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Geometry of Certain Almost Conformal Metrics in $f(\mathtt{R})$-Gravity

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Abstract. In this article, we explore certain almost conformal Ricci solitons in $f(\mathtt{R})$-gravity by assuming the potential vector field as a concircular vector field. We also study the almost conformal gradient-Ricci solitons and the almost conformal ω-Ricci solitons in $f(\mathtt{R})$-gravity. Furthermore, it is shown that an almost conformal ω-Ricci soliton and an almost conformal ω-Ricci-Yamabe soliton establish Poisson's equation. At the last, some examples are constructed.

1. **Introduction**

In 1982, Hamilton [1] gave the concept of Ricci flow. The Ricci soliton ($\mathcal{RS}$) is a natural generalization of Einstein metric, which are self-similar solutions of Hamilton's Ricci flow [2]. It often arises as limits of dialations of singularities in the Ricci flow. Sinha and Sharma [3] began the study of $\mathcal{RS}$ in contact manifolds. Later on, Bejan and Crasmareanu [4], Călin and Crasmareanu [5] examined $\mathcal{RS}$ in contact metric manifolds. The Ricci flow and $\mathcal{RS}$ equations are, respectively, mentioned below:

$$\frac{\partial g}{\partial t} + 2\mathcal{S} = 0, \tag{1.1}$$

and

$$\pounds_{\mathcal{F}} g + 2\mathcal{S} = -2\rho g, \tag{1.2}$$

where, $\pounds_{\mathcal{F}}$ is the Lie derivative along the soliton vector field $\mathcal{F}$, $\mathcal{S}$ is the Ricci tensor, g is the Riemannian metric and ρ is a real scalar. For further study, see [6–10].

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The idea of almost Ricci soliton ($\mathcal{ARS}$) was proposed by Pigola et.al. in 2011 [11], where the authors modified the definition of $\mathcal{RS}$ by imposing the restriction on ρ to be a variable function. In other words, we say that an n-dimensional Riemannian manifold (M^n, g) admits an $\mathcal{ARS}$, if there exists a potential vector field $\mathcal{F}$ and a smooth soliton function ρ: $M^n \rightarrow \mathbb{R}$ obeying:

$$\mathcal{S} + \frac{1}{2}\pounds_{\mathcal{F}} g = -\rho g. \tag{1.3}$$

Equation (1.3) is referred as the fundamental equation of an $\mathcal{ARS}$ $(M^n, g, \mathcal{F}, \rho)$. An $\mathcal{ARS}$ is expanding if $\rho > 0$, steady if $\rho = 0$, or shrinking if $\rho < 0$; otherwise, it is said to be indefinite. In case, $\mathcal{F}$ is gradient of a smooth function $-\psi$: $M^n \rightarrow \mathbb{R}$, the metric is named the gradient almost Ricci soliton ($\mathcal{GARS}$). In this case, (1.3) leads to

$$\mathcal{S} - \nabla^2\psi = -\rho g, \tag{1.4}$$

where, $\nabla^2\psi$ denotes the Hessian of ψ.

In 2005, Fischer [12] gave the idea of conformal Ricci flow by preserving the constant scalar curvature r of evolving metric and is presented by

$$\frac{\partial g}{\partial t} + 2\mathcal{S} = -(\mathcal{P} + \frac{2}{n})g, \ r = -1, \tag{1.5}$$

here, $\mathcal{P}$ is scalar non-dynamical field.

In [13], the authors proposed the idea of conformal Ricci soliton ($\mathcal{CRS}$). The equation of $\mathcal{CRS}$ is defined as:

$$\pounds_{\mathcal{F}} g + 2\mathcal{S} = -[2\rho - (\mathcal{P} + \frac{2}{n})]g. \tag{1.6}$$

The conformal Ricci flow equations are analogous to the Navier-Stokes equations in fluid mechanics and due to this analogy, $\mathcal{P}$ is named a conformal pressure, as for the real physical pressure it serves to maintain the incompressibility of the fluid. Equation (1.6) is the generalization of (1.2) and it also satisfies (1.5).

An advance class of geometric flows, namely, Ricci-Yambe flow of type (a, b) was proposed by the authors in [14], the solution of this flow is named Ricci-Yamabe soliton ($\mathcal{R}YS$) if it depends only on one parameter family of diffeomorphism and scaling and is defined as

$$\pounds_{\zeta} g + 2a\mathcal{S} + [2\rho - b\mathrm{R}]g = 0, \tag{1.7}$$

here a and b are scalars. A $\mathcal{R}YS$ is called a Yamabe soliton [2]; Ricci soliton [1]; σ-Einstein soliton [15]; or Einstein soliton [16] if $b = 1, a = 0$; $b = 0, a = 1$; $b = -2\sigma, a = 1$; or $b = -1, a = 1$, respectively.

Pseudo-Riemannian geometry is an extended case of Riemannian geometry. A Lorentzian manifold is an exclusive case of a pseudo-Riemannian manifold in which $(1, n-1)$ is the signature of metric. Spacetime is a 4-dimensional time-oriented Lorentzian manifold of signature $(-, +, +, +)$.

The formalization of Riemann's work appeared explicitly in 1913, the work of Weyl and the applications of these ideas were made to the theory of relativity in 1915 by Einstein, who used the idea of Riemannian manifolds to generate his theory of general relativity ($\mathcal{GTR}$).

$f(\mathtt{R})$-gravity generalizes $\mathcal{GTR}$. In fact, $f(\mathtt{R})$-gravity is a set of theories, each one is defined by a different function f of the Ricci scalar R. In 1970, $f(\mathtt{R})$-gravity was first introduced by Buchdahl [17]. It has now become a popular research field after Starobinsky on cosmic inflation. According to cosmic inflation theory, the early universe expanded exponentially fast for a fraction of a second after the Big Bang.

This paper is constructed in the following manner: Section 1 contains introduction, in which some useful concepts and their brief histories are given. Section 2 contains preliminaries, related to $f(\mathtt{R})$-gravity. Section 3 studies almost conformal $\mathcal{RS}$ in $\mathcal{PFST}$ under $f(\mathtt{R})$-gravity. Section 4 covers almost conformal gradient $\mathcal{RS}$ in $f(\mathtt{R})$-gravity. In Sections 5 and 6, we establish Poisson's equation through the almost conformal ω-$\mathcal{RS}$ and the almost conformal ω-Ricci-Yamabe solitons in $f(\mathtt{R})$-gravity. Examples are too added in Section 6. In the last Section 7, we have given some discussion on our study.

2. **Preliminaries**

In this section, we give 4-dimensional spacetime continuum satisfying $f(\mathtt{R})$-gravity [18,19]. We set

$$\mathcal{H} = \int \frac{1}{k^2}[\mathcal{L}_m + f(\mathtt{R})]\sqrt{-g}d^4x, \tag{2.1}$$

here, $\mathcal{H}$ and $\mathcal{L}_m$ denote modified Einstein-Hilbert action term and the scalar field's matter Lagrangian density, respectively. Also, $f(\mathtt{R})$ stands for the function of Ricci scalar, $k^2 = \frac{8\pi\mathcal{G}}{c^4}$, $\mathcal{G}$ is Newton's gravitational constant, c denotes the speed of light, g is determinant, and d^4x stands for the volume element.

The stress energy momentum tensor of matter is defined by

$$\mathcal{T}_{rs} = \frac{-2\delta(\sqrt{-g}\mathcal{L}_m)}{\sqrt{-g}\delta g^{rs}}. \tag{2.2}$$

The perfect fluid type $\mathcal{T}_{rs}$ for a unit time like vector ω_r is given by

$$\mathcal{T}_{rs} = pg_{rs} + (\gamma + p)\omega_r\omega_s, \tag{2.3}$$

where p is the isotropic pressure; g_{rs} is a metric tensor; r, s are constants; ω is a 1-form; and γ is the energy density. Assuming $\mathcal{L}_m$ depends on g_{rs} only, the field equations of $f(\mathtt{R})$-gravity after taking the variation of relation (2.1) w.r.t. g_{rs} is given by

$$\frac{\partial f(\mathtt{R})}{\partial \mathrm{R}}R_{rs} + g_{rs}\Box\frac{\partial f(\mathtt{R})}{\partial \mathrm{R}} = \frac{1}{2}f(\mathtt{R})g_{rs} + \nabla_r\nabla_s\frac{\partial f(\mathtt{R})}{\partial \mathrm{R}} + k^2\mathcal{T}_{rs}, \tag{2.4}$$

here $\mathtt{R}_{rs}$ denotes for the local components of $\mathcal{S}$ and $\Box \equiv \nabla_r\nabla^r$ indicates d'Alembert operator, ∇_r indicates the covariant derivative. The relation (2.4) can be weaken by changing $f(\mathtt{R})$ by R.
Choosing R = constant, relation (2.4) turns to

$$\mathrm{R}_{rs} - \frac{\mathrm{R}}{2}g_{rs} = \frac{k^2}{\frac{\partial f(\mathtt{R})}{\partial \mathrm{R}}}\mathcal{T}^{eff}_{rs}, \tag{2.5}$$

where,

$$\mathcal{T}_{rs}^{eff} = \mathcal{T}_{rs} + \frac{f(\mathtt{R}) - \mathtt{R}\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}{2k^2} g_{rs}.$$

The Ricci tensor in a perfect fluid spacetime ($\mathcal{PFST}$) satisfying $f(\mathtt{R})$-gravity is given by

$$\mathcal{S}(U,V) = \alpha\omega(U)\omega(V) + \beta g(U,V), \tag{2.6}$$

here, U, V are vector fields, $\alpha = \frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}$, and $\beta = \frac{f(\mathtt{R}) + 2k^2 p}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}$.

From (2.6), the Ricci operator Q is given by

$$QU = \alpha\omega(U)\zeta + \beta U, \tag{2.7}$$

where, 1-form ω is related to the velocity vector field ζ and $g(QU,V) = \mathcal{S}(U,V)$. Apart from the above, if p is the function of γ, then $\mathcal{PFST}$ is isentropic. Again, the $\mathcal{PFST}$ represents stiff matter era, when $p = \gamma$. The $\mathcal{PFST}$ is called the dust matter era, when $p = 0$. The $\mathcal{PFST}$ is dark matter era, when $p = -\gamma$. If $p = \frac{\gamma}{3}$, then it is radiation era [20].

3. Almost Conformal $\mathcal{RS}$ in $f(\mathtt{R})$-Gravity

In 1939, the concept of concircular vector field on $(\mathcal{M}^n, g)$ was given by Fialkow [21]. The concircular vector field φ is defined by the relation

$$\nabla_X \varphi = \nu X,$$

here, ∇: the Levi-Civita connection; ν: a non-trivial function on $(\mathcal{M}^n, g)$; and $X \in T\mathcal{M}$, $T\mathcal{M}$ is the tangent bundle of $(\mathcal{M}^n, g)$.

Theorem 3.1. *If $(g,\ \zeta,\ \rho)$ is an $\mathcal{ACRS}$ in a $\mathcal{PFST}$ obeying $f(\mathtt{R})$-gravity with a constant Ricci scalar. If $div\zeta = 0$, then $\mathcal{ACRS}$ is: expanding if* $p < \frac{(1+2\mathcal{P})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}} - 2f(\mathtt{R})}{4k^2}$, *shrinking if* $p > \frac{(1+2\mathcal{P})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}} - 2f(\mathtt{R})}{4k^2}$ *and steady if* $p = \frac{(1+2\mathcal{P})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}} - 2f(\mathtt{R})}{4k^2}$. *Apart from this, the $\mathcal{PFST}$ shows a dark matter era.*

Proof. We consider the velocity vector field ζ, equal to the potential vector field $\mathcal{F}$, therefore, $\mathcal{ACRS}$ is defined as

$$(\pounds_\zeta g)(U,V) + 2\mathcal{S}(U,V) + [2\rho - (\mathcal{P} + \frac{1}{2})]g(U,V) = 0, \tag{3.1}$$

where ρ is a real valued smooth function.

In view of explicit form of the Lie-derivative, the above relation takes the form

$$g(\nabla_U \zeta, V) + g(U, \nabla_V \zeta) + 2\mathcal{S}(U,V) + [2\rho - (\mathcal{P} + \frac{1}{2})]g(U,V) = 0. \tag{3.2}$$

Putting the value of $\mathcal{S}(U,V)$ from (2.6) into (3.2), we have

$$g(\nabla_U\zeta,V)+g(U,\nabla_V\zeta)+2\alpha\omega(U)\omega(V)+2\beta g(U,V)+[2\rho-(\mathcal{P}+\frac{1}{2})]g(U,V)=0. \tag{3.3}$$

Contracting the above relation w.r.t. U and V, we have

$$\begin{aligned}\sum\epsilon_i g(\nabla_{e_i}\zeta,e_i) \quad + \quad & \sum\epsilon_i g(e_i,\nabla_{e_i}\zeta)+2\alpha\sum\epsilon_i\omega(e_i)\omega(e_i)\\ & +2\beta\sum\epsilon_i g(e_i,e_i)+[2\rho-(\mathcal{P}+\frac{1}{2})]\sum\epsilon_i g(e_i,e_i)=0.\end{aligned}$$

On simplification, the above relation reduces to

$$2div\zeta-2\alpha+8\beta+4[2\rho-(\mathcal{P}+\frac{1}{2})]=0. \tag{3.4}$$

Setting $U=V=\zeta$ in (3.3) and using $\nabla_\zeta\zeta=0$, we obtain

$$2\alpha-2\beta-[2\rho-(\mathcal{P}+\frac{1}{2})]=0. \tag{3.5}$$

Adding equations (3.4) and (3.5), we have

$$2div\zeta+6\beta+3[2\rho-(\mathcal{P}+\frac{1}{2})]=0. \tag{3.6}$$

Since, $div\zeta=0$, then (3.6) leads to

$$\rho=\frac{\mathcal{P}}{2}+\frac{1}{4}-\beta. \tag{3.7}$$

Inserting the value of β from (2.6) into (3.7), we have

$$\rho=\frac{\mathcal{P}}{2}+\frac{1}{4}-\frac{f(\mathrm{R})+2k^2p}{2\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}}. \tag{3.8}$$

From the relation (3.8), we conclude that $\mathcal{A}CRS$ is: expanding if $p<\frac{(1+2\mathcal{P})\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}-2f(\mathrm{R})}{4k^2}$, steady if $p=\frac{(1+2\mathcal{P})\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}-2f(\mathrm{R})}{4k^2}$, and shrinking if $p>\frac{(1+2\mathcal{P})\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}-2f(\mathrm{R})}{4k^2}$.
Now, (3.7), together with (3.5) gives $\alpha=0$. The relation $\alpha=0$ implies that $p=-\gamma$, i.e., $\mathcal{PFST}$ is dark matter era. □

Theorem 3.2. *Let $(g,\mathcal{F},\rho)$ be an $\mathcal{A}CRS$ in $\mathcal{PFST}$ under $f(\mathrm{R})$-gravity. If the potential vector field is concircular, equal to the velocity vector field, then the $\mathcal{A}CRS$ is: steady if $p=\frac{(\mathcal{P}+\frac{1}{2}-2\nu)\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}-f(\mathrm{R})}{2k^2}$, expanding if $p<\frac{(\mathcal{P}+\frac{1}{2}-2\nu)\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}-f(\mathrm{R})}{2k^2}$, or shrinking if $p>\frac{(\mathcal{P}+\frac{1}{2}-2\nu)\frac{\partial f(\mathrm{R})}{\partial\mathrm{R}}-f(\mathrm{R})}{2k^2}$.*

Proof. We consider the potential vector field $\mathcal{F}$, equal to the velocity vector field ζ. Therefore, $\mathcal{A}CRS$ in $\mathcal{PFST}$ is given by

$$(\pounds_\zeta g)(U,V)+2\mathcal{S}(U,V)+[2\rho-(\mathcal{P}+\frac{1}{2})]g(U,V)=0, \tag{3.9}$$

which can be written as

$$g(\nabla_U\zeta, V) + g(U, \nabla_V\zeta) + 2\mathcal{S}(U, V) + [2\rho - (\mathcal{P} + \frac{1}{2})]g(U, V) = 0.$$

As, ζ is a concircular vector field, the above relation gives

$$\mathcal{S}(U, V) = [\frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \rho - \nu]g(U, V). \tag{3.10}$$

Now, from $f(\mathrm{R})$-gravity, we have the relation

$$\mathcal{S}(U, V) = \alpha\omega(U)\omega(V) + \beta g(U, V). \tag{3.11}$$

Comparing equations (3.10) and (3.11), we have

$$\alpha\omega(U)\omega(V) + \beta g(U, V) = [\frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \rho - \nu]g(U, V). \tag{3.12}$$

Contracting the relation (3.12) w.r.t. U and V, we have

$$-\alpha + 4\beta = 4[\frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \rho - \nu]. \tag{3.13}$$

Putting $U = V = \zeta$, in the relation (3.12), it yields

$$\alpha - \beta = -[\frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \rho - \nu]. \tag{3.14}$$

Solving (3.13) and (3.14) for ρ, we have

$$\rho = \frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \frac{f(\mathrm{R}) + 2k^2 p}{2\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}} - \nu. \tag{3.15}$$

Thus, the soliton is steady: if $p = \frac{(\mathcal{P} + \frac{1}{2} - 2\nu)\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}} - f(\mathrm{R})}{2k^2}$, expanding if $p < \frac{(\mathcal{P} + \frac{1}{2} - 2\nu)\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}} - f(\mathrm{R})}{2k^2}$, or shrinking if $p > \frac{(\mathcal{P} + \frac{1}{2} - 2\nu)\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}} - f(\mathrm{R})}{2k^2}$. □

Now, $P + \frac{1}{2} = 0$ gives the subsequent corollary:

Corollary 3.1. *Let $(g, \mathcal{F}, \rho)$ be an $\mathcal{RS}$ in $\mathcal{PFST}$ under $f(\mathrm{R})$-gravity. If the potential vector field is concircular, equal to the velocity vector field, then the $\mathcal{RS}$ is: expanding if $p < -\frac{2\nu\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}} + f(\mathrm{R})}{2k^2}$, steady if $p = -\frac{2\nu\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}} + f(\mathrm{R})}{2k^2}$, or shrinking if $p > -\frac{2\nu\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}} + f(\mathrm{R})}{2k^2}$.*

4. Almost Conformal Gradient $\mathcal{RS}$ ($\mathcal{A}CGRS$) in $f(\mathrm{R})$-Gravity

Theorem 4.1. *For a constant Ricci scalar, we assume that $\mathcal{PFST}$ satisfies $f(\mathrm{R})$-gravity and admits an $\mathcal{A}CGRS$. If the potential vector field, equal to the velocity vector field with $div\zeta = 0$ and $\zeta(-\beta - \rho + \frac{1}{2}\mathcal{P}) = 0$, then either energy density is constant; or the soliton is; steady if $\gamma = \frac{f(\mathrm{R}) - \frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}(\mathcal{P} + \frac{1}{2})}{2k^2}$, expanding if $\gamma > \frac{f(\mathrm{R}) - \frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}(\mathcal{P} + \frac{1}{2})}{2k^2}$, or shrinking if $\gamma < \frac{f(\mathrm{R}) - \frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}(\mathcal{P} + \frac{1}{2})}{2k^2}$.*

Proof. Let $\zeta = -D\psi$, then (3.9) becomes

$$g(\nabla_U D\psi, V) - \mathcal{S}(U,V) - [\rho - \frac{1}{2}(\mathcal{P} + \frac{1}{2})]g(U,V) = 0.$$

The above relation gives,

$$\nabla_U D\psi = QU + [\rho - \frac{1}{2}(\mathcal{P} + \frac{1}{2})]U, \tag{4.1}$$

for every U.
Since,

$$K(U,V)D\psi = [\nabla_U, \nabla_V]D\psi - \nabla_{[U,V]}D\psi,$$

where, K is Riemann curvature tensor.
Thus, in view of (4.1), we have

$$K(U,V)D\psi = (\nabla_U Q)V + [\nabla_U \rho - \frac{1}{2}(\nabla_U \mathcal{P})]V - (\nabla_V Q)U - [\nabla_V \rho - \frac{1}{2}(\nabla_V \mathcal{P})]U. \tag{4.2}$$

Since, from $f(R)$-gravity, we have

$$\mathcal{S}(U,V) = \alpha\omega(U)\omega(V) + \beta g(U,V). \tag{4.3}$$

From the above relation (4.3), it follows that

$$QV = \alpha\omega(V)\zeta + \beta V.$$

Differentiating covariantly above relation w.r.t. U, it gives

$$(\nabla_U Q)V = U(\alpha)\omega(V)\zeta + \alpha[(\nabla_U \omega)(V)\zeta + \omega(V)\nabla_U \zeta] + U(\beta)V.$$

Applying $U \leftrightarrow V$ in the above equation, we have

$$(\nabla_V Q)U = V(\alpha)\omega(U)\zeta + \alpha[(\nabla_V \omega)(U)\zeta + \omega(U)\nabla_V \zeta] + V(\beta)U.$$

Now, from the above last two relations, (4.2) gives

$$\begin{aligned} K(U,V)D\psi \; = \; & U(\alpha)\omega(V)\zeta + \alpha[(\nabla_U \omega)(V)\zeta + \omega(V)\nabla_U \zeta] \\ & + U(\beta)V + [\nabla_U \rho - \frac{1}{2}(\nabla_U \mathcal{P})]V - V(\alpha)\omega(U)\zeta \\ & - \alpha[(\nabla_V \omega)(U)\zeta + \omega(U)\nabla_V \zeta] - V(\beta)U - [\nabla_V \rho - \frac{1}{2}(\nabla_V \mathcal{P})]U. \end{aligned} \tag{4.4}$$

Contracting the above relation w.r.t. U

$$\begin{aligned}\sum \epsilon_i g(K(e_i,V)D\psi,e_i) &= \sum \epsilon_i g(D\beta,e_i)g(V,e_i)+\sum \epsilon_i g(D\alpha,e_i)g(\zeta,e_i)\omega(V)\\ &+\alpha[\sum \epsilon_i(\nabla_{e_i}\omega)(V)g(\zeta,e_i)+\omega(V)\sum \epsilon_i g(\nabla_{e_i}\zeta,e_i)]\\ &+\sum \epsilon_i g(D\rho,e_i)g(V,e_i)-\frac{1}{2}\sum \epsilon_i g(D\mathcal{P},e_i)g(V,e_i)\\ &-V(\beta)\sum \epsilon_i g(e_i,e_i)-V(\alpha)\sum \epsilon_i g(e_i,\zeta)g(e_i,\zeta)\\ &-\alpha[\sum \epsilon_i(\nabla_V\omega)(e_i)g(\zeta,e_i)+\sum \epsilon_i g(e_i,\zeta)g(\nabla_V\zeta,e_i)]\\ &-[\nabla_V\rho-\frac{1}{2}(\nabla_V\mathcal{P})]\sum \epsilon_i g(e_i,e_i).\end{aligned}$$

After simplification, the above relation gives

$$\mathcal{S}(V,D\psi)=-3V(\beta)+V(\alpha)+\zeta(\alpha)\omega(V)+\alpha(\nabla_\zeta\omega)(V)+\alpha\omega(V)div\zeta-3V(\rho)+\frac{3}{2}V(\mathcal{P}). \quad (4.5)$$

Replacing $U\to V$ and $V\to D\psi$ in equation (4.3), we have

$$\mathcal{S}(V,D\psi)=\alpha\omega(V)\omega(D\psi)+\beta g(V,D\psi). \quad (4.6)$$

Comparing equations (4.5) and (4.6), we get

$$\begin{aligned}\alpha\omega(V)\omega(D\psi)+\beta g(V,D\psi) &= -3V(\beta)+V(\alpha)+\zeta(\alpha)\omega(V)\\ &+\alpha(\nabla_\zeta\omega)(V)+\alpha\omega(V)div\zeta-3V(\rho)+\frac{3}{2}V(\mathcal{P}),\end{aligned}$$

which by taking $V=\zeta$, we lead to

$$(\beta-\alpha)\zeta(\psi)=\zeta(-3\beta-3\rho+\frac{3}{2}\mathcal{P})-\alpha div\zeta.$$

If, $\zeta(-\beta-\rho+\frac{1}{2}\mathcal{P})=0$ and $div\zeta=0$, then

$$(\beta-\alpha)\zeta(\psi)=0.$$

The above relation implies that either $\beta=\alpha$, or, $\zeta(\psi)=0$. Now, we have

Case I: If $\beta=\alpha$ = constant, then $\gamma=\dfrac{f(\mathrm{R})}{2k^2}$ = constant.

Case II: If $\beta\neq\alpha$, so $\zeta(\psi)=g(D\psi,\zeta)=0$.

Differentiating the relation $g(D\psi,\zeta)=0$ covariantly w.r.t. U, we have

$$(\nabla_U g)(D\psi,\zeta)+g(\nabla_U D\psi,\zeta)+g(D\psi,\nabla_U\zeta)=0.$$

Using $(\nabla_U g)(D\psi,\zeta)=0$, the above equation reduces to

$$g(\nabla_U\zeta,D\psi)=-g(\nabla_U D\psi,\zeta).$$

Relation (4.1), together with above relation, provides

$$g(\nabla_U\zeta,D\psi)=[(\alpha-\beta)+\frac{1}{2}(\mathcal{P}+\frac{1}{2})-\rho]\omega(U).$$

Replacing $U \to \zeta$ in the above relation, we have

$$g(\nabla_\zeta \zeta, D\psi) = [(\alpha - \beta) + \frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \rho]\omega(\zeta).$$

The equation (2.6) and above relation, taken together, we obtain

$$\rho = \frac{2k^2\gamma - f(\mathrm{R}) + \frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}(\mathcal{P} + \frac{1}{2})}{2\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}}.$$

This completes the proof. □

We know [22], the energy equation of the perfect fluid for the velocity vector field $\mathcal{F}$ is given as

$$\mathcal{F}(\gamma) = -(p + \gamma) div\mathcal{F}. \tag{4.7}$$

Hence, from case I, $\gamma = constant$. Therefore, either $p + \gamma = 0$ or $div\mathcal{F} = 0$.
This leads to the following corollary:

Corollary 4.1. *Let $\mathcal{PFST}$ obeying the $f(\mathrm{R})$-gravity and admit an AGC$\mathcal{RS}$ with the constant Ricci scalar such that $\zeta\psi = 0$. If potential vector field, equal to velocity vector field with div$\zeta = 0$ and $\zeta(-\beta - \rho + \frac{1}{2}\mathcal{P}) = 0$, then either perfect fluid has vanishing expansion scalar, or the spacetime is dark matter era.*

5. Almost Conformal ω-$\mathcal{RS}$ ($AC\omega\mathcal{RS}$) in $f(\mathrm{R})$-Gravity

An $AC\omega\mathcal{RS}$ is the generalization of $AC\mathcal{RS}$ and is defined by [23]

$$\pounds_{\mathcal{F}} g + 2\mathcal{S} + 2[\rho - \frac{1}{2}(\mathcal{P} + \frac{2}{n})]g + 2\mu\omega \otimes \omega = 0.$$

where ρ and μ are smooth functions. Please also see [24–26]

Theorem 5.1. *Let the $\mathcal{PFST}$ obeying $f(\mathrm{R})$-gravity with the constant Ricci scalar R and admit an $AC\omega\mathcal{RS}$ (g, ζ, ρ, μ). If the velocity vector ζ is equivalent to the potential vector field $\mathcal{F}$ and ω is dual of gradient ζ, then the Poisson equation satisfying by ψ is*

$$\Delta\psi = \frac{3}{2}[(\mathcal{P} - 2\rho + \frac{1}{2}) - \frac{2k^2 p + f(\mathrm{R})}{\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}}].$$

Moreover, if div ζ=0, then the soliton functions ρ and μ are given by

$$\rho = \frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \frac{f(\mathrm{R}) + 2k^2 p}{2\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}},$$

and

$$\mu = -\frac{k^2(p + \gamma)}{\frac{\partial f(\mathrm{R})}{\partial \mathrm{R}}},$$

respectively.

Proof. We consider the potential vector field $\mathcal{F}$, equivalent to the velocity vector field ζ. Therefore, an $AC\omega\mathcal{RS}$ in $\mathcal{PFST}$ is given by

$$\frac{1}{2}(\pounds_{\zeta}g)(U,V)+\mathcal{S}(U,V)+[\rho-\frac{1}{2}(\mathcal{P}+\frac{1}{2})]g(U,V)+\mu\omega(U)\omega(V)=0. \tag{5.1}$$

The above relation yields

$$\mathcal{S}(U,V)=-\frac{1}{2}[g(\nabla_U\zeta,V)+g(U,\nabla_V\zeta)]-[\rho-\frac{1}{2}(\mathcal{P}+\frac{1}{2})]g(U,V)-\mu\omega(U)\omega(V). \tag{5.2}$$

From $f(\mathtt{R})$-gravity,

$$\mathcal{S}(U,V)=\alpha\omega(U)\omega(V)+\beta g(U,V). \tag{5.3}$$

Comparing equations (5.2) and (5.3), it gives

$$\begin{aligned}\alpha\omega(U)\omega(V)+\beta g(U,V) &= -\frac{1}{2}[g(\nabla_U\zeta,V)+g(U,\nabla_V\zeta)]\\ &\quad -[\rho-\frac{1}{2}(\mathcal{P}+\frac{1}{2})]g(U,V)-\mu\omega(U)\omega(V).\end{aligned} \tag{5.4}$$

Assuming, $U=V=\zeta$, using $\omega(\zeta)=g(\zeta,\zeta)=-1$, the above relation yields

$$\alpha-\beta=[\rho-\frac{1}{2}(\mathcal{P}+\frac{1}{2})]-\mu. \tag{5.5}$$

Contracting over U and V, the relation (5.4) provides,

$$-\alpha+4\beta=-div\zeta-4[\rho-\frac{1}{2}(\mathcal{P}+\frac{1}{2})]+\mu. \tag{5.6}$$

Adding two preceding equations, we get

$$\rho=\frac{1}{2}(\mathcal{P}+\frac{1}{2})-\beta-\frac{1}{3}div\zeta. \tag{5.7}$$

Now, using $div\zeta$ = div (grad ψ) and $\beta=\frac{f(\mathtt{R})+2k^2p}{2\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}$, the above relation gives

$$\Delta\psi=\frac{3}{2}[(\mathcal{P}-2\rho+\frac{1}{2})-\frac{2k^2p+f(\mathtt{R})}{\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}]. \tag{5.8}$$

For $div\zeta=0$, the relations (5.5) and (5.7) provides, $\rho=\frac{1}{2}(\mathcal{P}+\frac{1}{2})-\frac{2k^2p+f(\mathtt{R})}{2\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}$ and $\mu=-\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}$, where $\alpha=\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}$ and $\beta=\frac{2k^2p+f(\mathtt{R})}{2\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}$ being used. This completes the proof. □

For $\mu=0$, an $AC\omega\mathcal{RS}$ reduces an $AC\mathcal{RS}$. In this case, from the relation $\mu=-\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial\mathtt{R}}}$, it follows that $p=-\gamma$. Thus, we state :

Corollary 5.1. *Let the $\mathcal{PFST}$ with $div\zeta=0$ obeying $f(\mathtt{R})$-gravity with the constant Ricci scalar $\mathtt{R}$ and admit an $AC\mathcal{RS}$. Then, the $\mathcal{PFST}$ is dark matter era.*

6. Almost Conformal ω-Ricci-Yamabe Solitons ($AC\omega\mathcal{RYS}$) in $f(\mathtt{R})$-Gravity

Theorem 6.1. *Let the $\mathcal{PFST}$ obeying $f(\mathtt{R})$-gravity with the constant Ricci scalar $\mathtt{R}$ and admit an $AC\omega\mathcal{RYS}$ (g,ζ,a,b,ρ,μ). If the velocity vector field ζ is equivalent to the potential vector field $\mathcal{F}$ and ω is the g-dual of the gradient vector field ζ = grad ψ, then the Poisson equation satisfying by ψ is*

$$\Delta\psi = -3[\mu + a\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}].$$

Proof. As a generalization of (1.7), we define an $AC\omega\mathcal{RYS}$ by the equation

$$\pounds_\zeta g + 2a\mathcal{S} + [2\rho - b\mathtt{R} - (\mathcal{P} + \frac{2}{n})]g + 2\mu\omega\otimes\omega = 0, \tag{6.1}$$

here a and b are scalars. Please also see [27–30].

For $n = 4$, equation (6.1) becomes

$$g(\nabla_U\zeta, V) + g(U, \nabla_V\zeta) + 2a\mathcal{S}(U,V) + [2\rho - b\mathtt{R} - (\mathcal{P} + \frac{1}{2})]g(U,V) + 2\mu\omega(U)\omega(V) = 0. \tag{6.2}$$

The relation (2.6), together with above relation, gives

$$g(\nabla_U\zeta, V) + g(U, \nabla_V\zeta) + [2a\beta + 2\rho - b\mathtt{R} - (\mathcal{P} + \frac{1}{2})]g(U,V) + 2(a\alpha + \mu)\omega(U)\omega(V) = 0. \tag{6.3}$$

Using $U = V = \zeta$, then the above equation reduces to

$$2a[\alpha - \beta] - [2\rho - b\mathtt{R} - (\mathcal{P} + \frac{1}{2})] + 2\mu = 0. \tag{6.4}$$

Contracting equation (6.3) over U and V, we have

$$2div\zeta + 2a[-\alpha + 4\beta] + 4[2\rho - b\mathtt{R} - (p + \frac{1}{2})] - 2\mu = 0. \tag{6.5}$$

Adding the relations (6.4) and (6.5), we obtain

$$\rho = \frac{1}{2}(\mathcal{P} + \frac{1}{2}) - \frac{1}{3}div\zeta + \frac{(3b-2a)k^2p + (2b-a)f(\mathtt{R}) - b\gamma}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}. \tag{6.6}$$

Putting the value of ρ in (6.4), we have

$$\mu = -\frac{1}{3}div\zeta - a\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}. \tag{6.7}$$

Since, $\Delta\psi = div(grad\psi)$, therefore, equation (6.8) gives

$$\Delta\psi = -3[\mu + a\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}]. \tag{6.8}$$

This completes the proof. □

For the smooth functions Ψ and θ in $f(\mathtt{R})$-gravity, the almost conformal ω-Ricci Yamabe soliton satisfies Poisson's equation if $\Psi = \theta$ holds. In case $\Psi = 0$, Poisson's equation transforms into Laplace's equation.

For $\mu = 0$, an almost conformal ω-Ricci Yamabe soliton $(g, \zeta, a, b, \rho, \mu)$ reduces to the almost conformal Ricci Yamabe soliton (g, ζ, a, b, ρ).

Now, for $\mu = 0$ and $a = 0$, (6.8) reduces to $\Delta\psi = 0$. Thus, we state:

Corollary 6.1. *The $\mathcal{PFST}$ obeying $f(\mathtt{R})$-gravity with the constant Ricci scalar $\mathtt{R}$ and admitting an almost conformal Yamabe solitons satisfies Laplace's equation.*

Next, for $\mu = 0$ and $p = -\gamma$, (6.8) reduces to $\Delta\psi = 0$. Thus, we state:

Corollary 6.2. *Let the $\mathcal{PFST}$ obeying $f(\mathtt{R})$-gravity with the constant Ricci scalar $\mathtt{R}$ and admitting an almost conformal Yamabe soliton. If $\mathcal{PFST}$ is the dark matter era, then the almost conformal Yamabe soliton satisfies Laplace's equation.*

If $div\zeta = 0$, then from (6.6) and (6.8), we have

$$\rho = \frac{1}{2}(\mathcal{P} + \frac{1}{2}) + \frac{(3b-2a)k^2 p + (2b-a)f(\mathtt{R}) - b\gamma}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}, \tag{6.9}$$

and

$$\mu = -a\frac{k^2(p+\gamma)}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}. \tag{6.10}$$

If, $a = 0$ and $b = 1$, then the relation (6.9) gives

$$\rho = \frac{1}{2}(\mathcal{P} + \frac{1}{2}) + \frac{3k^2 p + 2f(\mathtt{R}) - \gamma}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}. \tag{6.11}$$

Thus, we state:

Corollary 6.3. *If (g, ζ, ρ) is an almost conformal Yamabe soliton in a $\mathcal{PFST}$ obeying $f(\mathtt{R})$-gravity with constant $\mathtt{R}$, then the almost conformal Yamabe soliton is expanding if $\gamma > \frac{2f(\mathtt{R}) + (\mathcal{P} + \frac{1}{2})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}{3k^2+1}$; steady if $\gamma = \frac{2f(\mathtt{R}) + (\mathcal{P} + \frac{1}{2})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}{3k^2+1}$; or shrinking if $\gamma < \frac{2f(\mathtt{R}) + (\mathcal{P} + \frac{1}{2})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}{3k^2+1}$.*

If, $a = -1$ and $b = 1$, then the relation (6.9) gives

$$\rho = \frac{1}{2}(\mathcal{P} + \frac{1}{2}) + \frac{3f(\mathtt{R}) - (5k^2+1)\gamma}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}. \tag{6.12}$$

This leads to the following corollary:

Corollary 6.4. *Let (g, ζ, ρ) be almost conformal Yamabe solitons in a $\mathcal{PFST}$ obeying $f(\mathtt{R})$-gravity with constant $\mathtt{R}$, then the Einstein soliton is expanding if* $\gamma > \dfrac{(\mathcal{P}+\frac{1}{2})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}} + 3f(\mathtt{R})}{5k^2+1}$*; steady if* $\gamma = \dfrac{(\mathcal{P}+\frac{1}{2})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}} + 3f(\mathtt{R})}{5k^2+1}$*; or shrinking if* $\gamma < \dfrac{(\mathcal{P}+\frac{1}{2})\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}} + 3f(\mathtt{R})}{5k^2+1}$.

Example 6.1. *$\mathcal{PFST}$ is said to be radiation era, if* $p = \frac{\gamma}{3}$*. For radiation era, from equation* (6.6) *and* (6.7) *we have,* $\rho = \frac{1}{2}(\mathcal{P}+\frac{1}{2}) - \frac{1}{3}div\zeta + \dfrac{(2b-a)f(\mathtt{R}) - 2ak^2p}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}$*, and* $\mu = -[\frac{1}{3}div\zeta + a\dfrac{(k^2+3p)}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}]$.

Example 6.2. *$\mathcal{PFST}$ is said to be stiff matter era, if* $p = \gamma$*. For stiff matter era, from equation* (6.6) *and* (6.7)*, we have,* $\rho = \frac{1}{2}(\mathcal{P}+\frac{1}{2}) - \frac{1}{3}div\zeta + \dfrac{2(b-ak^2)p + (2b-a)f(\mathtt{R})}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}$*, and* $\mu = -[\frac{1}{3}div\zeta + 2a\dfrac{k^2p}{\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}]$.

Example 6.3. *$\mathcal{PFST}$ is said to be dark matter era, if* $p = -\gamma$*. For dark matter era, from equation* (6.6) *and* (6.7)*, we have,* $\rho = \frac{1}{2}(\mathcal{P}+\frac{1}{2}) - \frac{1}{3}div\zeta + \dfrac{2(ak^2-b)p + (2b-a)f(\mathtt{R})}{2\frac{\partial f(\mathtt{R})}{\partial \mathtt{R}}}$*, and* $\mu = -\frac{1}{3}div\zeta$.

7. **Discussion**

In the present paper, various metrics such as almost conformal $\mathcal{RS}$, almost conformal gradient $\mathcal{RS}$, almost conformal ω-Ricci-Yamabe solitons and Poisson's equation, obtained through almost conformal ω-Ricci solitons and almost conformal ω-Ricci-Yamabe solitons are discussed under $f(\mathcal{R})$-gravity. In this paper, we take $\mathcal{PFST}$ admitting $\mathcal{RS}$ with constant Ricci scalar satisfying $f(\mathcal{R})$-gravity and the potential vector field $\mathcal{F}$, equal to velocity vector field ξ and also observing that the spacetime represents a dark matter era under suitable condition on the vector field $\mathcal{F}$. It is also here noticed that under the same restriction, if the spacetime admits a gradient $\mathcal{RS}$, then either the spacetime represents a dark matter era, or the perfect fluid is vanishing expansion under suitable restriction. $\mathcal{RS}$ in $\mathcal{PFST}$ with the potential vector field as a concircular vector field, equal to the velocity vector field under $f(\mathcal{R})$-gravity, then the solitons in $\mathcal{PFST}$ shows dark matter era. For a constant Ricci scalar, $\mathcal{PFST}$ satisfying $f(\mathcal{R})$-gravity permits a gradient conformal $\mathcal{RS}$, then the $\mathcal{PFST}$ represents dark matter era. In an almost ω-Ricci solitons in $f(\mathcal{R})$-gravity, solitons reduces to almost conformal $\mathcal{RS}$. $\mathcal{PFST}$ obeying $f(\mathcal{R})$-gravity with the constant Ricci scalar admitting an almost conformal ω-Ricci-Yamabe solitons establishes Poisson's equation. Poisson's equation is a partial differential equation generally applicable in several fields like computer science, theoretical physics, electrostatics, mechanical engineering, chemistry, astronomy, and other fields.

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