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On (m,n)-Fuzzy Sets and Their Application in Ordered Semigroups

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Abstract. In this paper, we introduce the concepts of (m,n)-fuzzy subsemigroups, (m,n)-fuzzy left (right, two-sided, bi-, $(1,2)$-) ideals of an ordered semigroup and some their algebraic properties are studied, thereafter the relationship among their (m,n)-fuzzy ideals was investigated. Moreover, we characterize left (resp., right, two-sided, bi-) ideals by using (m,n)-fuzzy left (resp., right, two-sided, bi-) ideals. Finally, we characterize regular ordered semigroups and intra-regular ordered semigroups in terms of (m,n)-fuzzy left ideals, (m,n)-fuzzy right ideals, and (m,n)-fuzzy bi-ideals.

1. Introduction

The theory of fuzzy sets was first introduced by Zadeh [1] in 1965. Fuzzy sets are the most appropriate theory for dealing with uncertainty. After the introduction of the concept of fuzzy sets by Zadeh, several researchers conducted research on the generalizations of the notions of fuzzy sets with huge applications in computer science, artificial intelligence, control engineering, robotics, automata theory, decision theory, finite state machine, graph theory, logic, operations research, and many branches of pure and applied mathematics. For example, Xie et al. applied fuzzy set theory to the switching method [2]. The concept of fuzzy sets can also be applied to studying the properties of algebras. Kuroki [3,4] applied fuzzy set theory to semigroups. In 2007, Kehayopulu and Tsingelis [5] applied fuzzy set theory to ordered semigroups.

As a generalization of the fuzzy set, Atanassov [6] created an intuitionistic fuzzy set. An intuitionistic fuzzy set is widely used in all fields (See [7–10] for applications in algebraic structures). In 2013, Yager [11–13] introduced the Pythagorean fuzzy set and compared it with the intuitionistic

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fuzzy set. The Pythagorean fuzzy set is a new extension of the intuitionistic fuzzy set that conducts to simulate the vagueness originated by the real case that might arise in the sum of membership and non-membership is bigger than 1. Pythagorean fuzzy set is applied to groups (See [14]), *UP*-algebras (See [15]), and topological spaces (See [16]). Senapati et al. [17] introduced the Fermatean fuzzy set which is another extension of intuitionistic fuzzy sets and it is applied to groups (See [18]). Ibrahim et al. [19] introduced (3, 2)-fuzzy sets and applied them to topological spaces.

In 2022, Y. B. Jun and K. Hur [20] introduced the concept of the (m,n)-fuzzy set which is the supclass of intuitionistic fuzzy set, Pythagorean fuzzy set, $(3,2)$-fuzzy set, Fermatean fuzzy set and n-Pythagorean fuzzy set, and compared with them and applied (m,n)-fuzzy sets to *BCK*-algebras and *BCI*-algebras. Based on the concept of [20], we considered the algebraic structure of so-called ordered semigroups. In this paper, we introduce the concepts of (m,n)-fuzzy subsemigroups, (m,n)-fuzzy left (right, two-sided, bi-, $(1,2)$-) ideals of an ordered semigroup and some their algebraic properties are studied, thereafter the relationship among their (m,n)-fuzzy ideals was investigated. Moreover, we characterize left (resp., right, two-sided, bi-) ideals by using (m,n)-fuzzy left (resp., right, two-sided, bi-) ideals. Finally, we characterize regular ordered semigroups and intra-regular ordered semigroups in terms of (m,n)-fuzzy left (right, two-sided, bi-) ideals.

2. Preliminary

In this section, we will recall the basic terms and definitions from the ordered semigroup theory and the (m,n)-fuzzy set theory that we will use later in this paper.

A groupoid $\langle S;\cdot\rangle$ consists of a nonempty set S and a (binary) operation $\cdot$ on S. A semigroup $\langle S;\cdot\rangle$ is a groupoid in which the operation $\cdot$ is associative, that is,

$$x\cdot(y\cdot z)=(x\cdot y)\cdot z \text{ for all } x,y,z\in S.$$

Definition 2.1. *[21] The structure $\langle S;\cdot,\leq\rangle$ is called an ordered semigroup if the following conditions are satisfied:*

(1) *$\langle S;\cdot\rangle$ is a semigroup.*
(2) *$\langle S;\leq\rangle$ is a partially ordered set.*
(3) *For every $a,b,c\in S$ if $a\leq b$, then $a\cdot c\leq b\cdot c$ and $c\cdot a\leq c\cdot b$.*

For simplicity, we denoted an ordered semigroup $\langle S;\cdot,\leq\rangle$ by its carrier set as a bold letter **S** and if $a,b\in S$, we will instead of $a\cdot b$ by ab. Let A and B be two nonempty subsets of S. Then we define

$$AB:=\{ab : a\in A \text{ and } b\in B\}.$$

For $K\subseteq S$, we denote

$$(K]:=\{a\in S : a\leq k \text{ for some } k\in K\}.$$

Let **S** be an ordered semigroup and let A, B be subsets of S. It is observed that (1) $A\subseteq(A]$, (2) $(A](B]\subseteq(AB]$, (3) $((A](B]]=(AB]$ and (4) $S=(S]$.

Let **S** be an ordered semigroup. A nonempty subset A of S is called a *subsemigroup* of **S** [21] if $AA \subseteq A$.

Definition 2.2. *[21] Let* **S** *be an ordered semigroup. A nonemty subset A of S is called a left (resp., right) ideal of* **S** *if it satisfied the following conditions:*

(1) $SA \subseteq A$ *(resp.,* $AS \subseteq A$*).*
(2) *For* $x, y \in S$*, if* $x \preceq y$ *and* $y \in A$*, then* $x \in A$*.*

A nonempty subset of **S** is called a *two-sided ideal* (or an *ideal*) if it is both a left and a right ideal of **S**.

Definition 2.3. *[21] Let* **S** *be an ordered semigroup. A subsemigroup B of* **S** *is called a bi-ideal of* **S** *if it satisfied the following conditions:*

(1) $BSB \subseteq B$*.*
(2) *For* $x, y \in S$*, if* $x \preceq y$ *and* $y \in B$*, then* $x \in B$*.*

A fuzzy subset (or fuzzy set) of a nonempty subset X is a mapping $f : X \to [0,1]$ from X to a unit closed interval (see [1]).

Definition 2.4. *[20] Let* $f : X \to [0,1]$ *and* $g : X \to [0,1]$ *be fuzzy sets of a set X. If there exists* $(m,n) \in \mathbb{N} \times \mathbb{N}$ *such that*

$$(\forall x \in X)(0 \leq (f(x))^m + (g(x))^n \leq 1),$$

then the structure

$$F := \{(x, f(x), g(x)) : x \in X\}$$

is called the (m,n)*-fuzzy set of X.*

In what follows, we use the notations $f^m(x)$ and $g^n(x)$ instead of $(f(x))^m$ and $(g(x))^n$, respectively, and the (m,n)-fuzzy set of X is simply denoted by $F := (f, g)$. The collection of (m,n)-fuzzy sets of X is denoted by $\mathcal{F}_n^m(X)$.

Example 2.1. *[20] Let* $X = \{0, a, b, c, d\}$ *be a set and define fuzzy sets* $f : X \to [0,1]$ *and* $g : X \to [0,1]$ *as follows:*

X	$f(x)$	$g(x)$
0	0.93	0.87
a	0.74	0.43
b	0.92	0.79
c	0.55	0.66
d	0.67	0.58

Then $F = (f, g)$ *is a* $(5,n)$*-fuzzy set of X for* $n \geq 9$*. But it is not a* $(5,n)$*-fuzzy set of X for* $n \leq 8$ *because of* $(0.93)^5 + (0.87)^8 = 1.02390004084 \geq 1$.

The (m,n)-fuzzy set varies according to (m,n) as shown in the table below (for more detail see [20]).

(m,n)	(m,n)-fuzzy
$(1,1)$	Intuitionistic fuzzy set
$(2,2)$	Pythagorean fuzzy set
$(3,2)$	$(3,2)$-fuzzy set
$(3,3)$	Fermatean fuzzy set
(n,n)	n-Pythagorean fuzzy set

We, now define a binary relation $\sqsubseteq$ on $\mathcal{F}_n^m(X)$ and define a binary operation $\sqcap$ on $\mathcal{F}_n^m(X)$ as the following definition.

Definition 2.5. *Let $F_1 = (f_1, g_1)$ and $F_2 = (f_2, g_2)$ be elements of $\mathcal{F}_n^m(X)$. Then*

(1) *$F_1 \sqsubseteq F_2$ if $f_1^m(x) \leq f_2^m(x)$ and $g_1^n(x) \geq g_2^n(x)$ for all $x \in X$ and $F_1 = F_2$ means that $F_1 \sqsubseteq F_2$ and $F_2 \sqsubseteq F_1$.*

(2) *$F_1 \sqcap F_2 := (f_1 \cap f_2, g_1 \cup g_2)$ where for each $x \in X$,*

(2.1) $(f_1 \cap f_2)^m(x) := \min\{f_1^m(x), f_2^m(x)\}$,

(2.2) $(g_1 \cup g_2)^n(x) := \max\{g_1^n(x), g_2^n(x)\}$.

By Definition 2.5, we see that $\langle \mathcal{F}_n^m(X); \sqsubseteq\rangle$ is a partially ordered set. Let $\mathbf{S}$ be an ordered semigroup and $a \in S$. We set

$$\mathbf{S}_a := \{(x,y) \in S \times S : a \leq xy \text{ for some } x, y \in S\}.$$

Without the binary operation $\sqcap$ on $\mathcal{F}_n^m(X)$ that defined in Definition 2.5 (2), we also define new operation on $\mathcal{F}_n^m(S)$ as the following definition.

Definition 2.6. *Let $F_1 = (f_1, g_1)$ and $F_2 = (f_2, g_2)$ be elements of $\mathcal{F}_n^m(S)$. The product of F_1 and F_2 is an element of $\mathcal{F}_n^m(S)$, denoted by $F_1 \diamond F_2 = (f_1 \diamond f_2, g_1 \diamond g_2)$ and is defined as follows*

$$(f_1 \diamond f_2)^m(a) := \begin{cases} \bigvee_{(x,y)\in \mathbf{S}_a} \{\min\{f_1^m(x), f_2^m(y)\}\} & \text{if } \mathbf{S}_a \neq \emptyset \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(g_1 \diamond g_2)^n(a) := \begin{cases} \bigwedge_{(x,y)\in \mathbf{S}_a} \{\max\{g_1^n(x), g_2^n(y)\}\} & \text{if } \mathbf{S}_a \neq \emptyset \\ 1 & \text{otherwise,} \end{cases}$$

for all $a \in S$.

It is easy to see that the operation $\diamond$ satisfied associative property. Therefore the structure $\langle \mathcal{F}_n^m(S); \diamond\rangle$ is a semigroup. Let F, G and H be elements of $\mathcal{F}_n^m(S)$. If $F \sqsubseteq G$, then by Definition 2.5 (1) and Definition 2.6, it is easy to see that

$$F \diamond H \sqsubseteq G \diamond H \text{ and } H \diamond F \sqsubseteq H \diamond G.$$

Then the structure $\langle \mathcal{F}_n^m(S); \diamond, \sqsubseteq\rangle$ is an ordered semigroup and its called an *(m,n)-fuzzy ordered semigroup*.

3. Main Results

Firstly, we introduce the concepts of (m,n)-fuzzy subsemigroups, (m,n)-fuzzy left (right, two-sided, bi-, $(1,2)$-) ideals on (m,n)-fuzzy ordered semigroup $\langle \mathcal{F}_n^m(S);\diamond,\sqsubseteq\rangle$ and study some algebraic properties of (m,n)-fuzzy subsemigroups and their (m,n)-fuzzy ideals as follows.

Definition 3.1. *Let* $\mathbf{S}$ *be an ordered semigroup. An* (m,n)*-fuzzy subset* $F=(f,g)$ *of* S *is called an* (m,n)*-fuzzy subsemigroup of* $\mathbf{S}$ *if the following conditions are satisfied: For any* $x,y\in S$,

(1) $f^m(xy)\geq \min\{f^m(x),f^m(y)\}$.
(2) $g^n(xy)\leq \max\{g^n(x),g^n(y)\}$.

Example 3.1. *Let* $S=\{a,b,c,d\}$ *be a set with a binary operation "$*$" in the table below.*

$*$	a	b	c	d
a	a	a	a	a
b	b	a	a	b
c	c	b	a	c
d	d	d	d	a

and define a partial ordered $\leq$ *on* S *as follows.*

$$\leq := \Delta_S\cup\{(a,b)\},$$

where Δ_S *is an identity relation on* S*. Then it is easy to verify that* $\langle S;*,\leq\rangle$ *is an ordered semigroup. Define* $F=(f,g)$ *an* (m,n)*-fuzzy subset of* S *as follows:*

S	$f(x)$	$g(x)$
a	0.87	0.17
b	0.65	0.43
c	0.65	0.43
d	0.76	0.56

It is routine to verify that F *is an* (m,n)*-fuzzy subsemigroup of* $\langle S;*,\leq\rangle$ *for all* $(m,n)\in\mathbb{N}\times\mathbb{N}$ *with* $(m,n)\notin\{(1,1),(1,2),(2,1)\}$.

Proposition 3.1. *Let* $\mathbf{S}$ *be an ordered semigroup and let* F_1*,* F_2 *be* (m,n)*-fuzzy subsemigroups of* $\mathbf{S}$*. Then* $F_1\sqcap F_2$ *is an* (m,n)*-fuzzy subsemigroup of* $\mathbf{S}$.

Proof. Let $F_1=(f_1,g_1)$ and $F_2=(f_2,g_2)$ be (m,n)-fuzzy subsemigroups of $\mathbf{S}$ and $x,y\in S$. Let us consider as follows.

$$\begin{aligned}(f_1\cap f_2)^m(xy) &= \min\{f_1^m(xy),f_2^m(xy)\}\\ &\geq \min\{\min\{f_1^m(x),f_1^m(y)\},\min\{f_2^m(x),f_2^m(y)\}\}\\ &= \min\{\min\{f_1^m(x),f_2^m(x)\},\min\{f_1^m(y),f_2^m(y)\}\}\\ &= \min\{(f_1\cap f_2)^m(x),(f_1\cap f_2)^m(y)\},\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(xy) &= \max\{g_1^n(xy), g_2^n(xy)\} \\ &\leq \max\{\max\{g_1^n(x), g_1^n(y)\}, \max\{g_2^n(x), g_2^n(y)\}\} \\ &= \max\{\max\{g_1^n(x), g_2^n(x)\}, \max\{g_1^n(y), g_2^n(y)\}\} \\ &= \max\{(g_1 \cup g_2)^n(x), (g_1 \cup g_2)^n(y)\}.\end{aligned}$$

Therefore $F_1 \sqcap F_2$ is an (m,n)-fuzzy subsemigroup of $\mathbf{S}$. □

Definition 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup. An* (m,n)*-fuzzy subset* $F = (f,g)$ *of* S *is called an* (m,n)*-fuzzy left ideal of* $\mathbf{S}$ *if the following conditions are satisfied: For* $x, y \in S$*,*

(1) $f^m(xy) \geq f^m(y)$ *and* $g^n(xy) \leq g^n(y)$*.*
(2) *If* $x \leq y$*, then* $f^m(x) \geq f^m(y)$ *and* $g^n(x) \leq g^n(y)$*.*

Example 3.2. *Let* $S = \{a,b,c\}$ *be a set with a binary operation "$*$" in the table below.*

$*$	a	b	c
a	a	a	a
b	a	a	a
c	a	c	c

and define a partial ordered $\leq$ *on* S *as follows.*

$$\leq := \Delta_S \cup \{(a,b),(a,c)\},$$

where Δ_S *is an identity relation on* S*. Then it is easy to verify that* $\langle S; *, \leq \rangle$ *is an ordered semigroup. Define* $F = (f,g)$ *an* (m,n)*-fuzzy subset of* S *as follows:*

S	$f(x)$	$g(x)$
a	0.80	0.30
b	0.70	0.40
c	0.70	0.40

It is routine to verify that F *is an* (m,n)*-fuzzy left ideal of* $\langle S; *, \leq \rangle$ *for all* $(m,n) \in \mathbb{N} \times \mathbb{N}$ *with* $(m,n) \notin \{(1,1)\}$*.*

Definition 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup. An* (m,n)*-fuzzy subset* $F = (f,g)$ *of* S *is called an* (m,n)*-fuzzy right ideal of* $\mathbf{S}$ *if the following conditions are satisfied: For* $x, y \in S$*,*

(1) $f^m(xy) \geq f^m(x)$ *and* $g^n(xy) \leq g^n(x)$*.*
(2) *If* $x \leq y$*, then* $f^m(x) \geq f^m(y)$ *and* $g^n(x) \leq g^n(y)$*.*

Example 3.3. *By an ordered semigroup in Example 3.2. We define $F = (f, g)$ an (m, n)-fuzzy subset of S as follows:*

S	$f(x)$	$g(x)$
a	0.90	0.40
b	0.60	0.50
c	0.60	0.50

It is routine to verify that F is an (m, n)-fuzzy right ideal of $\langle S; *, \leq \rangle$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ with $(m, n) \notin \{(1, 1)\}$.

An (m, n)-fuzzy subset F of S is called an *(m, n)-fuzzy two-sided ideal* (or *(m, n)-fuzzy ideal*) of $\mathbf{S}$ if it is both an (m, n)-fuzzy left and an (m, n)-fuzzy right ideal of $\mathbf{S}$.

Example 3.4. *By an ordered semigroup in Example 3.2. We define $F = (f, g)$ an (m, n)-fuzzy subset of S as follows:*

S	$f(x)$	$g(x)$
a	0.80	0.40
b	0.70	0.50
c	0.70	0.50

It is routine to verify that F is an (m, n)-fuzzy ideal of $\langle S; *, \leq \rangle$ for all $(m, n) \in \mathbb{N} \times \mathbb{N}$ with $(m, n) \notin \{(1, 1)\}$.

Proposition 3.2. *Let $\mathbf{S}$ be an ordered semigroup and let F_1, F_2 be (m, n)-fuzzy left ideals of $\mathbf{S}$. Then $F_1 \sqcap F_2$ is an (m, n)-fuzzy left ideal of $\mathbf{S}$.*

Proof. Let $F_1 = (f_1, g_1)$ and $F_2 = (f_2, g_2)$ be (m, n)-fuzzy left ideals of $\mathbf{S}$ and $x, y \in S$. Let us consider as follows.

$$\begin{aligned}(f_1 \cap f_2)^m(xy) &= \min\{f_1^m(xy), f_2^m(xy)\}\\ &\geq \min\{f_1^m(y), f_2^m(y)\}\\ &= (f_1 \cap f_2)^m(y),\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(xy) &= \max\{g_1^n(xy), g_2^n(xy)\}\\ &\leq \max\{g_1^n(y), g_2^n(y)\}\\ &= (g_1 \cup g_2)^n(y).\end{aligned}$$

Let $x, y \in S$ be such that $x \leq y$. Then, we obtain

$$\begin{aligned}(f_1 \cap f_2)^m(x) &= \min\{f_1^m(x), f_2^m(x)\}\\ &\geq \min\{f_1^m(y), f_2^m(y)\}\\ &= (f_1 \cap f_2)^m(y),\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(x) &= \max\{g_1^n(x), g_2^n(x)\} \\ &\leq \max\{g_1^n(y), g_2^n(y)\} \\ &= (g_1 \cup g_2)^n(y).\end{aligned}$$

Therefore $F_1 \sqcap F_2$ is an (m,n)-fuzzy left ideal of $\mathbf{S}$. □

Similar to Proposition 3.2, we have the following proposition.

Proposition 3.3. *Let* $\mathbf{S}$ *be an ordered semigroup and let* F_1, F_2 *be* (m,n)*-fuzzy right ideals of* $\mathbf{S}$*. Then* $F_1 \sqcap F_2$ *is an* (m,n)*-fuzzy right ideal of* $\mathbf{S}$*.*

Combining Proposition 3.2 and Proposition 3.3, we obtain the following corollary.

Corollary 3.1. *Let* $\mathbf{S}$ *be an ordered semigroup and let* F_1, F_2 *be* (m,n)*-fuzzy ideals of* $\mathbf{S}$*. Then* $F_1 \sqcap F_2$ *is an* (m,n)*-fuzzy ideal of* $\mathbf{S}$*.*

Definition 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup. An* (m,n)*-fuzzy subsemigroup* $F = (f,g)$ *of* $\mathbf{S}$ *is called an* (m,n)*-fuzzy bi-ideal of* $\mathbf{S}$ *if the following conditions are satisfied: For any* $x, y, z \in S$*,*

(1) $f^m(xyz) \geq \min\{f^m(x), f^m(z)\}$ *and* $g^n(xyz) \leq \max\{g^n(x), g^n(z)\}$.
(2) *If* $x \leq y$*, then* $f^m(x) \geq f^m(y)$ *and* $g^n(x) \leq g^n(y)$*.*

Example 3.5. *By an ordered semigroup in Example 3.2. We define* $F = (f,g)$ *a* $(4,3)$*-fuzzy subsemigroup of* $\langle S; *, \preceq \rangle$ *as follows:*

S	$f(x)$	$g(x)$
a	0.85	0.42
b	0.62	0.55
c	0.62	0.55

It is routine to verify that F *is a* $(4,3)$*-fuzzy bi-ideal of* $\langle S; *, \preceq \rangle$*.*

Proposition 3.4. *Let* $\mathbf{S}$ *be an ordered semigroup and let* F_1, F_2 *be* (m,n)*-fuzzy bi-ideals of* $\mathbf{S}$*. Then* $F_1 \sqcap F_2$ *is an* (m,n)*-fuzzy bi-ideal of* $\mathbf{S}$*.*

Proof. Let $F_1 = (f_1, g_1)$ and $F_2 = (f_2, g_2)$ be (m,n)-fuzzy bi-ideals of $\mathbf{S}$. First, we shows that $F_1 \sqcap F_2$ is an (m,n)-fuzzy subsemigroup of $\mathbf{S}$. Let $x, y \in S$. Then, let us consider as follows.

$$\begin{aligned}(f_1 \cap f_2)^m(xy) &= \min\{f_1^m(xy), f_2^m(xy)\} \\ &\geq \min\{\min\{f_1^m(x), f_1^m(y)\}, \min\{f_2^m(x), f_2^m(y)\}\} \\ &\geq \min\{\min\{f_1^m(x), f_2^m(x)\}, \min\{f_1^m(y), f_2^m(y)\}\} \\ &= \min\{(f_1 \cap f_2)^m(x), (f_1 \cap f_2)^m(y)\},\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(xy) &= \max\{g_1^n(xy), g_2^n(xy)\}\\ &\leq \max\{\max\{g_1^n(x), g_1^n(y)\}, \max\{g_2^n(x), g_2^n(y)\}\}\\ &\leq \max\{\max\{g_1^n(x), g_2^n(x)\}, \max\{g_1^n(y), g_2^n(y)\}\}\\ &= \max\{(g_1 \cup g_2)^n(x), (g_1 \cup g_2)^n(y)\}.\end{aligned}$$

This complete to prove that $F_1 \sqcap F_2$ is an (m,n)-fuzzy subsemigroup of $\mathbf{S}$. Let $x, y, z \in S$. Then, let us consider as follows.

$$\begin{aligned}(f_1 \cap f_2)^m(xyz) &= \min\{f_1^m(xyz), f_2^m(xyz)\}\\ &\geq \min\{\min\{f_1^m(x), f_1^m(z)\}, \min\{f_2^m(x), f_2^m(z)\}\}\\ &= \min\{\min\{f_1^m(x), f_2^m(x)\}, \min\{f_1^m(z), f_2^m(z)\}\}\\ &= \min\{(f_1 \cap f_2)^m(x), (f_1 \cap f_2)^m(z)\},\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(xyz) &= \max\{g_1^n(xyz), g_2^n(xyz)\}\\ &\leq \max\{\max\{g_1^n(x), g_1^n(z)\}, \max\{g_2^n(x), g_2^n(z)\}\}\\ &= \max\{\max\{g_1^n(x), g_2^n(x)\}, \max\{g_1^n(z), g_2^n(z)\}\}\\ &= \max\{(g_1 \cup g_2)^n(x), (g_1 \cup g_2)^n(z)\}.\end{aligned}$$

Let $x, y \in S$ be such that $x \preceq y$. Then, we obtain

$$\begin{aligned}(f_1 \cap f_2)^m(x) &= \min\{f_1^m(x), f_2^m(x)\}\\ &\geq \min\{f_1^m(y), f_2^m(y)\}\\ &= (f_1 \cap f_2)^m(y),\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(x) &= \max\{g_1^n(x), g_2^n(x)\}\\ &\leq \max\{g_1^n(y), g_2^n(y)\}\\ &= (g_1 \cup g_2)^n(y).\end{aligned}$$

Therefore $F_1 \sqcap F_2$ is an (m,n)-fuzzy bi-ideal of $\mathbf{S}$. □

We, now study the relationship among such their (m,n)-fuzzy ideals. It is easy to verify that every (m,n)-fuzzy left (right, two-sided) ideal of $\mathbf{S}$ is an (m,n)-fuzzy subsemigroup of $\mathbf{S}$.

Lemma 3.1. *Let $\mathbf{S}$ be an ordered semigroup and let $F = (f, g)$ be an (m,n)-fuzzy left ideal of $\mathbf{S}$. Then F is an (m,n)-fuzzy bi-ideal of $\mathbf{S}$.*

Proof. Let $F = (f, g)$ be an (m, n)-fuzzy left ideal of **S** and $x, y, z \in S$. Then we obtain

$$\begin{aligned} f^m(xyz) &= f^m(x(yz)) \\ &\geq f^m(yz) \\ &\geq f^m(z), \end{aligned}$$

Similarly, we have $f^m(xyz) \geq f^m(x)$, which implies that $f^m(xyz) \geq \min\{f^m(x), f^m(z)\}$ and

$$\begin{aligned} g^n(xyz) &= g^n(x(yz)) \\ &\leq g^n(yz) \\ &\leq g^n(z), \end{aligned}$$

Similarly, we have $g^n(xyz) \leq g^n(x)$, which implies that $g^n(xyz) \leq \max\{g^n(x), g^n(z)\}$. Therefore F is an (m, n)-fuzzy bi-ideal of **S**. □

Similar to Lemma 3.1, we obtain the following lemma.

Lemma 3.2. *Let* **S** *be an ordered semigroup and let F be an (m, n)-fuzzy right ideal of* **S**. *Then F is an (m, n)-fuzzy bi-ideal of* **S**.

Combining Lemma 3.1 and Lemma 3.2, we obtain the following corollary.

Corollary 3.2. *Let* **S** *be an ordered semigroup and let F be an (m, n)-fuzzy ideal of* **S**. *Then F is an (m, n)-fuzzy bi-ideal of* **S**.

However, in general, (m, n)-fuzzy bi-ideals of **S** need not to be either (m, n)-fuzzy left or (m, n)-fuzzy right ideals of **S** as can be seen by the following example.

Example 3.6. *Let $S = \{a, b, c, d\}$ be a set with a binary operation "$*$" in the table below.*

$*$	a	b	c	d
a	a	a	a	a
b	a	a	a	c
c	a	a	a	c
d	c	c	c	d

and define a partial ordered $\leq$ on S as follows.

$$\leq := \Delta_S \cup \{(a, b)\},$$

where Δ_S is an identity relation on S. Then it is easy to verify that $\langle S; *, \leq\rangle$ is an ordered semigroup. Define $F = (f, g)$ an $(2, 3)$-fuzzy subset of S as follows:

S	$f(x)$	$g(x)$
a	0.87	0.17
b	0.75	0.43
c	0.65	0.56
d	0.65	0.56

It is routine to verify that F is a $(2,3)$-fuzzy bi-ideal of $\langle S;*,\preceq\rangle$ but F is not a $(2,3)$-fuzzy ideal of $\langle S;*,\preceq\rangle$ because of $f_1^2(d*b) = f_1^2(c) = (0.65)^2 = 4.225 \ngeq 0.5625 = (0.75)^2 = f_1^2(b)$ and $f_1^2(b*d) = f_1^2(c) = (0.65)^2 = 4.225 \ngeq 0.5625 = (0.75)^2 = f_1^2(b)$.

We, now give the concept of (m,n)-fuzzy ideals and (m,n)-fuzzy bi-ideals coinside. An ordered semigroup **S** is called *duo* if its one-sided ideal (left ideal or right ideal) is two-sided ideal and such ordered semigroup **S** is called *regular* if for each element $a \in S$, there exists $x \in S$ such that $a \preceq axa$.

Theorem 3.1. *Let **S** be a regular duo ordered semigroup and let F be an (m,n)-fuzzy subset of S. If F is an (m,n)-fuzzy bi-ideal of **S**, then F is an (m,n)-fuzzy ideal of **S**.*

Proof. Let $F = (f,g)$ be an (m,n)-fuzzy bi-ideal of **S** and $a,b \in S$. Then, it is easy to verify that $\{a\}S$ is a right ideal of **S** and then $\{a\}S$ is also a left ideal of **S**. Since **S** is regular, we obtain

$$ba \in S(aSa] \subseteq (SaSa] \subseteq (aSa].$$

This implies that there exists $x \in S$ such that $ba \preceq axa$ and we obtain

$$f^m(ba) \geq f^m(axa) \geq \min\{f^m(a), f^m(a)\} = f^m(a)$$

and

$$g^n(ba) \leq g^n(axa) \leq \max\{g^n(a), g^n(a)\} = g^n(a).$$

This means that F is an (m,n)-fuzzy left ideal of **S**. It can be seen in a similar way that F is also an (m,n)-fuzzy right ideal of **S**. Therefore F is an (m,n)-fuzzy ideal of **S**. This completes the proof. □

Combining Lemma 3.2 and Theorem 3.1 we obtain the following corollary.

Corollary 3.3. *Let **S** be a regular duo ordered semigroup and let F be an (m,n)-fuzzy subset of S. Then the following conditions are equivalent.*

(1) *F is an (m,n)-fuzzy ideal of **S**.*
(2) *F is an (m,n)-fuzzy bi-ideal of **S**.*

Definition 3.5. *Let **S** be an ordered semigroup. An (m,n)-fuzzy subsemigroup $F = (f,g)$ of S is called an (m,n)-fuzzy $(1,2)$-ideal of **S** if the following conditions are satisfied: For any $x,y,z,u \in S$,*

(1) *$f^m(xu(yz)) \geq \min\{f^m(x), f^m(y), f^m(z)\}$ and $g^n(xu(yz)) \leq \max\{g^n(x), g^n(y), g^n(z)\}$.*
(2) *If $x \preceq y$, then $f^m(x) \geq f^m(y)$ and $g^n(x) \leq g^n(y)$.*

Example 3.7. *Let $S = \{a,b,c,d\}$ be a set with a binary operation "$*$" in the table below.*

$*$	a	b	c	d
a	a	a	a	a
b	a	b	c	a
c	a	c	c	b
d	a	b	d	d

and define a partial ordered $\preceq$ on S as follows.

$$\preceq := \Delta_S \cup \{(b,d)\},$$

where Δ_S is an identity relation on S. Then it is easy to verify that $\langle S;*,\preceq\rangle$ is an ordered semigroup. Define $F=(f,g)$ a $(4,3)$-fuzzy subset of S as follows:

S	$f(x)$	$g(x)$
a	0.91	0.17
b	0.72	0.43
c	0.53	0.56
d	0.53	0.56

We obtain that F is a $(4,3)$-fuzzy $(1,2)$-ideal of $\langle S;*,\preceq\rangle$.

Proposition 3.5. *Let **S** be an ordered semigroup and let F_1, F_2 be (m,n)-fuzzy $(1,2)$-ideals of **S**. Then $F_1 \sqcap F_2$ is an (m,n)-fuzzy $(1,2)$-ideal of **S**.*

Proof. Let $F_1=(f_1,g_1)$ and $F_2=(f_2,g_2)$ be (m,n)-fuzzy $(1,2)$-ideals of **S** and $x,y\in S$. Then, let us consider as follows.

$$\begin{aligned}(f_1\cap f_2)^m(xy) &= \min\{f_1^m(xy), f_2^m(xy)\}\\ &\geq \min\{\min\{f_1^m(x), f_1^m(y)\}, \min\{f_2^m(x), f_2^m(y)\}\}\\ &= \min\{\min\{f_1^m(x), f_2^m(x)\}, \min\{f_1^m(y), f_2^m(y)\}\}\\ &= \min\{(f_1\cap f_2)^m(x), (f_1\cap f_2)^m(y)\},\end{aligned}$$

and

$$\begin{aligned}(g_1\cup g_2)^n(xy) &= \max\{g_1^n(xy), g_2^n(xy)\}\\ &\leq \max\{\max\{g_1^n(x), g_1^n(y)\}, \max\{g_2^n(x), g_2^n(y)\}\}\\ &= \max\{\max\{g_1^n(x), g_2^n(x)\}, \max\{g_1^n(y), g_2^n(y)\}\}\\ &= \max\{(g_1\cup g_2)^n(x), (g_1\cup g_2)^n(y)\}.\end{aligned}$$

Therefore $F_1 \sqcap F_2$ is an (m,n)-fuzzy subsemigroup of **S**. Let $u,x,y,z\in S$. Then, let us consider as follows.

$$\begin{aligned}(f_1\cap f_2)^m(xu(yz)) &= \min\{f_1^m(xu(yz)), f_2^m(xu(yz))\}\\ &\geq \min\{\min\{f_1^m(x), f_1^m(y), f_1^m(z)\}, \min\{f_2^m(x), f_2^m(y), f_2^m(z)\}\}\\ &= \min\{\min\{f_1^m(x), f_2^m(x)\}, \min\{f_1^m(y), f_2^m(y)\}, \min\{f_1^m(z), f_2^m(z)\}\}\\ &= \min\{(f_1\cap f_2)^m(x), (f_1\cap f_2)^m(y), (f_1\cap f_2)^m(z)\},\end{aligned}$$

and

$$\begin{aligned}(g_1 \cup g_2)^n(xu(yz)) &= \max\{g_1^n(xu(yz)), g_2^n(xu(yz))\}\\ &\leq \max\{\max\{g_1^n(x), g_1^n(y), g_1^n(z)\}, \max\{g_2^n(x), g_2^n(y), g_2^n(z)\}\}\\ &= \max\{\max\{g_1^n(x), g_2^n(x)\}, \max\{g_1^n(y), g_2^n(y)\}, \max\{g_1^n(z), g_2^n(z)\}\}\\ &= \max\{(g_1 \cup g_2)^n(x), (g_1 \cup g_2)^n(y), (g_1 \cup g_2)^n(z)\}.\end{aligned}$$

This completes to shows that $F_1 \sqcap F_2$ is an (m,n)-fuzzy $(1,2)$-ideal of $\mathbf{S}$. □

Theorem 3.2. *Let* $\mathbf{S}$ *be an ordered semigroup and let* F *be an* (m,n)*-fuzzy subset of* S*. If* F *is an* (m,n)*-fuzzy bi-ideal of* $\mathbf{S}$*, then* F *is an* (m,n)*-fuzzy* $(1,2)$*-ideal of* $\mathbf{S}$*.*

Proof. Let $F = (f, g)$ be an (m,n)-fuzzy bi-ideal of $\mathbf{S}$ and $x, y, z, u \in S$. Then, let us consider as follows.

$$\begin{aligned}f^m(xu(yz)) &\geq \min\{f^m(x), f^m(yz)\}\\ &\geq \min\{f^m(x), \min\{f^m(y), f^m(z)\}\}\\ &= \min\{f^m(x), f^m(y), f^m(z)\},\end{aligned}$$

and

$$\begin{aligned}g^n(xu(yz)) &\leq \max\{g^n(x), g^n(yz)\}\\ &\leq \max\{g^n(x), \max\{g^n(y), g^n(z)\}\}\\ &= \max\{g^n(x), g^n(y), g^n(z)\}.\end{aligned}$$

Therefore F is an (m,n)-fuzzy $(1,2)$-ideal of $\mathbf{S}$. This completes the proof. □

Theorem 3.3. *Let* $\mathbf{S}$ *be a regular ordered semigroup and let* F *be an* (m,n)*-fuzzy subset of* S*. If* F *is an* (m,n)*-fuzzy* $(1,2)$*-ideal of* $\mathbf{S}$*, then* F *is an* (m,n)*-fuzzy bi-ideal of* $\mathbf{S}$*.*

Proof. Let $F = (f, g)$ be an (m,n)-fuzzy $(1,2)$-ideal of $\mathbf{S}$ and $u, x, y, z \in S$. Since $\mathbf{S}$ is regular, we have $xu \in (xSx]S \subseteq (xSx]$, which implies that $xu \preceq xsx$ for some $s \in S$. Thus, we obtain

$$\begin{aligned}f^m(xuy) &\geq f^m((xsx)y)\\ &= f^m(xs(xy))\\ &\geq \min\{f^m(x), f^m(x), f^m(y)\}\\ &= \min\{f^m(x), f^m(y)\},\end{aligned}$$

and

$$\begin{aligned}g^n(xuy) &\leq g^n((xsx)y)\\ &= g^n(xs(xy))\\ &\leq \max\{g^n(x), g^n(x), g^n(y)\}\\ &= \max\{g^n(x), g^n(y)\}.\end{aligned}$$

Therefore F is an (m,n)-fuzzy bi-ideal of $\mathbf{S}$. □

Combining Theorem 3.2 and Theorem 3.3, we obtain the following result.

Corollary 3.4. *Let $\mathbf{S}$ be a regular ordered semigroup and let F be an (m,n)-fuzzy subset of S. Then the following conditions are equivalent.*

(1) *F is an (m,n)-fuzzy $(1,2)$-ideal of $\mathbf{S}$.*
(2) *F is an (m,n)-fuzzy bi-ideal of $\mathbf{S}$.*

Let $A \subseteq S$. We denote by $\chi_A := (\chi_A^*, \chi_A^+)$ the characteristic (m,n)-fuzzy subset of a subset A of S and is defined as follows:

$$(\chi_A^*(a))^m := \begin{cases} 1 & \text{if } a \in A \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(\chi_A^+(a))^n := \begin{cases} 0 & \text{if } a \in A \\ 1 & \text{otherwise.} \end{cases}$$

By definition of $\chi_A = (\chi_A^*, \chi_A^+)$, we see that $(\chi_A^*(a))^m = \chi_A^*(a)$ and $(\chi_A^+(a))^n = \chi_A^+(a)$ for all $m,n \in \mathbb{N}$, so we instead of $(\chi_A^*(a))^m$ and $(\chi_A^+(a))^n$ by $\chi_A^*(a)$ and $\chi_A^+(a)$, respectively, and we denoted by $\mathbf{1} = \chi_A$ if $A = S$ where $\mathbf{1} = (1^*, 0^+)$ such that $1^*(a) = 1$ and $0^+(a) = 0$ for all $a \in S$.

Next, we characterize left (resp., right, two-sided, bi-) ideals of an ordered semigroup $\mathbf{S}$ in terms of (m,n)-fuzzy left (resp., right, two-sided, bi-) ideals of $\mathbf{S}$ as follows.

Lemma 3.3. *Let $\mathbf{S}$ be an ordered semigroup. Then the following statements are equivalent.*

(1) *L is a left ideal of $\mathbf{S}$.*
(2) *$\chi_L = (\chi_L^*, \chi_L^+)$ is an (m,n)-fuzzy left ideal of $\mathbf{S}$.*

Proof. (1)⇒(2). Assume that (1) holds. Let $a,b \in S$. Then, we consider two cases. If $b \notin L$, then $\chi_L^*(ab) \geq 0 = \chi_L^*(b)$ and $\chi_L^+(ab) \leq 1 = \chi_L^+(b)$. If $b \in L$, since L is a left ideal of $\mathbf{S}$, we obtain $ab \in L$ and then $\chi_L^*(ab) = 1 = \chi_L^*(b)$ and $\chi_L^+(ab) = 0 = \chi_L^+(b)$. For any two cases, we obtain $\chi_L^*(ab) \geq \chi_L^*(b)$ and $\chi_L^+(ab) \leq \chi_L^+(b)$. Let $a,b \in S$ be such that $a \leq b$. Then, we consider two cases. If $b \in L$, since L is a left ideal of $\mathbf{S}$, we obtain $a \in L$ and then $\chi_L^*(a) = 1 = \chi_L^*(b)$ and $\chi_L^+(a) = 0 = \chi_L^+(b)$. If $b \notin L$, then $\chi_L^*(a) \geq 0 = \chi_L^*(b)$ and $\chi_L^+(a) \leq 1 = \chi_L^+(b)$. For any two cases, we obtain $\chi_L^*(a) \geq \chi_L^*(b)$ and $\chi_L^+(a) \leq \chi_L^+(b)$. This shows that (2) holds.

(2)⇒(1). Assume that (2) holds. Let $a,b \in S$ be such that $b \in L$. By hypothesis, we have $1 \geq \chi_L^*(ab) \geq \chi_L^*(b) = 1$ which means that $\chi_L^*(ab) = 1$ and $0 \leq \chi_L^+(ab) \leq \chi_L^+(b) = 0$ which means that $\chi_L^+(ab) = 0$, and then $ab \in L$. Let $a,b \in S$ be such that $a \leq b$. If $b \in L$, then by hypothesis, we obtain $1 \geq \chi_L^*(a) \geq \chi_L^*(b) = 1$, which means that $\chi_L^*(a) = 1$ and $0 \leq \chi_L^+(a) \leq \chi_L^+(b) = 0$, which means that $\chi_L^+(a) = 0$, and then $a \in L$. This completes to proof that (1) holds. □

Similar to Lemma 3.3, we obtain the following lemma.

Lemma 3.4. *Let $\mathbf{S}$ be an ordered semigroup. Then the following statements are equivalent.*

(1) *R is a right ideal of* $\mathbf{S}$.

(2) $\chi_R = (\chi^*_R, \chi^+_R)$ *is an* (m,n)*-fuzzy right ideal of* $\mathbf{S}$.

Combining Lemma 3.3 and Lemma 3.4, we have the following corollary.

Corollary 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent.*

(1) *I is an ideal of* $\mathbf{S}$.

(2) $\chi_I = (\chi^*_I, \chi^+_I)$ *is an* (m,n)*-fuzzy ideal of* $\mathbf{S}$.

Lemma 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following statements are equivalent.*

(1) *B is a bi-ideal of* $\mathbf{S}$.

(2) $\chi_B = (\chi^*_B, \chi^+_B)$ *is an* (m,n)*-fuzzy bi-ideal of* $\mathbf{S}$.

Proof. (1)⇒(2). Assume that (1) holds. Let $a,b \in S$. Then, we consider four cases. If $a,b \notin B$, then $\chi^*_B(ab) \geq 0 = \min\{\chi^*_B(a), \chi^*_B(b)\}$ and $\chi^+_B(ab) \leq 1 = \max\{\chi^+_B(a), \chi^+_B(b)\}$. If $a \in B$ and $b \notin B$, then $\chi^*_B(ab) \geq 0 = \min\{\chi^*_B(a), \chi^*_B(b)\}$ and $\chi^+_B(ab) \leq 1 = \max\{\chi^+_B(a), \chi^+_B(b)\}$. If $a \notin B$ and $b \in B$, then it is clear. If $a \in B$ and $b \in B$, then $\chi^*_B(ab) = 1 = \min\{\chi^*_B(a), \chi^*_B(b)\}$ and $\chi^+_B(ab) = 0 = \max\{\chi^+_B(a), \chi^+_B(b)\}$. For any four cases, we obtain $\chi^*_B(ab) \geq \min\{\chi^*_B(a), \chi^*_B(b)\}$ and $\chi^+_B(ab) \leq \max\{\chi^+_B(a), \chi^+_B(b)\}$. Thus χ_B is an (m,n)-fuzzy subsemigroup of $\mathbf{S}$. Let $a,b,c \in S$. Then, we consider four cases. If $a,c \in B$, then $\chi^*_B(abc) = 1 = \min\{\chi^*_B(a), \chi^*_B(c)\}$ and $\chi^+_B(abc) = 0 = \max\{\chi^+_B(a), \chi^+_B(c)\}$. If $a \in B$ and $c \notin B$, then $\chi^*_B(abc) \geq 0 = \min\{\chi^*_B(a), \chi^*_B(c)\}$ and $\chi^+_B(abc) \leq 1 = \max\{\chi^+_B(a), \chi^+_B(c)\}$. If $a \notin B$ and $c \in B$, then it is clear. If $a \notin B$ and $c \notin B$, then $\chi^*_B(abc) \geq 0 = \min\{\chi^*_B(a), \chi^*_B(c)\}$ and $\chi^+_B(abc) \leq 1 = \max\{\chi^+_B(a), \chi^+_B(c)\}$. For any four cases, we have $\chi^*_B(abc) \geq \min\{\chi^*_B(a), \chi^*_B(c)\}$ and $\chi^+_B(abc) \leq \max\{\chi^+_B(a), \chi^+_B(c)\}$. Let $a,b \in S$ be such that $a \leq b$. Then, we consider two cases. If $b \in B$, then $\chi^*_B(a) = 1 = \chi^*_B(b)$ and $\chi^+_B(a) = 0 = \chi^*_B(b)$. If $b \notin B$, then $\chi^*_B(a) \geq 0 = \chi^*_B(b)$ and $\chi^+_B(a) \leq 1 = \chi^*_B(b)$. For any two cases, we obtain $\chi^*_B(a) \geq \chi^*_B(b)$ and $\chi^+_B(a) \leq \chi^+_B(b)$. This completes to shows that (2) holds.

(2)⇒(1). Assume that (2) holds. Let $a,b \in S$ be such that $a,b \in B$. By hypothesis, we have $1 \geq \chi^*_B(ab) \geq \min\{\chi^*_B(a), \chi^*_B(b)\} = 1$ which means that $\chi^*_B(ab) = 1$ and $0 \leq \chi^+_B(ab) \leq \max\{\chi^+_B(a), \chi^+_B(b)\} = 0$ which means that $\chi^+_B(ab) = 0$. Then $ab \in B$. Therefore B is a subsemigroup of $\mathbf{S}$. Let $a,b,c \in S$ and $a,c \in B$. Then, by hypothesis, $1 \geq \chi^*_B(abc) \geq \min\{\chi^*_B(a), \chi^*_B(c)\} = 1$ which means that $\chi^*_B(abc) = 1$ and $0 \leq \chi^+_B(abc) \leq \max\{\chi^+_B(a), \chi^+_B(c)\} = 0$ which means that $\chi^+_B(abc) = 0$. Then $abc \in B$. Let $a,b \in S$ be such that $a \leq b$. If $b \in B$, then by hypothesis, we obtain $1 \geq \chi^*_B(a) \geq \chi^*_B(b) = 1$, which means that $\chi^*_L(a) = 1$ and $0 \leq \chi^+_B(a) \leq \chi^+_B(b) = 0$, which means that $\chi^+_L(a) = 0$ and then $a \in B$. This completes to proof that (1) holds. □

Lemma 3.6. *Let* $\mathbf{S}$ *be an ordered semigroup and let A,B be subsets of S. Then the following for each condition hold.*

(1) $\chi_A = \chi_B$ *if and only if* $A = B$.

(2) $\chi_A \diamond \chi_B = \chi_{(AB]}$.

(3) $\chi_A \sqcap \chi_B = \chi_{A \cap B}$.

Proof. Let A, B be subsets of S. We give only the proof of (2) since the rests are not difficult to verify, we consider two cases. If $a \in (AB]$, then there exist $x \in A$ and $y \in B$ such that $a \leq xy$. This means that $\mathbf{S}_a \neq \emptyset$. Thus, we obtain

$$\begin{aligned}(\chi^*_A \diamond \chi^*_B)(a) &\leq 1 \\ &= \chi^*_{(AB]}(a) \\ &= \min\{\chi^*_A(x), \chi^*_B(y)\} \\ &\leq \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{\chi^*_A(p), \chi^*_B(q)\}\} \\ &= (\chi^*_A \diamond \chi^*_B)(a),\end{aligned}$$

and

$$\begin{aligned}(\chi^+_A \diamond \chi^+_B)(a) &\geq 0 \\ &= \chi^+_{(AB]}(a) \\ &= \max\{\chi^+_A(x), \chi^+_B(y)\} \\ &\geq \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{\chi^+_A(p), \chi^+_B(q)\}\} \\ &= (\chi^+_A \diamond \chi^+_B)(a).\end{aligned}$$

This implies that $(\chi^*_A \diamond \chi^*_B)(a) = \chi^*_{(AB]}(a)$ and $(\chi^+_A \diamond \chi^+_B)(a) = \chi^+_{(AB]}(a)$. If $a \notin (AB]$, then there are no $x, y \in S$ such that $a \leq xy$. This implies that $\mathbf{S}_a = \emptyset$. Thus, we obtain

$$(\chi^*_A \diamond \chi^*_B)(a) = 0 = \chi^*_{(AB]}(a) \text{ and } (\chi^+_A \diamond \chi^+_B)(a) = 1 = \chi^+_{(AB]}(a).$$

For any two cases, we have $\chi_A \diamond \chi_B = \chi_{(AB]}$. □

Next, we characterization of regular and intra-regular ordered semigroups in terms of (m, n)-fuzzy left ideals, (m, n)-fuzzy right ideals and (m, n)-fuzzy bi-ideals but we need the following properties.

Lemma 3.7. *Let* $\mathbf{S}$ *be an ordered semigroup and let* $F = (f, g)$ *be an* (m, n)*-fuzzy subset of* S*. Then the following conditions are equivalent.*

(1) *F is an (m, n)-fuzzy left ideal of* $\mathbf{S}$*.*

(2) *F satisfies that*

 (2.1) *For $x, y \in S$, if $x \leq y$, then $f^m(x) \geq f^m(y)$ and $g^n(x) \leq g^n(y)$.*

 (2.2) $\mathbf{1} \diamond F \sqsubseteq F$*.*

Proof. (1)⇒(2). Assume that (1) holds, by hypothesis (2.1) holds. Let $a \in S$. Then, we consider two cases as follows. If $\mathbf{S}_a = \emptyset$, we obtain

$$(1^* \diamond f)^m(a) = 0 \leq f^m(a) \text{ and } (0^+ \diamond g)^n(a) = 1 \geq g^n(a).$$

If $\mathbf{S}_a \neq \emptyset$, we have

$$\begin{aligned}(1^* \diamond f)^m(a) &= \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{1^*(u), f^m(v)\}\} \\ &\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{1^*(uv), f^m(uv)\}\} \\ &\leq \bigvee_{(u,v)\in \mathbf{S}_a} \{\min\{1^*(a), f^m(a)\}\} \\ &= f^m(a),\end{aligned}$$

and

$$\begin{aligned}(0^+ \diamond g)^n(a) &= \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{0^+(u), g^n(v)\}\} \\ &\geq \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{0^+(uv), g^n(uv)\}\} \\ &\geq \bigwedge_{(u,v)\in \mathbf{S}_a} \{\max\{0^+(a), g^n(a)\}\} \\ &= g^n(a).\end{aligned}$$

For any two cases, we obtain $\mathbf{1} \diamond F \sqsubseteq F$. This completes to proof (2.2) holds.

(2)⇒(1). Assume that (2) holds. Let $a, b \in S$. Put $x = ab$. Then we obtain

$$\begin{aligned}f^m(ab) &= f^m(x) \\ &\geq (1^* \diamond f)^m(x) \\ &= \bigvee_{(u,v)\in \mathbf{S}_x} \{\min\{1^*(u), f^m(v)\}\} \\ &\geq \min\{1^*(a), f^m(b)\} \\ &= f^m(b),\end{aligned}$$

and

$$\begin{aligned}g^n(ab) &= g^n(x) \\ &\leq (0^+ \diamond g)^n(x) \\ &= \bigwedge_{(u,v)\in \mathbf{S}_x} \{\max\{0^+(u), g^n(v)\}\} \\ &\leq \max\{0^+(a), g^n(b)\} \\ &= g^n(b).\end{aligned}$$

It is completes to shows that (1) holds. □

Similar to Lemma 3.7, we obtain the following lemma.

Lemma 3.8. *Let* **S** *be an ordered semigroup and let* $F = (f, g)$ *be an* (m, n)*-fuzzy subset of* S*. Then the following conditions are equivalent.*

(1) F *is an* (m, n)*-fuzzy right ideal of* **S**.

(2) F *satisfies that*

(2.1) *For* $x, y \in S$*, if* $x \leq y$*, then* $f^m(x) \geq f^m(y)$ *and* $g^n(x) \leq g^n(y)$.

(2.2) $F \diamond \mathbf{1} \sqsubseteq F$.

Combining Lemma 3.7 and Lemma 3.8, we have the following result.

Lemma 3.9. *Let* **S** *be an ordered semigroup and let* $F = (f, g)$ *be an* (m, n)*-fuzzy subset of* S*. Then the following conditions are equivalent.*

(1) F *is an* (m, n)*-fuzzy ideal of* **S**.

(2) F *satisfies that*

(2.1) *For* $x, y \in S$*, if* $x \leq y$*, then* $f^m(x) \geq f^m(y)$ *and* $g^n(x) \leq g^n(y)$.

(2.2) $\mathbf{1} \diamond F \sqsubseteq F$ *and* $F \diamond \mathbf{1} \sqsubseteq F$.

Lemma 3.10. *[22] Let* **S** *be an ordered semigroup. Then the following conditions are equivalent.*

(1) **S** *is regular.*

(2) $R \cap L = (RL]$ *for every left ideal* L *and right ideal* R *of* **S**.

Theorem 3.4. *Let* **S** *be an ordered semigroup. Then the following conditions are equivalent.*

(1) **S** *is regular.*

(2) $F \sqcap G = F \diamond G$ *for every* (m, n)*-fuzzy left ideal* G *and* (m, n)*-fuzzy right ideal* F *of* **S**.

Proof. (1)⇒(2). Assume that (1) holds. Let $G = (g_1, g_2)$ and $F = (f_1, f_2)$ be an (m, n)-fuzzy left ideal of **S** and an (m, n)-fuzzy right ideal of **S**, respectively. Let $a \in S$. Then, since **S** is regular, there exists $x \in S$ such that $a \leq axa$, we obtain

$$\begin{aligned}(f_1 \diamond g_1)^m(a) &= \bigvee_{(p,q) \in \mathbf{S}_a} \{\min\{f_1^m(p), g_1^m(q)\}\} \\ &\geq \min\{f_1^m(a), g_1^m(xa)\} \\ &\geq \min\{f_1^m(a), g_1^m(a)\} \\ &= (f_1 \cap g_1)^m(a),\end{aligned}$$

and

$$\begin{aligned}(f_2 \diamond g_2)^n(a) &= \bigwedge_{(p,q) \in \mathbf{S}_a} \{\max\{f_2^n(p), g_2^n(q)\}\} \\ &\leq \max\{f_2^n(a), g_2^n(xa)\} \\ &\leq \max\{f_2^n(a), g_2^n(a)\} \\ &= (f_2 \cup g_2)^n(a).\end{aligned}$$

This means that $F \sqcap G \sqsubseteq F \diamond G$. In the other conclusion, by Lemma 3.7 and Lemma 3.8, we have $F \diamond G \sqsubseteq \mathbf{1} \diamond G \sqsubseteq G$ and $F \diamond G \sqsubseteq F \diamond \mathbf{1} \sqsubseteq F$ and then $F \diamond G \sqsubseteq F \sqcap G$. Altogether, we obtain $F \sqcap G = F \diamond G$. This shows that (2) holds.

(2)⇒(1). Assume that (2) holds. Let L and R be a left ideal of $\mathbf{S}$ and a right ideal of $\mathbf{S}$, respectively. By Lemma 3.3 and Lemma 3.4, we obtain χ_L and χ_R be an (m,n)-fuzzy left ideal of $\mathbf{S}$ and an (m,n)-fuzzy right ideal of $\mathbf{S}$, respectively. By hypothesis and Lemma 3.6, we have

$$\begin{aligned} \chi_{(R\cap L)} &= \chi_R \sqcap \chi_L \\ &= \chi_R \diamond \chi_L \\ &= \chi_{(RL]}. \end{aligned}$$

It follows that $R \cap L = (RL]$ and by Lemma 3.10, $\mathbf{S}$ is regular. This shows that (1) holds. □

An ordered semigroup $\mathbf{S}$ is called *intra-regular* if for each $a \in S$ there exist $x, y \in S$ such that $a \leq xa^2y$. We now characterize intra-regular ordered semigroups in terms of (m,n)-fuzzy left ideals and (m,n)-fuzzy right ideals.

Lemma 3.11. *[22] Let* $\mathbf{S}$ *be an ordered semigroup. Then the following conditions are equivalent.*

(1) $\mathbf{S}$ *is intra-regular.*
(2) $L \cap R \subseteq (LR]$ *for every left ideal* L *and right ideal* R *of* $\mathbf{S}$.

Theorem 3.5. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following conditions are equivalent.*

(1) $\mathbf{S}$ *is intra-regular.*
(2) $G \sqcap F \subseteq G \diamond F$ *for every* (m,n)*-fuzzy left ideal* G *and* (m,n)*-fuzzy right ideal* F *of* $\mathbf{S}$.

Proof. (1)⇒(2). Assume that (1) holds. Let $G = (g_1, g_2)$ and $F = (f_1, f_2)$ be an (m,n)-fuzzy left ideal of $\mathbf{S}$ and an (m,n)-fuzzy right ideal of $\mathbf{S}$, respectively. Let $a \in S$. Then, since $\mathbf{S}$ is intra-regular, there exist $x, y \in S$ such that $a \leq xa^2y = (xa)(ay)$, we obtain

$$\begin{aligned} (g_1 \diamond f_1)^m(a) &= \bigvee_{(p,q)\in \mathbf{S}_a} \{\min\{g_1^m(p), f_1^m(q)\}\} \\ &\geq \min\{g_1^m(xa), f_1^m(ay)\} \\ &\geq \min\{g_1^m(a), f_1^m(a)\} \\ &= (g_1 \cap f_1)^m(a), \end{aligned}$$

and

$$\begin{aligned} (g_2 \diamond f_2)^n(a) &= \bigwedge_{(p,q)\in \mathbf{S}_a} \{\max\{g_2^n(p), f_2^n(q)\}\} \\ &\leq \max\{g_2^n(xa), f_2^n(ay)\} \\ &\leq \max\{g_2^n(a), f_2^n(a)\} \\ &= (g_2 \cup f_2)^n(a). \end{aligned}$$

This means that $G \sqcap F \sqsubseteq G \diamond F$. This shows that (2) holds.

(2)⇒(1). Assume that (2) holds. Let L and R be a left ideal of $\mathbf{S}$ and a right ideal of $\mathbf{S}$, respectively. By Lemma 3.3 and Lemma 3.4, we obtain χ_L and χ_R be an (m,n)-fuzzy left ideal of $\mathbf{S}$ and an (m,n)-fuzzy right ideal of $\mathbf{S}$, respectively. By hypothesis and Lemma 3.6, we have

$$\begin{aligned} \chi_{(L\cap R)} &= \chi_L \sqcap \chi_R \\ &\sqsubseteq \chi_L \diamond \chi_R \\ &= \chi_{(LR]}. \end{aligned}$$

It follows that $L \cap R \subseteq (LR]$ and by Lemma 3.11, $\mathbf{S}$ is intra-regular. This shows that (1) holds. □

Finally, we characterize regular ordered semigroups and intra-regular ordered semigroups in terms of (m,n)-fuzzy bi-ideals.

Lemma 3.12. *[23] Let* $\mathbf{S}$ *be an ordered semigroup. Then the following conditions are equivalent.*

(1) $\mathbf{S}$ *is regular and intra-regular.*
(2) $B = (BB]$ *for every bi-ideal* B *of* $\mathbf{S}$*.*

Theorem 3.6. *Let* $\mathbf{S}$ *be an ordered semigroup. Then the following conditions are equivalent.*

(1) $\mathbf{S}$ *is regular and intra-regular.*
(2) $F = F \diamond F$ *for every* (m,n)*-fuzzy bi-ideal* F *of* $\mathbf{S}$*.*

Proof. (1)⇒(2). Assume that (1) holds. Let $F = (f,g)$ be an (m,n)-fuzzy bi-ideal of $\mathbf{S}$ and $a \in S$. Then, since $\mathbf{S}$ is regular and intra-regular, there exists $x \in S$ such that $a \preceq axa$ and there exist $y, z \in S$ such that $a \preceq ya^2z$. Thus

$$a \preceq axa \preceq ax(axa) \preceq ax(ya^2z)xa = (axya)(azxa).$$

Since F is (m,n)-fuzzy bi-ideal, we obtain

$$f^m(axya) \geq \min\{f^m(a), f^m(a)\} = f^m(a) \text{ and } g^n(axya) \leq \max\{g^n(a), g^n(a)\} = g^n(a),$$

similarly, we have

$$f^m(azxa) \geq \min\{f^m(a), f^m(a)\} = f^m(a) \text{ and } g^n(azxa) \leq \max\{g^n(a), g^n(a)\} = g^n(a).$$

Then

$$\begin{aligned} (f \diamond f)^m(a) &= \bigvee_{(u,v)\in \mathbf{S_a}} \{\min\{f^m(u), f^m(v)\}\} \\ &\geq \min\{f^m(axya), f^m(azxa)\} \\ &\geq \min\{f^m(a), f^m(a)\} \\ &= f^m(a), \end{aligned}$$

and

$$\begin{aligned}(g \diamond g)^n(a) &= \bigwedge_{(u,v)\in \mathbf{S_a}} \{\max\{g^n(u), g^n(v)\}\} \\ &\leq \max\{g^n(axya), g^n(azxa)\} \\ &\leq \max\{g^n(a), g^n(a)\} \\ &= g^n(a).\end{aligned}$$

Thus $F \sqsubseteq F \diamond F$. It is clear that $F \diamond F \sqsubseteq F$. Altogether, we obtain $F = F \diamond F$. It is completes to prove (2) holds.

(2)⇒(1). Assume that (2) holds. Let B be a bi-ideal of **S**. By Lemma 3.5, we have χ_B is an (m,n)-fuzzy bi-ideal of **S** and by hypothesis and Lemma 3.6, we have

$$\begin{aligned}\chi_B &= \chi_B \diamond \chi_B \\ &= \chi_{(BB]}.\end{aligned}$$

Then $B = (BB]$. By Lemma 3.12, **S** is regular and intra-regular. □

4. Conclusions

In this paper, we introduced the concepts of (m,n)-fuzzy subsemigroups, (m,n)-fuzzy left (right, two-sided, bi-, (1,2)-) ideals and some their algebraic properties are studied, thereafter the relationship among their (m,n)-fuzzy ideals was investigated. Moreover, we characterized left (resp., right, two-sided, bi-) ideals in terms of (m,n)-fuzzy left (resp., right, two-sided, bi-) ideals. Finally, we characterized regular ordered semigroups and intra-regular ordered semigroups by using (m,n)-fuzzy left, (m,n)-fuzzy right, and (m,n)-fuzzy bi-ideals. In our future work, we will use this idea to characterize other classes of ordered semigroups.

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