_q^{\delta}\psi(t,\zeta(t))\Big), \quad a.e \quad t, \ \beta \in (0,1], \\ \zeta(0) = 0, \ \int_0^1 \zeta(s) d_q s = \zeta_0. \end{cases} \tag{1.1}$$

Where $D^{\beta+1}$ and I_q^{δ} the Riemann-Liouville fractional derivative of order $\beta+1$ and The fractional q-integral of the Riemann-Liouville type of order $\delta \geq 0$ respectively. The existence of solution, under certain conditions, will be proved. The continuous dependence of the solution on ζ_0 and on the function g, will be studied.

The arrangement of the paper is as follows: In Section 2, Recalling some essential definitions and findings from the literature. In Section 3, We investigate whether there are ongoing solutions to the problem (1.1), We prove that there is just one solution to the problem (1.1), we prove the Hyers-Ulam stability of the problem, the continuous dependence of the solution is studied, and to illustrate the primary existence result, one example is provided. Finally, the conclusion of study in the research.

2. q-Calculus

This section provides some of the necessary resources for our research.We start with some basic definitions and findings from [2,3] for q-calculus, q-fractional integral, and q-derivatives. If $0 < q < 1$ are constants, the q-calculus relations are as follows.

- We define a q–real number $[a]_q$ by

$$[a]_q = \frac{1-q^a}{1-q}, \ a \in \mathbb{R}.$$

- A q-analog of the Pochhammer symbol is defined by

$$(a;q)_k = \begin{cases} 1, & k = 0, \\ \\ \prod_{i=1}^{k-1}(1-aq^i), & k \in \mathbb{N}. \end{cases}$$

- The q–gamma function is defined by

$$\Gamma_q(t) = \frac{E(q^r)}{(1-q)^{r-1}E(q)}, \ r \in \mathbb{R} - \{0,-1,-2,...\},$$

  where $E(q^t) = \frac{1}{(q^r;q)_\infty}$. Or, equivalently, $\Gamma_q(r) = \frac{1-q^{(r-1)}}{1-q^{r-1}}$, and satisfies $\Gamma_q(r+1) = [r]_q \Gamma_q(r)$.
- The q–integral of a function ϑ

$$(I_q\vartheta)(t) = \int_0^t \vartheta(s) d_q s = (1-q)t \sum_{i=0}^{\infty} q^i \vartheta(q^i t), \ t \in [0,b].$$

Definition 2.1. *[4] Let ϑ be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type of order $\delta \geq 0$ is given by*

$$(I_q^{\delta}\vartheta)(t) = \begin{cases} \varphi(t), & \delta = 0, \\ \\ \frac{1}{\Gamma_q(\delta)} \int_0^t (t-qs)^{\delta-1}\vartheta(s)d_qs = (1-q)^{\delta}t^{\delta}\sum_{i=0}^{\infty} q^i \frac{(q^{\delta};q)_i}{(q;q)_i}\vartheta(q^it). & \delta > 0. \end{cases}$$

Lemma 2.1. *[13] For $\delta > 0$, using q–integration by parts, we have*

$$(I_q^{\delta}1)(t) = \frac{t^{(\delta)}}{\Gamma_q(\delta+1)}.$$

3. Main results

The following lemma provides the equivalency between (1.1) and the integral equation.

Lemma 3.1. *The solution of the n-p* (1.1)*, if it exist, then the n-p* (1.1) *and the functional q–integral equation*

$$\begin{aligned} \zeta(t) &= [\beta]_q t^{\beta}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)d_qs\Big] \\ &\quad + I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big). \end{aligned} \tag{3.1}$$

are equivalent.

Proof. Let ζ be a solution of the n-p (1.1), we obtain

$$\begin{aligned} \frac{d^2}{dt^2} I^{2-(1+\beta)}\zeta(t) &= \varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big), \\ \frac{d^2}{dt^2} I^{1-\beta}\zeta(t) &= \varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big), \end{aligned}$$

integrating from 0 to t, we obtain

$$\frac{d}{dt} I^{1-\beta}\zeta(t) - (\frac{d}{dt} I^{1-\beta}\zeta(t))_{t=0} = I\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big),$$

we put $I^{1-\beta}\zeta(t))_{t=0} = \upsilon_1$ and integrating from 0 to t, we obtain

$$I^{1-\beta}\zeta(t) - (I^{1-\beta}\zeta(t))_{t=0} - t\upsilon_1 = I^2\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big),$$

we put $(I^{1-\beta}\zeta(t))_{t=0} = \upsilon_2$ and integrating I^{β}, we obtain

$$I\zeta(t) - I^{\beta}\upsilon_2 - I^{\beta}t\upsilon_1 = I^{2+\beta}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big), \tag{3.2}$$

differentiation (3.2), we get

$$\zeta(t) = \frac{t^{\beta-1}}{\Gamma(\beta)}\upsilon_2 + \frac{t^{\beta}}{\Gamma(1+\beta)}\upsilon_1 + I^{1+\beta}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big).$$

Using the nonlocal q–condition, we get

$$\zeta(t) = \frac{t^\beta}{\Gamma(1+\beta)}v_1 + I^{1+\beta}\varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big), \tag{3.3}$$

$$\int_0^1 \zeta(s)d_qs = v_1\int_0^1 \frac{s^\beta}{\Gamma(1+\beta)}d_qs + \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^\delta\psi(s,\zeta(s))\Big)d_qs,$$

then

$$v_1 = \frac{1}{\int_0^1 \frac{s^\beta}{\Gamma(1+\beta)}d_qs}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^\delta\psi(s,\zeta(s))\Big)d_qs\Big]. \tag{3.4}$$

Using (3.3) and (3.4), we obtain (3.1). To complete the proof, suppose that ζ satisfies equation(3.1), integrating $I^{1-\beta}$ and differentiation (3.1), we obtain

$$I^{1-\beta}\zeta(t) = \frac{t}{\Gamma(2)}v_1 + I^2\varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big),$$

$$\frac{d}{dt}I^{1-\beta}x = v_1 + I\varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big),$$

$$D^{\beta+1}\zeta(t) = \varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big),$$

and

$$\int_0^1 \zeta(\tau)d_q\tau = \frac{1}{\int_0^1 s^\beta d_qs}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^\delta\psi(s,\zeta(s))\Big)d_qs\Big]\int_0^1 s^\beta d_q\tau + \int_0^1 I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big)d_q\tau.$$

Then

$$\int_0^1 \zeta(\tau)d_q\tau = \zeta_0,\ \zeta(0) = 0.$$

□

3.1. **Existence of solution.** The next theorem establishes the existence of at least one solution to (1.1) using Schauder fixed point theorem [14].

Theorem 3.1. *Let $\varphi : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ and $\psi : [0,1]\times\mathbb{R} \to \mathbb{R}$ be fulfills Carathéodory condition, if There exist functions $v_{1,2} \in L^1[0,1]$ and positive constants $p_{1,2} > 0$ such that*

$$|\varphi(t,\zeta,y)| \le v_1(t) + p_1|\zeta| + p_1|y|,$$

$$|\psi(t,\zeta)| \le v_2(t) + p_2|\zeta|,$$

$$\sup_{t\in[0,1]}\int_0^t v_1(s)ds \le V_1, \quad \sup_{t\in[0,1]}\int_0^t I_q^\delta v_2(s)ds \le V_2,$$

and

$$((1+[\beta]_q)[\frac{p_1}{\Gamma(\beta+2)} + \frac{p_1|\lambda|p_2}{\Gamma(\beta+2).\Gamma_q(\delta+1)}]) \in [0,1),$$

then the n-p (1.1) *has at least one solution.*

Proof. Provide a definition for the operator Ξ associated to integral equation (3.1) by

$$\begin{aligned}\Xi\zeta(t) &= [\beta]_q t^{\beta}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)d_qs\Big] \\ &\quad + I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big).\end{aligned}$$

Let $\Omega_r = \{\zeta(t)\in\mathbb{R} : \|x\|\le r\}$, where $r \ge \frac{[\beta]_q|\zeta_0|+(1+[\beta]_q)[V_1+\frac{p_1|\lambda|V_2}{\Gamma(\beta+2)}]}{1-((1+[\beta]_q)[\frac{p_1}{\Gamma(\beta+2)}+\frac{p_1|\lambda|p_2}{\Gamma(\beta+2).\Gamma_q(\delta+1)}])}$.

Then we have, for $\zeta\in\Omega_r$

$$\begin{aligned}|\Xi\zeta(t)| &\le \Big|[\beta]_q t^{\beta}\Big|\Big[|\zeta_0| + \int_0^1 I^{\beta+1}\Big|\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)\Big|d_qs\Big] \\ &\quad + I^{\beta+1}\Big|\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big)\Big| \\ &\le \Big|[\beta]_q t^{\beta}\Big|\Big[|\zeta_0| + \int_0^1 I^{\beta+1}(|v_1(s)| + p_1|\zeta(s)| + p_1|\lambda|I_q^{\delta}(v_2(s)+p_2|\zeta(s)|))d_qs\Big] \\ &\quad + I^{\beta+1}\Big(|v_1(t)| + p_1|\zeta(t)| + p_1|\lambda|I_q^{\delta}(v_2(t)+p_2|\zeta(t)|)\Big) \\ &\le \Big|[\beta]_q t^{\beta}\Big|\Big[|\zeta_0| + \int_0^1\Big[V_1 + \int_0^t \frac{(t-\theta)^{\beta}}{\Gamma(\beta+1)}\Big(p_1\|x\| + p_1|\lambda|(V_2 + I_q^{\delta}p_2\|x\|)\Big)d\theta\Big]d_qs\Big] \\ &\quad + V_1 + \int_0^t \frac{(t-\theta)^{\beta}}{\Gamma(\beta+1)}\Big(p_1\|x\| + p_1|\lambda|(V_2 + I_q^{\delta}p_2\|x\|)\Big)d\theta \\ &\le \Big|[\beta]_q t^{\beta}\Big|\Big[|\zeta_0| + \int_0^1\Big[V_1 + \frac{1}{\Gamma(\beta+2)}(p_1\|x\| + p_1|\lambda|V_2) \\ &\quad + p_1|\lambda|p_2\|x\|\int_0^t \frac{(t-\theta)^{\beta}}{\Gamma(\beta+1)}.\frac{\theta^{\delta}}{\Gamma_q(\delta+1)}d\theta\Big]d_qs\Big] \\ &\quad + V_1 + \frac{1}{\Gamma(\beta+2)}(p_1\|x\| + p_1|\lambda|V_2) + p_1|\lambda|p_2\|x\|\int_0^t \frac{(t-\theta)^{\beta}}{\Gamma(\beta+1)}.\frac{\theta^{\delta}}{\Gamma_q(\delta+1)}d\theta \\ &\le \Big|[\beta]_q t^{\beta}\Big|\Big[|\zeta_0| + \int_0^1\Big[V_1 + \frac{1}{\Gamma(\beta+2)}(p_1\|x\| + p_1|\lambda|V_2) \\ &\quad + p_1|\lambda|p_2\|x\|\frac{1}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\Big]d_qs\Big] \\ &\quad + V_1 + \frac{1}{\Gamma(\beta+2)}(p_1\|x\| + p_1|\lambda|V_2) + p_1|\lambda|p_2\|x\|\frac{1}{\Gamma(\beta+2).\Gamma_q(\delta+1)} \\ &\le [\beta]_q\Big[|\zeta_0| + V_1 + \frac{p_1\|x\| + p_1|\lambda|V_2}{\Gamma(\beta+2)} + \frac{p_1|\lambda|p_2\|x\|}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\Big] \\ &\quad + V_1 + \frac{p_1\|x\| + p_1|\lambda|V_2}{\Gamma(\beta+2)} + \frac{p_1|\lambda|p_2\|x\|}{\Gamma(\beta+2).\Gamma_q(\delta+1)} \\ &= [\beta]_q|\zeta_0| + (1+[\beta]_q)\Big[V_1 + \frac{p_1r + p_1|\lambda|V_2}{\Gamma(\beta+2)} + \frac{p_1|\lambda|p_2r}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\Big]\le r.\end{aligned}$$

This demonstrates that $\Xi : \Omega_r \to \Omega_r$ and the class of functions $\{\Xi\zeta\}$ is uniformly bounded in Ω_r.
Now, let $t_1, t_2 \in (0,1)$ such that $|t_2 - t_1| < \delta$, then

$$
\begin{aligned}
&|\Xi\zeta(t_2) - \Xi\zeta(t_1)| = \Bigg|[\beta]_q t_2^\beta\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)d_q s\Big] \\
&+ \int_0^{t_2} \frac{(t_2-\theta)^\beta}{\Gamma(\beta+1)}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)ds \\
&- [\beta]_q t_1^\beta\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)d_q s\Big] \\
&- \int_0^{t_1} \frac{(t_1-\theta)^\beta}{\Gamma(\beta+1)}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)ds\Bigg| \\
&\le |t_2^\beta - t_1^\beta|[\beta]_q\Bigg|\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)d_q s\Bigg| \\
&+ \Bigg|\int_{t_1}^{t_2} \frac{(t_2-\theta)^\beta}{\Gamma(\beta+1)}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)ds\Bigg| \\
&+ \Bigg|\int_0^{t_1} \frac{(t_2-\theta)^\beta - (t_1-\theta)^\beta}{\Gamma(\beta+1)}\varphi\Big(s, \zeta(s), \lambda I_q^\delta \psi(s,\zeta(s))\Big)ds\Bigg| \\
&\le |t_2^\beta - t_1^\beta|[\beta]_q\Bigg[|\zeta_0| + V_1 + \frac{p_1 r + p_1|\lambda|V_2}{\Gamma(\beta+2)} + \frac{p_1|\lambda|p_2 r}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\Bigg] \\
&+ \Bigg|\int_{t_1}^{t_2}\Bigg|\frac{(t_2-\theta)^\beta}{\Gamma(\beta+1)}\Bigg|\Bigg[|v_1(t)| + p_1 r + p_1|\lambda| I_q^\delta(v_2(t) + p_2 r)\Bigg] \\
&+ \int_0^{t_1}\Bigg|\frac{(t_2-\theta)^\beta - (t_1-\theta)^\beta}{\Gamma(\beta+1)}\Bigg|\Bigg[|v_1(t)| + p_1 r + p_1|\lambda| I_q^\delta(v_2(t) + p_2 r)\Bigg]
\end{aligned}
$$

This implies that the class of functions $\{\Xi\zeta\}$ is equi-continuous in Ω_r.
Let $\zeta_n \in \Omega_r$, $\zeta_n \to \zeta (n \to \infty)$, then from continuity of the functions φ and $\psi($, we obtain $\varphi(t, \zeta_n(t), y_n(t)) \to \varphi(t, \zeta(t), y(t))$ and $\psi(t, \zeta_n(t)) \to \psi(t, \zeta(t))$ as $n \to \infty$. Also

$$
\begin{aligned}
|\Xi\varphi_n(t) - \Xi\zeta(t)| &= \Bigg|[\beta]_q t^\beta[\zeta_0 - \int_0^1 I^{\beta+1}\varphi(s, \varphi_n(s), \lambda I_q^\delta \psi(s, \varphi_n(s)))d_q s] \\
&\quad + I^{\beta+1}\varphi(t, \varphi_n(t), \lambda I_q^\delta \psi(t, \varphi_n(t))) \\
&\quad - [\beta]_q t^\beta[\zeta_0 - \int_0^1 I^{\beta+1}\varphi(s, \zeta(s), \lambda I_q^\delta \psi(s, \zeta(s)))d_q s] \\
&\quad - I^{\beta+1}\varphi(t, \zeta(t), \lambda I_q^\delta \psi(t, \zeta(t)))\Bigg| \\
&\le [\beta]_q t^\beta \int_0^1 I^{\beta+1}\Bigg|\varphi(s, \varphi_n(s), \lambda I_q^\delta \psi(s, \varphi_n(s)))d_q s \\
&\quad - \varphi(s, \zeta(s), \lambda I_q^\delta \psi(s, \zeta(s)))d_q s\Bigg|
\end{aligned}
$$

$$+I^{\beta+1}\Big|\varphi(t,\varphi_n(t),\lambda I_q^{\delta}\psi(t,\varphi_n(t)))-\varphi(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t)))\Big|.$$

Using assumptions (1)-(2), the operator Ξ is continuous. Then by Schauder fixed point Theorem [14], at least one solution exists $\zeta\in C[0,1]$ of the n-p (1.1). □

3.2. **Uniqueness of the solution.** We prove the existence of exactly one solution to (1.1) in the following theorem.

Theorem 3.2. *Let $\varphi:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ is continuous in ζ, y $\forall t\in[0,1]$ and measurable in t for any $\zeta,y\in\mathbb{R}$ and $\psi:[0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous in ζ $\forall t\in[0,1]$ and measurable in t for any $\zeta\in\mathbb{R}$. If there exists $p_{1,2}>0$ with*

$$|\varphi(t,\zeta,y)-\varphi(t,u,v)|\le p_1|\zeta-u|+p_1|y-v|, \tag{3.5}$$

and

$$|\psi(t,\zeta)-\psi(t,u)|\le p_2|\zeta-u|, \tag{3.6}$$

then the solution of the n-p (1.1) *is unique.*

Proof. Let ζ,y be two the solution of (1.1), then

$$\begin{aligned}
|\zeta(t)-y(t)| &= \Big|[\beta]_q t^{\beta}\Big[\zeta_0-\int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)d_qs\Big]\\
&\quad +I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big)\\
&\quad -[\beta]_q t^{\beta}\Big[\zeta_0-\int_0^1 I^{\beta+1}\varphi\Big(s,y(s),\lambda I_q^{\delta}\psi(s,y(s))\Big)d_qs\Big]\\
&\quad -I^{\beta+1}\varphi\Big(t,y(t),\lambda I_q^{\delta}\psi(t,y(t))\Big)|\\
&\le |[\beta]_q t^{\beta}|\int_0^1 I^{\beta+1}\Big|\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)\\
&\quad -\varphi\Big(s,y(s),\lambda I_q^{\delta}\psi(s,y(s))\Big)\Big|d_qs\\
&\quad +I^{\beta+1}\Big|\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big)-\varphi\Big(t,y(t),\lambda I_q^{\delta}\psi(t,y(t))\Big)\Big|\\
&\le |[\beta]_q t^{\beta}|\int_0^1 I^{\beta+1}\Big[p_1|\zeta(s)-y(s)|+p_1\lambda I_q^{\delta}|\psi(s,\zeta(s))-\psi(s,y(s))|\Big]d_qs\\
&\quad +I^{\beta+1}\Big[p_1|\zeta(t)-y(s)|+p_1\lambda I_q^{\delta}|\psi(t,\zeta(t))-\psi(t,y(t))|\Big]\\
&\le ([\beta]_q+1)[\frac{p_1}{\Gamma(\beta+2)}+\frac{p_1p_2\lambda}{\Gamma(\beta+2).\Gamma_q(\delta+1)}]||\zeta-y||.
\end{aligned}$$

Hence

$$(1-([\beta]_q+1)[\frac{p_1}{\Gamma(\beta+2)}+\frac{p_1p_2\lambda}{\Gamma(\beta+2).\Gamma_q(\delta+1)}])||\zeta-y||\le 0.$$

since $([\beta]_q+1)[\frac{p_1}{\Gamma(\beta+2)}+\frac{p_1p_2\lambda}{\Gamma(\beta+2).\Gamma_q(\delta+1)}]<1$, then $\zeta=y$ and the solution of the n-p (1.1) is unique. □

3.3. Hyers–Ulam Stability.

Definition 3.1. *Let the solution to* (1.1) *exist. The n-p* (1.1) *is Hyers–Ulam-stable if* $\forall\epsilon>0,\ \exists\ \delta(\epsilon)>0$ *such that, for any solution* $\zeta_s\in C[0,1]$ *of* (1.1) *satisfying*

$$\left|D^{\beta+1}\zeta_s(t)-\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\right|\leq\delta. \tag{3.7}$$

Then

$$\|\zeta-\zeta_s\|\leq\epsilon.$$

Theorem 3.3. *Assume that the hypothesis of Theorem 2 is satisfied; then, problem* (1.1) *is Hyers–Ulam-stable.*

Proof. Let the condition of Equation (3.7) be satisfied; then, we have

$$-\delta\leq D^{\beta+1}\zeta_s(t)-\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq\delta$$

$$-\delta\leq\frac{d^2}{dx^2}I^{1-\beta}\zeta_s(t)-\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq\delta$$

$$-I\delta\leq\frac{d}{dx}I^{1-\beta}\zeta_s(t)-\upsilon_1-I\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq\delta$$

$$-I^2\delta\leq I^{1-\beta}\zeta_s(t)-\upsilon_2-t\upsilon_1-I^2\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq I^2\delta$$

$$-I^{\beta+2}\delta\leq I\zeta_s(t)-I^{\beta}\upsilon_2-I^{\beta}t\upsilon_1-I^{\beta+2}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq I^{\beta+2}\delta$$

$$-I^{\beta+1}\delta\leq\zeta_s(t)-\frac{t^{\beta}-1}{\Gamma(\beta)}\upsilon_2-\frac{t^{\beta}-1}{\Gamma(\beta+1)}\upsilon_1-I^{\beta+2}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq I^{\beta+1}\delta$$

$$-I^{\beta+1}\delta=-\delta_1\leq\zeta_s(t)-[\beta]_qt^{\beta}\Big[\zeta_0-\int_0^1 I^{\beta+1}\varphi\Big(s,\zeta_s(s),\lambda I_q^{\delta}\psi(s,\zeta_s(s))\Big)d_qs\Big]$$

$$-I^{\beta+1}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\leq I^{\beta+1}=\delta_1.$$

Now,

$$|\zeta-\zeta_s|=\Big|[\beta]_qt^{\beta}\Big[\zeta_0-\int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)d_qs\Big]$$

$$+I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big)-\zeta_s$$

$$-[\beta]_qt^{\beta}\Big[\zeta_0-\int_0^1 I^{\beta+1}\varphi\Big(s,\zeta_s(s),\lambda I_q^{\delta}\psi(s,\zeta_s(s))\Big)d_qs\Big]$$

$$-I^{\beta+1}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)$$

$$
\begin{aligned}
&+ [\beta]_q t^{\beta}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta_s(s),\lambda I_q^{\delta}\psi(s,\zeta_s(s))\Big)d_q s\Big] \\
&+ I^{\beta+1}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\Big| \\
&\leq \Big|[\beta]_q t^{\beta}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big)d_q s\Big] \\
&+ I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big) \\
&- [\beta]_q t^{\beta}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta_s(s),\lambda I_q^{\delta}\psi(s,\zeta_s(s))\Big)d_q s\Big] \\
&- I^{\beta+1}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\Big| \\
&+ \Big|[\beta]_q t^{\beta}\Big[\zeta_0 - \int_0^1 I^{\beta+1}\varphi\Big(s,\zeta_s(s),\lambda I_q^{\delta}\psi(s,\zeta_s(s))\Big)d_q s\Big] \\
&+ I^{\beta+1}\varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big) - \zeta_s\Big| \\
&\leq |[\beta]_q t^{\beta}| \int_0^1 I^{\beta+1}\Big|\varphi\Big(s,\zeta(s),\lambda I_q^{\delta}\psi(s,\zeta(s))\Big) - \varphi\Big(s,\zeta_s(s),\lambda I_q^{\delta}\psi(s,\zeta_s(s))\Big)\Big| d_q s \\
&+ I^{\beta+1}\Big|\varphi\Big(t,\zeta(t),\lambda I_q^{\delta}\psi(t,\zeta(t))\Big) - \varphi\Big(t,\zeta_s(t),\lambda I_q^{\delta}\psi(t,\zeta_s(t))\Big)\Big| + \delta_1 \\
&\leq |[\beta]_q t^{\beta}| \int_0^1 I^{\beta+1}\Big[p_1|\zeta(s)-\zeta_s(s)| + p_1\lambda I_q^{\delta}|\psi(s,\zeta(s)) - \psi(s,\zeta_s(s))|\Big]d_q s \\
&+ I^{\beta+1}\Big[p_1|\zeta(t)-\zeta_s(s)| + p_1\lambda I_q^{\delta}|\psi(t,\zeta(t)) - \psi(t,\zeta_s(t))|\Big] + \delta_1 \\
&\leq ([\beta]_q + 1)\Big[\frac{p_1}{\Gamma(\beta+2)} + \frac{p_1 p_2 \lambda}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\Big]\|\zeta - \zeta_s\| + \delta_1.
\end{aligned}
$$

Hence

$$
\|\zeta - \zeta_s\| \leq \frac{\delta_1}{1 - \big(\frac{([\beta]_q+1)p_1}{\Gamma(\beta+2)} + \frac{(q+2)p_1 p_2|\lambda|}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\big)} = \epsilon.
$$

Then, the problem (1.1) is Hyers–Ulam-stable. □

3.4. **Continuous dependence.** The meaning of continuous depends on the initial data ζ_0 and the parameter λ is provided below.

Definition 3.2. *The solution $\zeta \in C[0,1]$ of the n-p (1.1) depends continuously on ζ_0 and λ, if*

$$
\forall \epsilon > 0, \quad \exists \quad \delta_1(\epsilon) \quad s.t \quad |\zeta_0 - \zeta_0^*| < \delta_2,\ |\lambda - \lambda^*| < \delta_3 \Rightarrow \|\zeta - \zeta^*\| < \epsilon_1,
$$

where ζ^* is the solution of the n-p

$$\begin{cases} D^{\beta+1}\zeta^*(t)=\varphi\Big(t,\zeta^*(t),\lambda^* I_q^\delta\psi(t,\zeta^*(t))\Big), \quad a.e \quad t,\ \beta\in(0,1], \\ \zeta^*(0)=0,\ \int_0^1\zeta^*(s)d_qs=\zeta_0^*. \end{cases} \tag{3.8}$$

Theorem 3.4. *Assume that the assumptions in Theorem 3.2 are met, then the solution of the n-p* (1.1) *depends continuously on ζ_0 and λ.*

Proof. Let ζ,ζ^* be two solutions of the n-p (1.1) and (3.8) respectively. Then

$$\begin{aligned}
&|\zeta(t)-\zeta^*(t)| \\
= \ & \Bigg|[\beta]_q t^\beta\Big[\zeta_0-\int_0^1 I^{\beta+1}\varphi\Big(s,\zeta(s),\lambda I_q^\delta\psi(s,\zeta(s))\Big)d_qs\Big] \\
&+I^{\beta+1}\varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big) \\
&-[\beta]_q t^\beta\Big[\zeta_0^*-\int_0^1 I^{\beta+1}\varphi\Big(s,\zeta^*(s),\lambda^* I_q^\delta\psi(s,\zeta^*(s))\Big)d_qs\Big] \\
&-I^{\beta+1}\varphi\Big(t,\zeta^*(t),\lambda^* I_q^\delta\psi(t,\zeta^*(t))\Big)\Bigg| \\
\leq \ & [\beta]_q[|\zeta_0-\zeta_0^*|+\int_0^1 I^{\beta+1}\Big|\varphi\Big(s,\zeta(s),\lambda I_q^\delta\psi(s,\zeta(s))\Big)-\varphi\Big(s,\zeta^*(s),\lambda^* I_q^\delta\psi(s,\zeta^*(s))\Big)\Big|d_qs] \\
&+I^{\beta+1}\Big|\varphi\Big(t,\zeta(t),\lambda I_q^\delta\psi(t,\zeta(t))\Big)-\varphi\Big(t,\zeta^*(t),\lambda^* I_q^\delta\psi(t,\zeta^*(t))\Big)\Big| \\
\leq \ & [\beta]_q[|\zeta_0-\zeta_0^*|+\int_0^1 I^{\beta+1}p_1[|\zeta(s)-\zeta^*(s)|+|\lambda I_q^\delta\psi(s,\zeta(s))-\lambda^* I_q^\delta\psi(s,\zeta^*(s))|]d_qs] \\
&+I^{\beta+1}p_1[|\zeta(t)-\zeta^*(t)|+|\lambda I_q^\delta\psi(t,\zeta(t))-\lambda^* I_q^\delta\psi(t,\zeta^*(t))|] \\
\leq \ & [\beta]_q[\delta_2+\int_0^1 I^{\beta+1}p_1[\|\zeta-\zeta^*\|+|\lambda-\lambda^*|I_q^\delta|\psi(s,\zeta(s))|+|\lambda^*|I_q^\delta|\psi(s,\zeta(s))-\psi(s,\zeta^*(s))|]d_qs] \\
&+I^{\beta+1}p_1[\|\zeta-\zeta^*\|+|\lambda-\lambda^*|I_q^\delta|\psi(t,\zeta(t))|+|\lambda^*|I_q^\delta|\psi(t,\zeta(t))-\psi(t,\zeta^*(t))|] \\
\leq \ & [\beta]_q[\delta_2+\frac{p_1}{\Gamma(\beta+2)}\|\zeta-\zeta^*\|+\frac{p_1\delta_3}{\Gamma(\beta+2).\Gamma_q(\delta+2)}+\frac{p_1p_2|\lambda^*|}{\Gamma(\beta+2).\Gamma_q(\delta+1)}\|\zeta-\zeta^*\|] \\
&+\frac{p_1}{\Gamma(\beta+2)}\|\zeta-\zeta^*\|+\frac{p_1\delta_3}{\Gamma(\beta+2).\Gamma_q(\delta+1)}+\frac{p_1p_2|\lambda^*|}{\Gamma(\beta+2).\Gamma_q(\delta+2)}\|\zeta-\zeta^*\| \\
= \ & [\beta]_q\delta_2+\frac{([\beta]_q+1)p_1\delta_3}{\Gamma([\beta]_q+1).\Gamma_q(\delta+2)}+(\frac{([\beta]_q+1)p_1}{\Gamma(\beta+2)}+\frac{(q+2)p_1p_2|\lambda^*|}{\Gamma(\beta+2).\Gamma_q(\delta+1)})\|\zeta-\zeta^*\|.
\end{aligned}$$

Hence

$$\|\zeta-\zeta^*\|\leq\frac{[\beta]_q\delta_2+\frac{([\beta]_q+1)p_1\delta_3}{\Gamma(\beta+2).\Gamma_q(\delta+2)}}{1-(\frac{([\beta]_q+1)p_1}{\Gamma(\beta+2)}+\frac{(q+2)p_1p_2|\lambda^*|}{\Gamma(\beta+2).\Gamma_q(\delta+1)})}=\epsilon_1.$$

This implies that the solution of the n-p (1.1) depends continuously on ζ_0 and λ. □

3.5. **Examples:** In this section, we highlight our findings with an example.

Example 3.1. *Consider the following n-p*

$$\begin{cases} D^{\frac{3}{2}}\zeta = t^2e^{-t} + \frac{1+\zeta(t)}{3+t^2} + I_{0.5}^{\frac{2}{5}}\frac{1}{9}(\cos(2t+2) \\ +t^3\cos\zeta(t) + e^{-t}\zeta(t)), \quad a.e \quad t \in (0,1], \\ \zeta(0) = \zeta_0, \int_0^1 \zeta(s)d_{0.5}s = \zeta_0. \end{cases} \tag{3.9}$$

Set

$$\begin{aligned} \varphi(t,\zeta(t), I_q^\delta\psi(t,\zeta(t))) &= t^2e^{-t} + \frac{1+\zeta(t)}{3+t^2} + \frac{1}{9}I_{0.5}^{\frac{2}{5}}(\cos(2t+2) \\ &\quad +t^3\cos\zeta(t) + e^{-t}\zeta(t)). \end{aligned}$$

Then

$$\begin{aligned} |\varphi(t,\zeta(t), I_q^\delta\psi(t,\zeta(t)))| &\leq t^2e^{-t} + \frac{1}{3}(|x| + \frac{1}{3}I_{0.5}^{\frac{2}{5}}\frac{1}{3}|(\cos(2t+2) \\ &\quad +t^3\cos\zeta(t) + e^{-t}\zeta(t))|), \end{aligned}$$

and also

$$|\psi(t,\zeta(t))| \leq \frac{1}{3}|\cos(2t+2)| + \frac{1}{3}|\zeta(t)|.$$

It is evident that the assumptions (1)-(4) of Theorem 3.1 are satisfied with
$\upsilon_1(t) = t^2e^{-t} \in L^1[0,1], \quad \upsilon_2(t) = \frac{1}{2}|\cos(2t+2)| \in L^1[0,1], p_1 = \frac{1}{3}, p_2 = \frac{1}{3},$

$$([\beta]_q + 1)[\frac{p_1}{\Gamma(\beta+2)} + \frac{p_1|\lambda|p_2}{\Gamma(\beta+2).\Gamma_q(\delta+1)}] = (1.585786)[\frac{\frac{1}{3}}{\Gamma(2.5)} + \frac{\frac{1}{3}.\frac{1}{3}}{\Gamma(2.5).\Gamma_0.5(\frac{7}{5})}] < 1,$$

by applying to Theorem 3.1, the given n-p (3.9) *has a continuous solution.*

Conclusion

This work has examined the continuous dependence of the q-fractional functional integro-differential equation on initial data and parameter, existence solutions using the Schauder fixed point theorem, and uniqueness solutions. We demonstrated that the problem is Hyers-Ulam stable. To demonstrate the utility of our findings, a few examples are presented.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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