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Interconnections between μ-Value and D-Stable, $D(\alpha)$-Stable Matrices from Economic Models

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Abstract. In this paper, we review a number of well established methods to study the interconnection between D-stability and μ-values. The D-stability in economic, and dynamic systems plays a crucial role for maintaining equilibrium under proportional changes in parameters, for instance, prices, production levels, or financial flows. The computation of structured singular value a.k.a μ-value is a well-known mathematical tool for analysis of systems appearing in robust control. The μ-value provides the quantitative measure of linear systems stability subject to structured uncertainties. The approximation of an upper bounds of μ-value plays a critical role for ensuring robust stability and performance which guarantees in practical linear control systems. This article also presents the state-of-the-art mathematical methods for approximating upper bounds of μ-values. The μ-value is deeply interconnected with D-stability theory of economic models. The key methods includes the computation of upper bounds of μ-values for mixed real and complex uncertainties, optimization based methods, linear matrix inequalities (LMI)-based techniques.

1. Introduction

In designing and mathematical analysis of linear control systems, the robustness and performance plays an important role and are among the fundamental requirements. Furthermore, these must operate in a reliable manner in the presence of structured or unstructured uncertainties. The structured singular value, known as μ-value, establishes most powerful mathematical framework

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to deal with the difficulties by quantifying the robustness, performance of a linear control system subject to structured uncertainties.

The structured singular value was introduced by [5,6] in order to study and analyze the robustness, performance, and structured uncertainties in the linear models. The mathematical framework for the μ-value is developed in such a way that it must characterize the robust stability and performance in sense of H_∞. The most of the research on computation of *mu*-value problem involves only the complex uncertainties which represent uncertainties due to unmodeled dynamics. The complex uncertainties arises for the mathematical problems of robust performance. Furthermore, almost all of practical applications of μ involve at least one complex uncertainty.

The exact computation of μ-value is an NP-hard problem, see [3]. The well-known example is Kharitonov's result for the polynomials having coefficients in the intervals, see [20]. The computations against most general cases of uncertainties are very expensive, see [1,12].

In [8], an upper bound of μ-value is approximated for the case of mixed real parametric, and complex parametric uncertainties. The results for an upper bound are obtained by using the local search methods. But, unfortunately, these local search methods are costly to implement, and hence fail to determine the global solutions. The local methods yields global solutions, and as a result one might get the tighter bounds.

In [5], an improved upper bound was obtained, but unfortunately no practical way for its computation was presented. It was shown that the maximization to $\rho_{\mathbb{R}}$ is exactly equal to the computation of μ_{Δ} at its global maximum, but not the local maximum.

At Honeywell's Systems and Research Center, J. Wall began using the generalization of Osborne's routine [7] to approximate an upper bounds to μ-values. He was able to find the local maximum for the lower bounds of μ-values by using the gradient method. The Osborne's algorithm is to minimize the Frobenius norm instead of the minimization of the maximum singular value, and then right scaling yields an approximation to an upper bounds. A less general approximation to an upper bounds based upon Perron eigenvectors was suggested by Safonov [14].

The computational issue related with the computation of μ has been studied by very many authors, see [18], and references therein. A Matlab Toolbox is also available to determine the bounds of μ-values, see [10]. In [3], a best known result for the approximation to an upper bound of μ-value was proposed by Fan, Tits, and Doyle [15]. The results for an upper bounds were generalized from an earlier results to an upper bound of μ-value by Doyle [6], and this upper bound in convex.

2. Interconnections between D-stability and μ-values

In this section, we review a number of well established methods to study the interconnections between D-stability and μ-values. Furthermore, we present the methods which study and analyze the stability, D-stability, strong D-stability for linear models from economics.

2.1. D**-stability, Strong** D**-stability and** μ**-values:** In [19], some new and interesting theoretical results were presented on interconnections between D-stability, and μ-values. Furthermore, the interconnections between strong D-stability, and μ-value were analyzed and presented. The following theorem [19] show that $M \in \mathbb{R}^{n,n}$ is stable matrix, and $0 \leq \mu_{\mathbb{B}}(\frac{1}{M^2}) < 1$.

Theorem 2.1. *Let $M \in \mathbb{R}^{n,n}$. Then, M is D-stable if and only if M is stable, and $0 \leq \mu_{\mathbb{B}}(\frac{1}{M^2}) < 1$.*

The following theorem [19], gives the results on the interconnections between D-stability and μ-values.

Theorem 2.2. *Let $M \in \mathbb{C}^{n,n}$. Then, M is D-stable if and only if $Re(\lambda_k(PM + M^*P)) > 0$, $\forall\ k = 1 : n$, $\forall\ P \in \Omega$, and $0 \leq \mu_{\mathbb{B}}(A) < 1$, where $A = (iI_n + PM + M^*P)^{-1}(iI_n - PM - M^*P)$, Ω is the set of positive diagonal matrices.*

The following theorem [19] results in the interconnections between strong D-stability and μ-values.

Theorem 2.3. *Let $M \in \mathbb{C}^{n,n}$. For a Hermitian matrix $A \in \mathbb{C}^{n,n}$, $M = e^A$. Then M is strongly D-stable if M is stable and for $\gamma > 0$, the matrix $M + G$ is D-stable, with*

$$G = ((M \otimes A)^*\Delta + \Delta(M \otimes A)),\ \Delta \in \mathbb{B},\ \|G\| < \gamma.$$

Following theorem [19] gives the interconnection between μ-values and strong D-stability when the given matrix can be decomposed in term of a skew-symmetric matrix.

Theorem 2.4. *Let $M \in \mathbb{R}^{n,n}$. Then $M = S + A$, a strongly D-stable matrix if $S^T = -S$, and for $z \neq 0$, $z^TAy < 0$, and*

$$0 \leq \mu_{\mathbb{B}}(iI_n + S + A)^{-1}(iI_n - S - A) < 1.$$

Theorem 2.5. *Let $M \in \mathbb{R}^{n,n}$. Then $M = S + A$, $S^T = -S$, and for $z \neq 0$, $z^TAy < 0$ if and only if $0 \leq \mu_{\mathbb{B}}(\hat{A})$ with*

$$\hat{A} = \begin{pmatrix} 0 & S + A \\ I_n & 0 \end{pmatrix}.$$

2.2. **Interconnections between** H**-stable,** $D(\alpha)$**-stable matrices and** μ**-values:** The new theoretical results on the interconnection between H-stable matrices, and μ-values for a family of squared real or complex valued matrices were analyzed and presented in [16].

Following Definition 2.1 is taken from [2].

Definition 2.1. *The n-dimensional real-valued matrix $A \in \mathbb{R}^{n\times n}$ is known as (multiplicative) H-stable if HA is a stable matrix for each symmetric positive-definite matrix H.*

Theorem 2.6 shows that $A \in \mathbb{C}^{n\times n}$ is H-stable matrix whenever $H > 0$, a positive definite matrix.

Theorem 2.6. *Consider that $A, H \in \mathbb{C}^{n\times n}, H > 0$, a positive definite matrix. If A is H-stable matrix for each $H > 0$, then $H > 0$, a positive definite matrix whenever A is a H-stable matrix.*

The following Theorem 2.7 gives an interaction between H-stability and μ-values.

Theorem 2.7. *Let $A \in \mathbb{R}^{n\times n}$. Let A be a H-stable matrix. Then for each $H \in H^+, 0 \leq \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$, with*

$$H^+ = \{H : \lambda_i(H) > 0, \forall i = 1 : n\},$$

and $\mathbb{B}$ represent set of block-diagonal uncertainties.

Theorem 2.8. *Let $A, H \in \mathbb{C}^{n\times n}, H \geq 0$, a Hermitian positive semi-definite matrix. If $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H))$ for each $H \geq 0$, then $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ and*

$$0 \leq \mu_{\mathbb{B}}\left((\boldsymbol{i}I_n + A)^{-1}(\boldsymbol{i}I_n - A)\right) < 1,\ \boldsymbol{i} = \sqrt{-1}.$$

Following theorem acts as a bridge between $D(\alpha)$-stable matrices and μ-values.

Theorem 2.9. *Let $A \in \mathbb{R}^{n\times n}$. Then given matrix A is $D(\alpha)$-stable iff*

$$\mathrm{Re}\left(\lambda_i(\mathrm{diag}(d_{kk}I[\alpha_k]))A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right) > 0,\ \ \forall i = 1 : n, \forall k = 1 : p, \quad 1 \leq p \leq n,$$

and $0 \leq \mu_{\mathbb{B}}(M) < 1$, where matrix M is obtained from A as

$$M = \left(\boldsymbol{i}I_n + \mathrm{diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right)^{-1}\left(\boldsymbol{i}I_n - \mathrm{diag}(d_{kk}I[\alpha_k])A - A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right).$$

Theorem 2.10 yields the necessary conditions for given matrix $A \in \mathbb{C}^{n,n}$ to be $D(\alpha)$-stable.

Theorem 2.10. *Let $A \in \mathbb{C}^{n,n}$. A necessary condition for given matrix A to be $D(\alpha)$-stable matrix is that $A = e^B$ (Hermitian matrix $B \in \mathbb{C}^{n,n}$) is a stable matrix, and $0 \leq \mu_{\mathbb{B}}\left((\boldsymbol{i}I_n + e^B)^{-1}(\boldsymbol{i}I_n - e^B)\right) < 1$.*

Theorem 2.11. *Let $A \in \mathbb{C}^{n,n}$. Then matrix A is D-semistable iff A is semi-stable matrix, and $\lambda_k(A + v\omega^*) \neq 0, \forall k$, with $v\omega^*$, a rank-1.*

The following theorem 2.12 implies that a given a square complex-valued matrix is D-semistable matrix if and only if it is a semi-stable matrix, and the μ-value is bounded by 1.

Theorem 2.12. *Let $A \in \mathbb{C}^{n,n}$. Then A is D-semistable matrix iff A is semi-stable matrix, and $0 \leq \mu_{\mathbb{B}}(M) < 1$, where*

$$M = (\boldsymbol{i}I_n + \widehat{A})^{-1}(\boldsymbol{i}I_n - \widehat{A}),$$

with $\widehat{A} = A + v\omega^*$ for $v, \omega \in \mathbb{C}^{n,1}$.

3. Mathematical methods to approximate upper bounds of μ-values

3.1. **The μ-values for complex and real perturbations:** The basic methods for the analysis, performance, and robustness properties of feedback systems were reviewed in a much greater detail in [5]. For (G, Δ) interaction, the objective could be to determine whether the error e is contain by a set for the sets of inputs v, and an admissible perturbation Δ. The transfer function from v to e maybe expressed as the linear fractional transformation

$$e = F_u(G, \Delta)v = \left(G_{22} + G_{21}\Delta(I_n - G_{11}\Delta)^{-1}G_{12}\right)v.$$

In the robust stability analysis of feedback system, the plant uncertainties can be used to destabilize the nominal stable system. The term stability is taken in the sense that all the real parts of eigenvalues are strictly negative. These setting allows to have the following result [5] for the robust stability.

Theorem 3.1. *(Robust stability, unstructured) The system is stable for* Δ, $\bar{\sigma}(\Delta) < 1 \Leftrightarrow \|G_{11}\|_\infty \leq 1$.

In the above Theorem 3.1, the term unstructured refers to the fact that the admissible perturbation Δ is bounded, otherwise it is unknown. For the practical problems, the uncertainties consists of a number of parameter variations, and a number of multiple norm bounded perturbations. The uncertain coefficients in differential equation models of some physical systems generates the parameter variations, and the effect of unmodeled dynamics cause the norm bounded perturbations.

The uncertainty set $\mathbf{\Delta}$ has a block-diagonal structure

$$\mathbf{\Delta} := \{diag(\delta_1, \delta_2, \cdots, \delta_m; \Delta_1, \Delta_2, \cdots, \Delta_n) : \ \delta_i \in \mathbb{R}, \ \Delta_j \in \mathbb{C}^{k_j, k_j}\}.$$

It can also be of the form of a bounded subset

$$\mathbf{B\Delta} := \{\Delta \in \mathbf{\Delta} : \ \bar{\sigma}(\Delta) < 1\}.$$

One can easily generalized the uncertainty set while taking various number of repeated real or repeated complex scalar blocks, and a number of full real or full complex blocks. The non-square type of uncertainties maybe overcome by augmenting the interconnection structure with a fixed number of rows or a fixed number of columns having all zero elements.

The sufficient conditions on the robust stability can be obtained for a given $\Delta \in \mathbf{B\Delta}$. The computation of structured singular values (μ-values) is to obtain the precise generalization of Theorem 3.1. The μ-value is a positive real-valued function, and satisfies property

$$\prod_{i=1}^{n} \lambda_i(I_n - M\Delta) \neq 0, \ \forall \ \Delta \in \mathbf{\Delta}, \ \bar{\sigma}(\Delta) < \alpha \ \ \Leftrightarrow \ \ \alpha\mu(M) \leq 1.$$

On the other hand if $\mu(M) \neq 0$, means that $\exists \ \Delta \in \mathbf{\Delta}$ such that $\prod_{i=1}^{n} \lambda_i(I_n - M\Delta) = 0$, then

$$\mu_{\mathbf{\Delta}}(M) := min\{\bar{\sigma}(\Delta) : \ det(I_n - M\Delta) = 0\},$$

with **min** is taken over $\Delta \in \mathbf{\Delta}$. A generalization of Theorem 3.1 is given as below, (see [5]).

Theorem 3.2. *(Robust stability, structured) The system is stable for all* $\Delta \in \mathbf{B\Delta} \Leftrightarrow \|G_{11}\|_\mu \leq 1$, *where*

$$\|G\|_\mu := sup \ \mu[G(j\omega)],$$

the ***sup*** *is taken over* ω.

In Theorem 3.2, $\|G_{11}\|_\mu$ is not a norm, but a notation which depends not only on G but also also on the structure of $\mathbf{\Delta}$.

3.1.1. *The μ-value for complex uncertainties:* The robust performance and stability subject to structured uncertainty reduce down to the problem of computing μ-value for the constant matrices $G(j\omega)$. The **sup** taken over all ω for the approximation of μ-values. In this section, we begin with outlining some of the fundamental properties of μ-values against complex perturbations. We consider the approximation of μ-values as a natural extension to the spectral radius, and the spectral norm (the computation of the largest singular value $\bar{\sigma}$). Then, we focus on the characterization of the μ-values in term of ρ, the spectral radius, and $\bar{\sigma}$, the largest singular values.

We start with the consideration that $\mathbf{\Delta}$ as the sub-algebra of structured matrices satisfying $\{\lambda I_n : \lambda \in \mathbf{C}\} \subset \mathbf{\Delta} \subset \mathbf{C}^{n,n}$. The spectrum of $M \in \mathbf{C}^{n,n}$ w.r.t $\mathbf{\Delta}$ is defined as

$$Spec(M) := \{\Delta \in \mathbf{\Delta} : det(I_n - M\Delta) = 0\},$$

and the inverse spectrum is defined as

$$InvSpec(M) := \{\Delta \in \mathbf{\Delta} : det(I_n - M\Delta) = 0\}.$$

The μ-value as a natural generalization to spectral radius can be easily verified that

$$\mu_{\mathbf{\Delta}}(M) = sup\ \sigma_{min}(\Delta),$$

where **sup** is taken over $\Delta \in Spec(M)$. On the other side, if $\mu_{\mathbf{\Delta}(M)} \neq 0$, then we have

$$\mu_{\mathbf{\Delta}}(M) = sup\ \frac{1}{\sigma_{min}(\Delta)},$$

with **sup** taken over $\Delta \in InvSpec(M)$.

The characterization as a result yields that the approximation of μ-values is the generalization to the computation of $\bar{\sigma}$, the largest singular value. The μ-value also acts as the spectral radius ρ, or the computation of $\bar{\sigma}$, the largest singular value.

For $\mathbf{\Delta} = \{\lambda I_n : \lambda \in \mathbf{C}\} \Rightarrow \mu_{\mathbf{\Delta}}(M) = \rho(M)$, and for $\mathbf{\Delta} = \mathbf{C}^{n,n} \Rightarrow \mu_{\mathbf{\Delta}}(M) = \bar{\sigma}(M)$.

Remark 3.1. *For the set of block-diagonal matrices $\mathbf{\Delta}$, one can show that following results holds true*

$$\rho(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \bar{\sigma}(M).$$

Remark 3.2. *The bounds for $\rho(M) \leq \mu_{\mathbf{\Delta}}(M) \leq \bar{\sigma}(M)$ can be improved to*

$$sup\ \rho(MU) \leq \mu_{\mathbf{\Delta}}(M) \leq inf\ \bar{\sigma}(DMD^{-1}),$$

*with **sup** taken over $U \in \hat{U}$, and **inf** taken over $D \in \hat{D}$.*

Note: In remark-2, $\hat{U}$, and $\hat{D}$ are the sets such that for $\Delta \in \mathbf{\Delta}$, we have that

$$U \in \hat{U} \Rightarrow \bar{\sigma}(U\Delta) = \bar{\sigma}(\Delta),$$

and

$$D \in \hat{D} \Rightarrow D^{-1}\Delta D = \Delta.$$

Further,

$$U \in \hat{U} \Rightarrow \mu_{\mathbf{\Delta}}(MU) = \mu_{\mathbf{\Delta}}(M),$$

and

$$D \in \hat{D} \Rightarrow \mu_{\mathbf{\Delta}}(DMD^{-1}) = \mu_{\mathbf{\Delta}}(M).$$

The lower bounds of μ-value in term of $\rho(MU)$ has the property of achieving μ-values which is independent of the choice of number of blocks. The quantity $\rho(MU)$ can have only multiple local maximums but not the global maximum. The approximation to upper bounds of μ-value is relatively easy since the computation of $\bar{\sigma}(DMD^{-1})$ have only global minimum. The upper bound offers a better alternative to the approximation of μ-values.

3.1.2. *The μ-value for real uncertainties:* Consider the block-diagonal structure $\mathbf{\Delta}$ with only pure repeated real uncertainties, that is,

$$\mathbf{\Delta} = \{diag(\delta_1, \delta_2, \cdots, \delta_m; \Delta_1, \Delta_2, \cdots, \Delta_n) : \delta_i \in \mathbf{R}, \Delta_j \in \mathbf{C}^{k_j, k_j}\}.$$

For such a choice of Δ, it is possible that $Spec(M)$, and $InvSpec(M)$ maybe an empty set. We consider

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}.$$

and we assume that $det(I_n - T_{22}M) \neq 0$ such that $F_l(T, M) = T_{11} + T_{12}M(I_n - T_{22}M)^{-1}T_{21}$ is well-defined. The following Lemma-1 [5] gives a useful scaling.

Lemma 3.1. *Consider that $\exists$ T so that $B\mathbf{\Delta} \subset \{F_u(T, \Delta) : \bar{\sigma}(\Delta) < 1\}$, then*

$$\bar{\sigma}(F_l(T, M)) \leq 1 \Rightarrow \mu_{\mathbf{\Delta}}(M) \leq 1.$$

According to Lemma-1, we can obtained an upper bounds of the μ-values by using T. The next step involve the identification of set containing $T's$ which must satisfy Lemma-1. The use of Lemma-1 results that $\mu(M) \leq \hat{\mu}(M)$, where $\hat{\mu}(M)$ yields an upper bounds to μ. For the case when there are no real parameters, then

$$\hat{\mu}(M) = inf\ \bar{\sigma}(DMD^{-1}),$$

where **inf** is taken over $D \in \hat{D}$.

3.2. **An improved upper bound of μ-values [8]:** For a given matrix $M \in \mathbb{C}^{n,n}$, and $\mathbf{\Delta}$, the set of block-diagonal matrices having mixed real and complex uncertainties. For $x \neq 0$, we have

$$\Delta Mx = x, \quad \Delta \in \mathbf{\Delta}, \quad \|x\| = 1.$$

Consider that $Q_r \in \mathbb{R}^{m_r, n}$, $Q_c \in \mathbb{R}^{m_c, n}$ defined as

$$Q_r = \begin{bmatrix} I_r & 0 \end{bmatrix}, \quad Q_c = \begin{bmatrix} 0 & I_c \end{bmatrix}.$$

Further, we may have that

$$\delta_r Q_r Mx = Q_r x, \quad \text{and} \quad \Delta_c Q_c Mx = Q_c x,$$

where $\delta_r \in \mathbb{R}$, $\Delta_c \in \mathbb{C}^{m_c,m_c}$. The necessary and sufficient conditions to achieve that for some Δ_c, $\bar{\sigma}(\Delta_c) \leq a$, $a > 0$, is that

$$a||Q_cMx|| \geq ||Q_cx||.$$

Also, the necessary condition is to achieve that for some $|\delta_r| \leq a$, $\delta_r Q_r Mx = Q_r x$ is that

$$a||Q_rMx|| \geq ||Q_rx||.$$

The equation $\Delta Mx = x$ implies additional constraints, that is, $x_i \neq 0,\ \forall i$,

$$\frac{(Mx)_i}{x_i} = \frac{(Mx)_j}{x_j}, \quad i, j = 1:n.$$

For $i = j$, there exists $\eta \in \mathbb{R}$ such that

$$\eta(Mx)_i = x_i, \quad i = 1:n.$$

This further implies that $\mu_{\mathbf{\Delta}}(M) = a_*^{-1}$, where a_* is the smallest a for which some x does exist satisfying constraints. Take $\theta = a^{-1}$, then definition of μ-value can be reformulated as follows:

$$\mu_{\mathbf{\Delta}}(M) := \begin{cases} 0 & \text{if } \gamma_{\mathbf{\Delta}}(M) = \emptyset, \\ \max\{\theta, ||Q_rMx|| \geq \theta||Q_rx||, ||Q_cMx|| \geq \theta||Q_cx||\} & \text{otherwise,} \end{cases}$$

with

$$\gamma_{\mathbf{\Delta}}(M) = \{x \in \partial B : \bar{x}_j(Mx)_i = x_i(Mx)_j,\ i, j = 1:n\}$$

and

$$x \in \partial B := \{x \in \mathbb{C}^n : ||x|| = 1\}.$$

Remark 3.3. *The constraint $\bar{x}_j(Mx)_i = x_i(\bar{Mx})_j$, $i, j = 1:n$ can be written equivalently as $x^H M^H E^{ij} x = x^H E^{ij} Mx$, $i, j = 1:n$, with E^{ij} being (n, n)-matrix having (i, j)-th non-zero entry.*

For a number of repeated complex scalar blocks, we have that

$$\frac{(Mx)_i}{x_i} = \frac{(Mx)_j}{x_j}, \quad i \text{ and } j$$

ranging over indexes corresponding to the blocks under consideration. The above results can be generalized by the following Theorem 3.3.

Theorem 3.3. *For a given $M \in \mathbb{C}^{n,n}$, and Δ, the set of block-diagonal matrices, the μ-value is*

$$\mu_{\mathbf{\Delta}}(M) := \begin{cases} 0 & \textit{if } \tilde{\gamma}_{\mathbf{\Delta}}(M) = \emptyset, \\ \max\left\{\theta, ||Q_qMx|| \geq \theta||Q_qx||,\ q = 1:n\right\}, & \textit{otherwise} \end{cases}$$

where $Q_q \in \mathbf{\Delta}$, and $\tilde{\gamma}_{\mathbf{\Delta}}(M)$ given as

$$\left\{x \in \partial B : x_i(Mx)_j = \bar{x}_j(Mx)_i;\ (i, j) \in \bigcup_{q=1}^{m_r} J_q \times J_q;\ x_i(Mx)_j = x_j(Mx)_i,\ (i, j) \in \bigcup_{q=m_r+1}^{m_r+m_c} J_q \times J_q\right\}.$$

Remark 3.4. *The last theorem yields local maxima but not the global. Unfortunately, it does not provide enough information whether global maximum can be easily determined or not.*

3.2.1. *Computation of upper bound of μ-value:* The following proposition is a consequence of Theorem 3.3.

Proposition 3.1. *Let M be an n-dimensional matrix, and let Δ be the set of block-diagonal matrices, then*

$$\mu_{\mathbf{\Delta}}(M) \le \eta_{\mathbf{\Delta}}(M) \le \nu_{\mathbf{\Delta}}(M) \le \bar{\sigma}_{\mathbf{\Delta}}(M),$$

where

$$\eta_{\mathbf{\Delta}}(M) = \begin{cases} 0 & \text{if } \tilde{\gamma}_{\mathbf{\Delta}}(M) = \emptyset, \\ \max \|Mx\|, & \text{otherwise} \end{cases}$$

and

$$\nu_{\mathbf{\Delta}}(M) = \left(\max\left\{0,\ \inf \bar{\lambda}\left(MM^H + i(GM - M^H G)\right)\right\}\right)^{1/2}.$$

Here, max *is taken over $x \in \tilde{\gamma}_{\mathbf{\Delta}}(M)$, and* inf *is taken over G.*

Theorem 4: Let M be an n-dimensional matrix, and let $\mathbf{\Delta}$ be the set of block-diagonal matrices, then

$$\mu_{\mathbf{\Delta}}(M) \le \inf \eta_{\mathbf{\Delta}}(DMD^{-1}) \le \inf \nu_{\mathbf{\Delta}}(DMD^{-1}) \le \inf \bar{\sigma}_{\mathbf{\Delta}}(DMD^{-1}),$$

where **inf** is taken over D.

3.3. **An upper bound of μ-value via Linear Matrix Inequalities methodology [9]:** A multiplier approach was constructed to determine the upper bounds of the μ-value for a mixed structural singular value problem. It was proven that new bounds (upper) are convex and were computed by using Linear Matrix Inequality (LMI) method, and also it is numerically efficient.

The set of block-diagonal matrices are defined as:

$$\hat{D} := \left\{\mathrm{diag}(D_1, D_2, \ldots, D_{m_c+m_r}, d_1 I, \ldots, d_{m_c+m_C} I) : 0 < D_i = D_i^H \in \mathbb{C}^{k_i,k_i}, d_i > 0\right\}.$$

$$\hat{G} := \left\{\mathrm{diag}(G_1, G_2, \ldots, G_{m_r}, O, \ldots, O) : G_i = G_i^H \in \mathbb{C}^{k_i,k_i}\right\},$$

with O denotes a $k \times k$ zero-matrix. We define a matrix-valued function Φ_a as:

$$\Phi_a(D,G) := M^H DM + i\left(GM - M^H G\right) - a^2 D,$$

where $a \in \mathbb{R}$.

The following Lemma 3.2 [4] gives an upper bound of μ-values.

Lemma 3.2. *Let $M \in \mathbb{C}^{n,n}$ and let $\mathbf{\Delta}$ be the set of block-diagonal matrices, then*

$$\mu_{\mathbf{\Delta}}(M) \le \nu_{\mathbf{\Delta}}(M), \quad \text{where}$$

$$\nu_{\mathbf{\Delta}}(M) = \inf\left\{a : \exists D \in \hat{D}, G \in \hat{G} : \Phi_a(D,G) < 0\right\},$$

*with **inf** taken over $a > 0, a \in \mathbb{R}$.*

Three new upper bounds for a mixed real and complex perturbations were presented in [9]. The first upper bound to μ-value was obtained by using multiplier approach. This upper bound is convex and numerically approximated only for real perturbation. The well-known S-procedure of Yakubovich [11] is used to determine a looser upper bound of μ-value. A third looser new upper bound is computed for μ-value for real blocks.

3.3.1. *First New Upper Bound:* The results on the computation of an upper bound of μ-value are derived by using the following fact that:

$A \in \mathbb{C}^{n,n}$, a matrix family is non-singular if $\exists\ C \in \mathbb{C}^{n,n}$, a multiplier-matrix, such that Hermitian part of AC or CA is negative-definite matrix. By making use of this fact to μ-value problem yields that

$$0 \leq \mu_{\mathbf{\Delta}}(M) \leq a, \quad a > 0, a \in \mathbb{R},$$

if $\exists$ C $\in \mathbb{C}^{n,n}$, a multiplier-matrix so that

$$C\Big(I_n - M\Delta\Big) + \Big(I_n - M\Delta\Big)^H C^H < 0, \quad \forall \Delta \in \mathbf{\Delta},\ \bar{\sigma}(\Delta) \leq \frac{1}{a}.$$

Remark 3.5. *For $M = AB^H$, $A,\ B \in \mathbb{C}^{n,q}$, $q \leq n$, we have*

$$\prod_i \lambda_i\Big(I_n - M\Delta\Big) = \prod_i \lambda_i\Big(I_n - \Delta M\Big) = \prod_i \Big(I_q - B^H \Delta A\Big).$$

Remark 3.6. *The sufficient condition for μ-value problem reduces to determine $C \in \mathbb{C}^{q,q}$, a multiplier-matrix and to verify that*

$$\Big(I_q - B^H \Delta A\Big)^H C + C^H\Big(I_q - B^H \Delta A\Big) < 0, \quad \forall \Delta \in \mathbf{\Delta},\ \bar{\sigma}(\Delta) \leq \frac{1}{a}.$$

This is further equivalent to (for replacing Δ^H by Δ),

$$C^H\Big(I_q - A^H \Delta B\Big)^H + \Big(I_q - A^H \Delta B\Big)C < 0, \quad \forall \Delta \in \mathbf{\Delta},\ \bar{\sigma}(\Delta) \leq \frac{1}{a}.$$

Based on this analysis, the new upper bound to μ-value is proposed as: For a given $M \in \mathbb{C}^{n,n}$, and a block-diagonal structure $\mathbf{\Delta}$,

$$\overline{\mu}_{\mathbf{\Delta}}(M) = \inf\Big\{a : \exists\ C \in \mathbb{C}^{q,q} : E(C,\Delta) > 0, \forall \Delta \in \mathbf{\Delta},\ \bar{\sigma}(\Delta) \leq \frac{1}{a}\Big\},$$

where inf is taken over $a > 0, a \in \mathbb{R}$, and

$$E(C,\Delta) = C^H\Big(I - A^H \Delta B\Big)^H + \Big(I - A^H \Delta B\Big)C.$$

Thus, finally we have that

$$\mu_{\mathbf{\Delta}}(M) \leq \overline{\mu}_{\mathbf{\Delta}}(M).$$

3.3.2. *Second New Upper Bound:* Since the analysis for the computation of new upper bound of μ-value is not useful for complex perturbations. The looser upper bounds of μ-value were derived by making use of S-procedure [11]. This procedure was employed to complex-blocks in $\mathbf{\Delta}$. Δ_R, Δ_C are the notations used for real and complex sub-blocks of $\mathbf{\Delta}$, respectively. Thus, we have that

$$\Delta_R = \left\{\mathrm{diag}\left(\delta_1^r I, \delta_2^r I, \ldots, \delta_{m_r}^r I\right)\right\};$$

$$\Delta_C = \left\{\mathrm{diag}\left(\delta_1^C I, \delta_2^C I, \ldots, \delta_{m_c}^C I, \Delta_1^C, \Delta_2^C, \ldots, \Delta_{m_c}^C\right)\right\}.$$

Then $M = AB^H = \begin{bmatrix} A_R \\ A_C \end{bmatrix} \begin{bmatrix} B_R^H & B_C^H \end{bmatrix}$. Furthermore, the expression for $E(C, \Delta)$ can be re-written as:

$$E(C, \Delta) = E(C, \Delta_R) - C^H B_C^H A_C - A_C^H \Delta B_C C,$$

where

$$E(C, \Delta_R) = C^H \left(I - A_R^H \Delta_R B_R\right)^H + \left(I - A_R^H \Delta_R B_R\right) C.$$

The use of S-procedure to Complex Blocks of $E(C, \Delta)$ yields the following result.

Theorem 3.4. *Let $E(C, \Delta)$, and let $a > 0$, $a \in \mathbb{R}$, then*

$$E(C, \Delta) < 0, \forall \Delta \in \mathbf{\Delta}, \quad \bar{\sigma}(\Delta) \leq \frac{1}{a} \text{ if } \exists\, d_i, i = 1 : m_c + m_C \text{ to produce } D_C \in \hat{D} \text{ so that}$$

$$F(C, D_C, \Delta_R) = E(C, \Delta_R) + \frac{1}{a^2} C^H B_C^H D_C^{-1} B_C C + A_C^H D_C A_C < 0, \quad \forall\ \delta_i = \pm\frac{1}{a},\ 1 \leq i \leq m_r,$$

also,

$$\hat{F}(C, D_C, \Delta_R) = \begin{pmatrix} E(C, \Delta_R) + A_C^H D_C A_C & C^H B_C^H \\ B_C C & -a^2 D_C \end{pmatrix} < 0,$$

an LMI implying that $F(C, D_C, \Delta_R)$ is convex in C and D_C.

The second upper bound to μ-value is based on Theorem 4.

$$\bar{\mu}_{\mathbf{\Delta}}(M) = \inf\left\{a : \exists\ C \in \mathbb{C}^{q \times q},\ D_C \in \hat{D}; F(C, D_C, \Delta_R) < 0, \forall \delta_i{}^r = \pm\frac{1}{a},\ 1 \leq i \leq m_r\right\},$$

with inf taken over $a > 0$, $a \in \mathbb{R}$. The equivalence between $F(C, D_C, \Delta_R)$ and $\hat{F}(C, D_C, \Delta_R)$ allows a new upper bound to μ-value by using the LMI method.

Theorem 3.5. *$F(C, D_C, \Delta_R)$ holds iff $\exists\ K = K^H \in \mathbb{C}^{v,v}$ so that the following LMI hold:*

$$F_1(C, K, \Delta_R) = E(C, \Delta_R) + K < 0, \quad \forall \delta_i{}^r = \pm\frac{1}{a},\ 1 \leq i \leq m_r,$$

$$F_2(C, K, D_C) = \begin{pmatrix} -K + A_C^H D_C A_C & C^H B_C^H \\ B_C C & -a^2 D_C \end{pmatrix} \leq 0.$$

Consequently, $\bar{\mu}_{\mathbf{\Delta}}(M)$ can be expressed as

$$\bar{\mu}_{\mathbf{\Delta}}(M) = \inf\left\{a : \exists\ C, K = K^H \in \mathbb{C}^{q,q},\ D_C \in \hat{D} : F_2(C, K, D_C) < 0,\ F_1(C, K, \Delta_R) < 0,\right.$$

$$\left.\forall \delta_i{}^r = \pm\frac{1}{a},\ 1 \leq i \leq m_r\right\}.$$

3.3.3. *Third New Upper Bound:* The aim was to derive an even more looser upper bound of μ-values than $\bar{\bar{\mu}}$. Consider

$$\Delta_R = \text{diag}(\Delta_{R_1}, \Delta_{R_2}),$$

where Δ_{R_1} contains m_1 repeated real-blocks, while Δ_{R_2} contains the remaining $m_r - m_1$ blocks of Δ. In this case, $E(C,\Delta)$ is defined as,

$$E(C,\Delta) = E(C,\Delta_R) - C^H B_{R2}^H \Delta_{R2}^H A_R - A_R^H \Delta_{R2} B_{R2} C - C^H B_C^H \Delta_C^H A_C - A_C^H \Delta_C B_C C,$$

with

$$E(C,\Delta_{R_1}) = C^H \left(I - A_{R_1}^H \Delta_{R_1} B_{R_1}\right)^H + \left(I - A_{R_1}^H \Delta_{R_1} B_{R_1}\right) C.$$

This partition allows us to have the following results:

Theorem 3.6. *Let $F(C, D_C, \Delta_R)$, and let $a > 0$. Then $F(C, D_C, \Delta_R) < 0$ $\forall \Delta_R \in \mathbf{\Delta}$, $\bar{\sigma}(\Delta) \le \frac{1}{a}$ if $\exists$ $D_{R_2} \in \hat{D}$, $G_{R_2} \in \hat{G}$ such that the following LMI hold:*

$$L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2}) = \begin{pmatrix} L_{11} & C^H B_C^H & L_{13} \\ B_C C & -a^2 D_C & 0 \\ L_{13}^H & 0 & -a^2 D_{R_2} \end{pmatrix} < 0, \quad \forall \delta_i{}^r = \pm\frac{1}{a},\ i = 1 : m_r,$$

with

$$L_{11} = E(C,\Delta_{R_1}) + A_C^H D_C A_C + A_{R_2}^H \Delta_{R_2} A_{R_2}, \quad L_{13} = C^H B_{R_2}^H + i A_{R_2}^H G_{R_2}.$$

Furthermore, the conditions in $L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2})$ are equivalent to that $\exists$ $D_{R_2} \in \hat{D}$, $G_{R_2} \in \hat{G}$, $K = K^H \in \mathbb{C}^{q,q}$ such that the following LMI holds:

$$L_1(C, K, \Delta_{R_1}) = E(C,\Delta_{R_1}) + K < 0, \quad \forall \delta_i{}^r = \pm\frac{1}{a},\ i = 1 : m_1, \quad \textit{and}$$

$$L_2(C, K, D_C, D_{R_2}, G_{R_2}) = \begin{pmatrix} -\hat{K} & C^H B_C^H & L_{13} \\ B_C C & -a^2 D_C & 0 \\ L_{13}^H & 0 & -a^2 \Delta_{R_2} \end{pmatrix} < 0, \quad \textit{with}$$

$$\hat{K} = K - A_C^H D_C A_C + A_{R_2}^H D_{R_2} A_{R_2}.$$

Theorem 6 allows us to have a third new upper bound to μ-value as:

$$\bar{\bar{\mu}}_{\Delta}(M) = \inf\Big\{a : \exists\ C \in \mathbb{C}^{q,q}, D_{R_2} \in \hat{D}, G_{R_2} \in \hat{G}, D_C \in \hat{D};\ L(C, D_C, \Delta_{R_1}, D_{R_2}, G_{R_2}) < 0,$$

$$\forall \delta_i{}^r = \pm\frac{1}{a},\ i = 1 : m_1\Big\},$$

with **inf** is taken over $a > 0, a \in \mathbb{R}$.

3.4. **A fast algorithm for approximation of upper bound on μ-norm [13]:** In [13], a fast algorithm for the approximation of H_∞-norm of a finite dimension linear time-invariant system is presented. This quantity appears as an upper bound to μ-value. For the computation of the upper bound of μ-value, the set $\hat{D}$ is defined as

$$\hat{D} = \{\text{diag}(\delta_1 I, \delta_2 I, \ldots, \delta_{n_f-1} I, I) : 0 < \delta_j \in \mathbb{R}\}.$$

For a given $M \in \mathbb{C}^{n,n}$, an upper bound for $\mu_{\mathbf{\Delta}}(M)$ is given by

$$\hat{\mu}_{\mathbf{\Delta}}(M) = \inf \sigma_1(DMD^{-1}),$$

with **inf** defined over $D \in \hat{D}$. Inserting $\hat{\mu}_{\mathbf{\Delta}}(M)$ into $\|P\|_{\mathbf{\Delta}} = \sup_{\omega\in\mathbb{R}} \mu_{\mathbf{\Delta}}(P(j\omega))$ yields an upper bound on μ-value as

$$\|P\|_{\hat{\mu}} = \sup_{\omega\in\mathbb{R}} \hat{\mu}_{\mathbf{\Delta}}(P(j\omega)).$$

The system described by $P(s) = C(sI - A)^{-1}B$ is stable if $\|P\|_{\mathbf{\Delta}} < 1$, which is a necessary and sufficient condition for the robust stability of $P - \mathbf{\Delta}$ system. The main loop theorem [17] suggests to develop an approach for the computation of an upper bound of μ-value. For discrete time systems, the quantity

$$P(z) = C(zI - A)^{-1}B + D,$$

where A is a stable matrix. For γ, a positive scalar, the μ-norm of $P < \gamma$ is

$$0 \leq \mu_{\mathbf{\Delta}_p}(M(\gamma)) < 1,$$

where

$$M(\gamma) = \begin{pmatrix} A & B \\ \frac{1}{\gamma}C & \frac{1}{\gamma}D \end{pmatrix}, \quad \text{and } \mathbf{\Delta}_P = \{\text{diag}(\delta I_n, \triangle) : \delta \in \mathbb{C}, \triangle \in \mathbf{\Delta}\}.$$

An algorithm was constructed [13] for the fast computation of $\|P\|_{\hat{\mu}}$, whose key ideas are given as follows. Consider that

$$\hat{\mu}_{\mathbf{\Delta}}(\omega) = \inf_{D\in\hat{D}} \sigma_1(DP(j\omega)D^{-1}).$$

The objective was to maximize $\hat{\mu}_{\mathbf{\Delta}}(\omega)$ over $\omega \in \mathbb{R}$. For this purpose, $\hat{\mu}^*$ is defined as

$$\hat{\mu}^* = \sup_{\omega\in\mathbb{R}} \hat{\mu}_(\omega), \quad \omega^* = \arg\max_{\omega\in\mathbb{R}} \hat{\mu}_(\omega).$$

Let $v_D(\omega)$, a curve defined as

$$v_D(\omega) = \sigma_1(DP(j\omega)D^{-1}), \quad D \in \hat{D}.$$

Suppose that ξ_k, at kth iteration, is the best known lower bound to $\hat{\mu}^*$, and let ω_k be the frequency. Further, suppose that ω^* lies in Ω_k, an open set. Then, the problem is to compute

$$D_{k+1} = \arg\min_{D\in\hat{D}} \sigma_1(DP(j\omega_k)D^{-1}).$$

Let $\xi'_{k+1} = \hat{\mu}(\omega_k) = v_{D_{k+1}}(\omega_k)$. If $\xi'_{k+1} > \xi_k$, then take $\xi_{k+1} = \xi'_{k+1}$, a new estimate to $\hat{\mu}^*$.

Remark 3.7. $\hat{\mu}_{\mathbf{\Delta}}(\omega) \leq v_{D_{k+1}}(\omega), \quad \forall\omega \in \mathbb{R}$. *The inequality becomes an equality for* $\omega = \omega_k$.

The quantity $v_{D_{k+1}}(\omega)$ be the largest singular value curve for $D_{k+1}P(s)D_{k+1}^{-1}$ at $s = j\omega$. Let Ω'_{k+1} denote the open set of frequencies, and $v_{D_{k+1}}(\omega) > \xi_{k+1}$. From $\hat{\mu}_{\mathbf{\Delta}}(\omega) \leq v_{D_{k+1}}(\omega), \forall \omega \in \mathbb{R}$, and ξ_{k+1} being as the lower bound to μ^*, we have that

(a) If $\Omega'_{k+1} = \phi$, then $\hat{\mu}^* = \xi_{k+1}$; and

(b) If $\Omega'_{k+1} \neq \phi$, then $\omega^* \in \Omega'_{k+1}$.

Thus, it follows that

$$\omega^* \in \Omega_{k+1} = \mathcal{V}_k \cap \Omega'_{k+1},$$

then we choose next frequency ω_{k+1}. An outline to above algorithm is given as follows:

Data: $P(s) = C(sI - A)^{-1}B, \mathbf{\Delta}, \epsilon > 0$.
Initialize: $k = 0$, choose $\omega_0 > 0$, $\xi_0 = 0$, $\Omega_0 = (0, \infty)$.
Step 1: Compute D_k, ξ_k.
Step 2: Compute Ω_k.
Step 3: Compute ω_{k+1}.
Step 4: $k \leftarrow k + 1$. If stopping criterion is satisfied, stop.
Otherwise, return back to step 1.

4. Conclusion

In this paper, we have surveyed and discussed various mathematical methods for interconnection between D-stability, H-stability, $D(\alpha)$-stability, and μ-values. We also present a number of mathematical methods for the approximation of an upper bounds of structured singular values. We have considered those methods which are helpful for the approximation of an upper bounds of μ-values for a mixture of real and complex uncertainties. The main advantage of an approximation of an upper to μ-value is to discuss the stability of linear time invariant system appearing in system theory. We have presented the methods based on mathematical optimization, linear matrix inequalities for the approximation of an upper bounds of μ-values.

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