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Bipolar Fuzzy Magnified Translation of Γ-Near Rings

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Abstract. This study introduces the concept of bipolar fuzzy magnified translations of Γ-near rings (BF-MT-GNRs), extending the application of bipolar fuzzy set theory within Γ-near rings. The research establishes a one-to-one correspondence between BF-MT-GNRs and bipolar fuzzy sub-GNRs, ideals, and bi-ideals, offering a deeper understanding of these algebraic structures. Furthermore, homomorphisms on BF-MT-GNRs are explored to demonstrate their structural properties and theoretical consistency. These findings contribute significantly to the ongoing development of bipolar fuzzy set theory and its applications in advanced algebraic frameworks. In alignment with Sustainable Development Goal 4 (SDG 4) on Quality Education, this study promotes mathematical literacy and critical thinking by providing new perspectives on algebraic structures that can be incorporated into school and university curricula. By making abstract mathematical concepts more accessible to students, this research fosters inclusive and equitable learning opportunities, empowering both educators and learners in their pursuit of higher-level mathematical knowledge. Moreover, the results serve as a valuable resource for researchers, facilitating further studies in algebraic systems with applications in computational mathematics, cryptography, and decision-making models. Ultimately, this work supports the global effort to enhance education at all levels, ensuring that students acquire the skills necessary for future academic and professional success.

1. Introduction

The proposal of a bipolar-valued fuzzy set (BF-set) was conferred by Zhang [16], which is the broadening of the theory of Zadeh's [15] fuzzy sets to BF-sets. Γ-Near rings (GNRs) were

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characterized, and the ideal theory in Γ-near rings was extensively explored by Satyanarayana [10] and Booth [2]. Likewise, various authors examined numerous algebraic structures on GNRs, like ideals, bi-ideals, weak ideals, normal ideals, and quasi-ideals. Several researchers, like Ragamayi [7–9, 13, 14], have done their scrutiny of the progress of BF-theory on diverse algebraic structures such as groups, subgroups, rings, semirings, etc. Initially, Majumder and Sardar [6] studied fuzzy magnified translation. The notion of intuitionistic fuzzy magnified translation in groups has been discussed by Sharma [11]. Presented herein, we introduce the BF-MT-GNRs and draw a few significant results.

2. Preliminaries

To establish a robust foundation for the concept of BF-MT-GNRs, it is imperative to first revisit the essential definitions and properties that underpin GNRs and bipolar fuzzy set theory. This section outlines the fundamental algebraic structures, including GNRs, fuzzy sets, and bipolar fuzzy sets, along with their associated operations and properties. These preliminaries not only contextualize the proposed framework but also provide the necessary mathematical tools for understanding the subsequent development and results. By bridging these foundational concepts, the groundwork is laid for a comprehensive exploration of BF-MT-GNRs and their algebraic implications.

Definition 2.1. *[10] A Γ-near ring (GNR) is a triple $(M_R, +, \Gamma)$ whereas*
(i) $(M_R, +)$ is a group,
(ii) $\Gamma \neq \emptyset$, is a binary operator set on M_R such that $(M_R, +, \alpha)$ is a near-ring for each $\alpha \in \Gamma$,
(iii) $\varphi\alpha(\varpi\kappa\varkappa) = (\varphi\alpha\varpi)\kappa\varkappa$ for all $\varphi, \varpi, \varkappa \in M_R$ and $\alpha, \kappa \in \Gamma$.

Remark 2.1. *A GNR M_R is said to be zero-symmetric if $\varphi\alpha 0 = 0$ for all $\varphi \in M_R$ and $\alpha \in \Gamma$. All over this paper, M_R denotes a zero-symmetric right GNR consisting of at least two elements.*

Definition 2.2. *[3] A fuzzy set ξ of M_R is a fuzzy sub-GNR if*
(i) $\xi(\varphi - \varpi) \geq \min\{\xi(\varphi), \xi(\varpi)\}$,
(ii) $\xi(\varphi\alpha\varpi) \geq \min\{\xi(\varphi), \xi(\varpi)\}, \forall \varphi, \varpi \in M_R, \alpha \in \Gamma$.

Definition 2.3. *[5] Consider a set $\mathcal{D}$ over the universal set $\mathcal{U}$ defined by the positive and negative membership functions, $\xi^+_{\mathcal{D}} : \mathcal{U} \to [0,1]$ and $\xi^-_{\mathcal{D}} : \mathcal{U} \to [-1,0]$. Then $\mathcal{D}$ is claimed to be a bipolar fuzzy set (BF-set) of $\mathcal{U}$, and represented as $\mathcal{D} = (\xi^+_{\mathcal{D}}, \xi^-_{\mathcal{D}})$.*

Definition 2.4. *[4, 16] A BF-set $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a BF-sub-GNR of M_R, provided M_R is a GNR and B_R a BF-set of M_R if*
(i) $\xi^+_{B_R}(\varphi - \varpi) \geq \min\{\xi^+_{B_R}(\varphi), \xi^+_{B_R}(\varpi)\}$ and $\xi^-_{B_R}(\varphi - \varpi) \leq \max\{\xi^-_{B_R}(\varphi), \xi^-_{B_R}(\varpi)\}$,
(ii) $\xi^+_{B_R}(\varphi\alpha\varpi) \geq \min\{\xi^+_{B_R}(\varphi), \xi^+_{B_R}(\varpi)\}$ and $\xi^-_{B_R}(\varphi\alpha\varpi) \leq \max\{\xi^-_{B_R}(\varphi), \xi^-_{B_R}(\varpi)\}$, $\forall \varphi, \varpi \in M_R, \alpha \in \Gamma$.

Definition 2.5. *[3] A fuzzy set ξ of M_R is called a fuzzy left (resp., right) ideal of M_R if*
(i) $\xi(\varphi - \varpi) \geq \min\{\xi(\varphi), \xi(\varpi)\}$,
(ii) $\xi(\varpi + \varphi - \varpi) \geq \xi(\varphi)$,

(iii) $\xi(a_l\alpha(\varphi+b_l)-a_l\alpha b_l)\geq\xi(\varphi)$ *(resp.,* $\xi(\varphi\alpha a_l)\geq\xi(\varphi)$*),* $\forall\varphi,\varpi,a_l,b_l\in M_R,\alpha\in\Gamma$.

Definition 2.6. *[13] A BF-set* $B_R=(\xi^+_{B_R},\xi^-_{B_R})$ *of* M_R *is called a BF-ideal of* M_R *if*
(i) $\xi^+_{B_R}(\varphi-\varpi)\geq\min\{\xi^+_{B_R}(\varphi),\xi^+_{B_R}(\varpi)\}$,
(ii) $\xi^+_{B_R}(\varpi+\varphi-\varpi)\geq\xi^+_{B_R}(\varphi)$,
(iii) $\xi^+_{B_R}(a_l\alpha(\varphi+b_l)-a_l\alpha b_l)\geq\xi^+_{B_R}(\varphi)$,
(iv) $\xi^+_{B_R}(\varphi\alpha\varpi)\geq\xi^+_{B_R}(\varphi)$,
(v) $\xi^-_{B_R}(\varphi-\varpi)\leq\max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varpi)\}$,
(vi) $\xi^-_{B_R}(\varpi+\varphi-\varpi)\leq\xi^-_{B_R}(\varphi)$,
(vii) $\xi^-_{B_R}(a_l\alpha(\varphi+b_l)-a_l\alpha b_l)\leq\xi^-_{B_R}(\varphi)$,
(viii) $\xi^-_{B_R}(\varphi\alpha\varpi)\leq\xi^-_{B_R}(\varphi),\forall\varphi,\varpi,a_l,b_l\in M_R,\alpha\in\Gamma$.

Definition 2.7. *[14] A BF-set* $B_R=(\xi^+_{B_R},\xi^-_{B_R})$ *of* M_R *is called a BF-bi-ideal of* M_R *if*
(i) $\xi^+_{B_R}(\varphi-\varpi)\geq\min\{\xi^+_{B_R}(\varphi),\xi^+_{B_R}(\varpi)\}$,
(ii) $\xi^+_{B_R}(\varpi+\varphi-\varpi)\geq\xi^+_{B_R}(\varphi)$,
(iii) $\xi^+_{B_R}(\varphi\alpha\varpi\kappa\varkappa\wedge(\varphi\alpha(\varpi+\varkappa)-(\varphi\alpha\varpi)))\geq\min\{\xi^+_{B_R}(\varphi),\xi^+_{B_R}(\varkappa)\}$, *where* $\wedge$ *is the* min *operation,*
(iv) $\xi^-_{B_R}(\varphi-\varpi)\leq\max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varpi)\}$,
(v) $\xi^-_{B_R}(\varpi+\varphi-\varpi)\leq\xi^-_{B_R}(\varphi)$,
(vi) $\xi^-_{B_R}(\varphi\alpha\varpi\kappa\varkappa\wedge(\varphi\alpha(\varpi+\varkappa)-(\varphi\alpha\varpi)))\leq\max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varkappa)\},\forall\varphi,\varpi,\varkappa\in M_R,\alpha,\kappa\in\Gamma$.

Definition 2.8. *[1] Let* ξ *be a fuzzy set of* M_R *and* $\alpha\in[0,1]$*. A mapping* $\xi^T_\alpha:M_R\to[0,1]$ *is said to be a fuzzy* α*-translation of* ξ *if it satisfies* $\xi^T_\alpha(\varphi)=\xi(\varphi)+\alpha,\forall\varphi\in M_R$.

Definition 2.9. *[1] Let* ξ *be a fuzzy set of* M_R *and* $\alpha\in[0,1]$*. A mapping* $\xi^M_\alpha:M_R\to[0,1]$ *is said to be a fuzzy* α*-multiplication of* ξ *if it satisfies* $\xi^M_\alpha(\varphi)=\alpha\xi(\varphi),\forall\varphi\in M_R$.

Remark 2.2. *[12] For any BF-set* $B_R=(\xi^+_{B_R},\xi^-_{B_R})$ *of* M_R*, we denote* $\bot=-1-\inf_{\varphi\in M_R}\xi^-_{B_R}(\varphi)$ *and* $\top=1-\sup_{\varphi\in M_R}\xi^+_{B_R}(\varphi)$*, and* $(\theta,\phi)\in[\bot,0]\times[0,\top]$*. The BF-*$(\theta,\phi)$*-translation of* B_R*,* $B^T_{R(\theta,\phi)}=(\xi^+_{B,\theta},\xi^-_{B,\phi})$*, where* $\xi^+_{B,\theta}(\varphi)=\xi^+_{B_R}(\varphi)+\theta$ *for* $\xi^+_{B,\theta}:M_R\to[0,1]$ *and* $\xi^-_{B,\phi}(\varphi)=\xi^-_{B_R}(\varphi)+\phi$ *for* $\xi^-_{B,\phi}:M_R\to[-1,0]$.

3. Bipolar Fuzzy Magnified Translation of Γ-Near Rings

Building upon the foundational concepts of GNRs and bipolar fuzzy set theory established in the previous section, we now delve into the notion of BF-MT-GNRs within the context of GNRs. This section formalizes the definition of BF-MT-GNRs and explores its unique algebraic characteristics, which distinguish it from conventional fuzzy translations. By introducing key operations and presenting illustrative examples, we aim to provide a comprehensive understanding of how BF-MT-GNRs integrate with the structural framework of GNRs. These insights set the stage for the subsequent analysis of its correspondence with BF-sub-GNRs, ideals, and bi-ideals.

Definition 3.1. *For any BF-set* $B_R=(\xi^+_{B_R},\xi^-_{B_R})$ *and* $\alpha,\kappa\in(0,1]$*, we denote* $\bot=-1-\inf_{\varphi\in M_R}\xi^-_{B_R}(\varphi)$ *and* $\top=1-\sup_{\varphi\in M_R}\xi^+_{B_R}(\varphi)$*, and* $(\theta,\phi)\in[\bot,0]\times[0,\top]$*. The BF-*$(\alpha_\theta,\kappa_\phi)$*-MT of* B_R*,* $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}=$

$(\xi^{+}_{B_{R(\alpha,\theta)}}, \xi^{-}_{B_{R(\kappa,\phi)}})$, where $\xi^{+}_{B_{R(\alpha,\theta)}}(\varphi) = \alpha\xi^{+}_{B_R}(\varphi) + \theta$ for $\xi^{+}_{B_{R(\alpha,\theta)}} : M_R \to [0,1]$ and $\xi^{-}_{B_{R(\kappa,\phi)}}(\varphi) = \kappa\xi^{-}_{B_R}(\varphi) + \phi$ for $\xi^{-}_{B_{R(\kappa,\phi)}} : M_R \to [-1,0]$.

Example 3.1. *M_R and Γ are additive commutative groups, provided M_R the set of all real numbers and $\Gamma = M_R$. Establish $M_R * \Gamma * M_R \to M_R$ by $\varphi\alpha\varpi$ the usual product of φ, α, ϖ for every $\varphi, \varpi \in M_R, \alpha \in \Gamma$. Then M_R is a GNR with zero-symmetric. Let $B_R = (\xi^{+}_{B_R}, \xi^{-}_{B_R})$, where $\xi^{+}_{B_R} : M_R \to [0,1]$ and $\xi^{-}_{B_R} : M_R \to [-1,0]$ defined by*

$$\xi^{+}_{B_R}(\varphi) = \begin{cases} 0.3 \text{ if } \varphi = 0 \\ 0.6 \text{ if } \varphi > 0 \\ 0.7 \text{ if } \varphi < 0 \end{cases}$$

$$\xi^{-}_{B_R}(\varphi) = \begin{cases} -0.2 \text{ if } \varphi = 0 \\ -0.5 \text{ if } \varphi > 0 \\ -0.6 \text{ if } \varphi < 0 \end{cases}$$

Let $\theta \in [0, 0.3], \phi \in [0, -0.4]$ and let $\alpha = 0.2, \kappa = 0.1$. Hence, BF-$(\alpha_\theta, \kappa_\phi)$-MT of B_R is

$$\xi^{+}_{B_{R(\alpha,\theta)}}(\varphi) = \begin{cases} 0.26 \text{ if } \varphi = 0 \\ 0.32 \text{ if } \varphi > 0 \\ 0.34 \text{ if } \varphi < 0 \end{cases}$$

$$\xi^{-}_{B_{R(\kappa,\phi)}}(\varphi) = \begin{cases} -0.32 \text{ if } \varphi = 0 \\ -0.35 \text{ if } \varphi > 0 \\ -0.36 \text{ if } \varphi < 0 \end{cases}$$

Theorem 3.1. *Let $B^{MT}_{R(\alpha_\theta,\kappa_\phi)} = (\xi^{+}_{B_{R(\alpha,\theta)}}, \xi^{-}_{B_{R(\kappa,\phi)}})$ be the BF-$(\alpha_\theta, \kappa_\phi)$-MT of a BF-sub-GNR B_R of M_R. Then*
(i) $\xi^{+}_{B_{R(\alpha,\theta)}}(-\varphi) = \xi^{+}_{B_{R(\alpha,\theta)}}(\varphi)$ and $\xi^{-}_{B_{R(\kappa,\phi)}}(-\varphi) = \xi^{-}_{B_{R(\kappa,\phi)}}(\varphi)$,
(ii) $\xi^{+}_{B_{R(\alpha,\theta)}}(\varphi) \leq \xi^{+}_{B_{R(\alpha,\theta)}}(0)$ and $\xi^{-}_{B_{R(\kappa,\phi)}}(\varphi) \geq \xi^{-}_{B_{R(\kappa,\phi)}}(0)$.

Proof. (i) $\xi^{+}_{B_{R(\alpha,\theta)}}(-\varphi) = \alpha\xi^{+}_{B_R}(-\varphi) + \theta = \alpha\xi^{+}_{B_R}(\varphi) + \theta = \xi^{+}_{B_{R(\alpha,\theta)}}(\varphi)$ and $\xi^{-}_{B_{R(\kappa,\phi)}}(-\varphi) = \kappa\xi^{-}_{B_R}(-\varphi) + \phi = \kappa\xi^{-}_{B_R}(\varphi) + \phi = \xi^{-}_{B_{R(\kappa,\phi)}}(\varphi)$.

(ii) $\xi^{+}_{B_{R(\alpha,\theta)}}(0) = \alpha\xi^{+}_{B_R}(0) + \theta \geq \alpha\xi^{+}_{B_R}(\varphi) + \theta = \xi^{+}_{B_{R(\alpha,\theta)}}(\varphi)$ and $\xi^{-}_{B_{R(\kappa,\phi)}}(0) = \kappa\xi^{-}_{B_R}(0) + \phi \leq \kappa\xi^{-}_{B_R}(\varphi) + \phi = \xi^{-}_{B_{R(\kappa,\phi)}}(\varphi)$. □

Theorem 3.2. *Let $B_R = (\xi^{+}_{B_R}, \xi^{-}_{B_R})$ be a BF-set of M_R. Then B_R is a BF-sub-GNR of M_R if and only if the BF-$(\alpha_\theta, \kappa_\phi)$-MT of B_R, $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-sub-GNR of M_R.*

Proof. Let B_R be a BF-sub-GNR of M_R, and $\varphi, \varpi \in M_R, \gamma \in \Gamma$. Then

$$\begin{aligned} \xi^{+}_{B_{R(\alpha,\theta)}}(\varphi - \varpi) &= \alpha\xi^{+}_{B_R}(\varphi - \varpi) + \theta \\ &\geq \alpha \min\{\xi^{+}_{B_R}(\varphi), \xi^{+}_{B_R}(\varpi)\} + \theta \\ &= \min\{\alpha\xi^{+}_{B_R}(\varphi) + \theta, \alpha\xi^{+}_{B_R}(\varpi) + \theta\} \\ &= \min\{\xi^{+}_{B_{R(\alpha,\theta)}}(\varphi), \xi^{+}_{B_{R(\alpha,\theta)}}(\varpi)\}, \end{aligned}$$

$$\begin{aligned}\xi^-_{B_{R(\kappa,\phi)}}(\varphi-\varpi) &= \kappa\xi^-_{B_R}(\varphi-\varpi)+\phi\\ &\leq \kappa\max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varpi)\}+\phi\\ &= \max\{\kappa\xi^-_{B_R}(\varphi)+\phi,\kappa\xi^-_{B_R}(\varpi)+\phi\}\\ &= \max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi),\xi^-_{B_{R(\kappa,\phi)}}(\varpi)\},\end{aligned}$$

$$\begin{aligned}\xi^+_{B_{R(\alpha,\theta)}}(\varphi\gamma\varpi) &= \alpha\xi^+_{B_R}(\varphi\gamma\varpi)+\theta\\ &\geq \alpha\min\{\xi^+_{B_R}(\varphi),\xi^+_{B_R}(\varpi)\}+\theta\\ &= \min\{\alpha\xi^+_{B_R}(\varphi)+\theta,\alpha\xi^+_{B_R}(\varpi)+\theta\}\\ &= \min\{\xi^+_{B_{R(\alpha,\theta)}}(\varphi),\xi^+_{B_{R(\alpha,\theta)}}(\varpi)\},\end{aligned}$$

$$\begin{aligned}\xi^-_{B_{R(\kappa,\phi)}}(\varphi\gamma\varpi) &= \kappa\xi^-_{B_R}(\varphi\gamma\varpi)+\phi\\ &\leq \kappa\max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varpi)\}+\phi\\ &= \max\{\kappa\xi^-_{B_R}(\varphi)+\phi,\kappa\xi^-_{B_R}(\varpi)+\phi\}\\ &= \max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi),\xi^-_{B_{R(\kappa,\phi)}}(\varpi)\}.\end{aligned}$$

Conversely, suppose that $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-sub-GNR of M_R. Let $\varphi,\varpi\in M_R,\gamma\in\Gamma$. Then

$$\begin{aligned}\xi^+_{B_R}(\varphi-\varpi) &= \frac{1}{\alpha}[\xi^+_{B_{R(\alpha,\theta)}}(\varphi-\varpi)-\theta]\\ &\geq \frac{1}{\alpha}[\min\{\xi^+_{B_{R(\alpha,\theta)}}(\varphi),\xi^+_{B_{R(\alpha,\theta)}}(\varpi)\}-\theta]\\ &= \min\{\frac{1}{\alpha}(\xi^+_{B_{R(\alpha,\theta)}}(\varphi)-\theta),\frac{1}{\alpha}(\xi^+_{B_{R(\alpha,\theta)}}(\varpi)-\theta)\}\\ &= \min\{\xi^+_{B_R}(\varphi),\xi^+_{B_R}(\varpi)\},\end{aligned}$$

$$\begin{aligned}\xi^-_{B_R}(\varphi-\varpi) &= \frac{1}{\kappa}[\xi^-_{B_{R(\kappa,\phi)}}(\varphi-\varpi)-\phi]\\ &\leq \frac{1}{\kappa}[\max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi),\xi^-_{B_{R(\kappa,\phi)}}(\varpi)\}-\phi]\\ &= \max\{\frac{1}{\kappa}(\xi^-_{B_{R(\kappa,\phi)}}(\varphi)-\phi),\frac{1}{\kappa}(\xi^-_{B_{R(\kappa,\phi)}}(\varpi)-\phi)\}\\ &= \max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varpi)\}.\end{aligned}$$

Similarly, we can show that $\xi^-_{B_R}(\varphi\alpha\varpi)\geq\min\{\xi^+_{B_R}(\varphi),\xi^+_{B_R}(\varpi)\}$ and $\xi^-_{B_R}(\varphi-\varpi)\leq\max\{\xi^-_{B_R}(\varphi),\xi^-_{B_R}(\varpi)\}$. Hence, B_R is a BF-sub-GNR of M_R. □

The proofs of the following two theorems are similar and follow the same principle as the proofs of Theorem 3.2.

Theorem 3.3. *Let $B_R=(\xi^+_{B_R},\xi^-_{B_R})$ be a BF-set of M_R. Then B_R is a BF-ideal of M_R if and only if the BF-$(\alpha_\theta,\kappa_\phi)$-MT of B_R, $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-ideal of M_R.*

Theorem 3.4. *Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BF-set of M_R. Then B_R is a BF-bi-ideal of M_R if and only if the BF-$(\alpha_\theta, \kappa_\phi)$-MT of B_R, $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-bi-ideal of M_R.*

Theorem 3.5. *If B_R is a BF-ideal of M_R, then the BF-$(\alpha_\theta, \kappa_\phi)$-MT of B_R, $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-bi-ideal of M_R.*

Proof. Let B_R be a BF-ideal of M_R and let $\varphi, \varpi, \varkappa \in M_R, \gamma_1, \gamma_2 \in \Gamma$. Then

$$\begin{aligned}
\xi^+_{B_{R(\alpha,\theta)}}(\varphi - \varpi) &= \alpha\xi^+_{B_R}(\varphi - \varpi) + \theta \\
&\geq \alpha \min\{\xi^+_{B_R}(\varphi), \xi^+_{B_R}(\varpi)\} + \theta \\
&= \min\{\alpha\xi^+_{B_R}(\varphi) + \theta, \alpha\xi^+_{B_R}(\varpi) + \theta\} \\
&= \min\{\xi^+_{B_{R(\alpha,\theta)}}(\varphi), \xi^+_{B_{R(\alpha,\theta)}}(\varpi)\},
\end{aligned}$$

$$\begin{aligned}
\xi^+_{B_{R(\alpha,\theta)}}(\varphi + \varpi - \varphi) &= \alpha\xi^+_{B_R}(\varphi + \varpi - \varphi) + \theta \\
&\geq \alpha\xi^+_{B_R}(\varpi) + \theta \\
&= \xi^+_{B_{R(\alpha,\theta)}}(\varpi),
\end{aligned}$$

$$\begin{aligned}
&\xi^+_{B_{R(\alpha,\theta)}}[\varphi\gamma_1\varpi\gamma_2\varkappa \wedge (\varphi\gamma_1(\varpi + \varkappa) - \varphi\gamma_1\varpi)] \\
&= \alpha\xi^+_{B_R}[\varphi\gamma_1\varpi\gamma_2\varkappa \wedge (\varphi\gamma_1(\varpi + \varkappa) - \varphi\gamma_1\varpi)] + \theta \\
&= \alpha[\min\{\xi^+_{B_R}(\varphi\gamma_1\varpi\gamma_2\varkappa), \xi^+_{B_R}(\varphi\gamma_1(\varpi + \varkappa) - \varphi\gamma_1\varpi)\}] + \theta \\
&\geq \alpha \min\{\xi^+_{B_R}(\varphi), \xi^+_{B_R}(\varkappa)\} + \theta \\
&= \min\{\alpha\xi^+_{B_R}(\varphi) + \theta, \alpha\xi^+_{B_R}(\varkappa) + \theta\} \\
&= \min\{\xi^+_{B_{R(\alpha,\theta)}}(\varphi), \xi^+_{B_{R(\alpha,\theta)}}(\varkappa)\}.
\end{aligned}$$

Similarly, we can establish that

$$\xi^-_{B_{R(\kappa,\phi)}}(\varphi - \varpi) \leq \max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi), \xi^-_{B_{R(\kappa,\phi)}}(\varpi)\},$$
$$\xi^-_{B_{R(\kappa,\phi)}}(\varphi + \varpi - \varphi) \leq \xi^-_{B_{R(\kappa,\phi)}}(\varpi),$$
$$\xi^-_{B_{R(\kappa,\phi)}}[\varphi\gamma_1\varpi\gamma_2\varkappa \wedge (\varphi\gamma_1(\varpi + \varkappa) - \varphi\gamma_1\varpi)] \leq \max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi), \xi^-_{B_{R(\kappa,\phi)}}(\varkappa)\}.$$

Hence, $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-bi-ideal of M_R. □

Theorem 3.6. *A GNR homomorphic image of a BF-$(\alpha_\theta, \kappa_\phi)$-MT of a BF-sub-GNR of M_R is a BF-sub-GNR of N_R.*

Proof. Let $f : M_R \to N_R$ be a GNR homomorphism, and $A_R = (\xi^+_{A_R}, \xi^-_{A_R})$ and $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be BF-sub-GNRs of M_R and N_R, respectively. Let $\varphi, \varpi \in M_R, \gamma \in \Gamma$. Then

$$\begin{aligned}
\xi^+_{B_R}(f(\varphi - \varpi)) &= \xi^+_{B_{R(\alpha,\theta)}}(\varphi - \varpi) \\
&= \alpha\xi^+_{B}(\varphi - \varpi) + \theta \\
&\geq \alpha \min\{\xi^+_{B_R}(\varphi), \xi^+_{B_R}(\varpi)\} + \theta
\end{aligned}$$

$$
\begin{aligned}
&= \min\{\alpha\xi^+_{B_R}(\varphi)+\theta, \alpha\xi^+_{B_R}(\varpi)+\theta\}\\
&= \min\{\xi^+_{B_{R(\alpha,\theta)}}(\varphi), \xi^+_{B_{R(\alpha,\theta)}}(\varpi)\},
\end{aligned}
$$

$$
\begin{aligned}
\xi^-_{B_R}(f(\varphi-\varpi)) &= \xi^-_{B_{R(\kappa,\phi)}}(\varphi-\varpi)\\
&= \kappa\xi^-_{B_R}(\varphi-\varpi)+\phi\\
&\le \kappa\max\{\xi^-_{B_R}(\varphi), \xi^-_{B_R}(\varpi)\}+\phi\\
&= \max\{\kappa\xi^-_{B_R}(\varphi)+\phi, \kappa\xi^-_{B_R}(\varpi)+\phi\}\\
&= \max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi), \xi^-_{B_{R(\kappa,\phi)}}(\varpi)\},
\end{aligned}
$$

$$
\begin{aligned}
\xi^+_{B_R}(f(\varphi\gamma\varpi)) &= \xi^+_{B_{R(\alpha,\theta)}}(\varphi\gamma\varpi)\\
&= \alpha\xi^+_{B_R}(\varphi\gamma\varpi)+\theta\\
&\ge \alpha\min\{\xi^+_{B_R}(\varphi), \xi^+_{B_R}(\varpi)\}+\theta\\
&= \min\{\alpha\xi^+_{B_R}(\varphi)+\theta, \alpha\xi^+_{B_R}(\varpi)+\theta\}\\
&= \min\{\xi^+_{B_{R(\alpha,\theta)}}(\varphi), \xi^+_{B_{R(\alpha,\theta)}}(\varpi)\},
\end{aligned}
$$

$$
\begin{aligned}
\xi^-_{B_R}(f(\varphi\gamma\varpi)) &= \xi^-_{B_{R(\kappa,\phi)}}(\varphi\gamma\varpi)\\
&= \kappa\xi^-_{B_R}(\varphi\gamma\varpi)+\phi\\
&\le \kappa\max\{\xi^-_{B_R}(\varphi), \xi^-_{B_R}(\varpi)\}+\phi\\
&= \max\{\kappa\xi^-_{B_R}(\varphi)+\phi, \kappa\xi^-_{B_R}(\varpi)+\phi\}\\
&= \max\{\xi^-_{B_{R(\kappa,\phi)}}(\varphi), \xi^-_{B_{R(\kappa,\phi)}}(\varpi)\}.
\end{aligned}
$$

Hence, $B^{MT}_{R(\alpha_\theta,\kappa_\phi)}$ is a BF-sub-GNR of N_R. □

The proofs of the following two theorems are similar and follow the same principle as the proofs of Theorem 3.6.

Theorem 3.7. *A GNR homomorphic image of a BF-$(\alpha_\theta,\kappa_\phi)$-MT of a BF-ideal of M_R is a BF-ideal of N_R.*

Theorem 3.8. *A GNR homomorphic image of a BF-$(\alpha_\theta,\kappa_\phi)$-MT of a BF-bi-ideal of M_R is a BF-bi-ideal of N_R.*

4. Conclusion

In this study, the concept of BF-MT-GNRs was introduced to extend the scope of BF-set theory within GNRs. A one-to-one correspondence between BF-MT-GNRs and BF-sub-GNRs, ideals, and bi-ideals was established, providing a more profound understanding of the interplay between these algebraic structures. Additionally, the investigation of homomorphisms on BF-MT-GNRs confirmed their structural integrity and theoretical coherence. These results underscore the versatility and potential of BF-MT-GNRs in enriching the mathematical framework of BF-set theory. The insights gained from this research are anticipated to inspire further exploration and application of these concepts in advanced algebraic studies and related fields.

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