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Woven Continuous Generalized Frames in Hilbert C^*-Modules

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Abstract. The aim of this paper is to study woven c-g-frames for Hilbert C^*-modules. We begin by providing some definitions and key properties that are essential for studying this concept. Additionally, we present several properties of woven c-g-frames. Finally, we explore the perturbation theory related to woven c-g- frames.

1. Introduction and Preliminaries

Frames for Hilbert spaces were introduced by Duffin and Schaefer [7] in 1952 to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [9] for signal processing. In fact, in 1946 Gabor, showed that any function $f \in L^2(\mathbb{R})$ can be reconstructed via a Gabor system $\{g(x-ka)e^{2\pi imbx} : k, m \in \mathbb{Z}\}$ where g is a continuous compact support function. These ideas did not generate much interest outside of nonharmonic Fourier series and signal processing until the landmark paper of Daubechies, Grossmann, and Meyer [6] in 1986, where they developed the class of tight frames for signal reconstruction and they showed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to the expansions using orthonormal bases. After this innovative work the theory of frames began to be widely studied. While orthonormal bases have been widely used for many applications, it is the redundancy that makes frames useful in applications.

Formally, a frame in a separable Hilbert space $\mathcal{H}$ is a sequence $\{f_i\}_{i\in I}$ for which there exist positive constants $A, B > 0$ called frame bounds such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, f_i\rangle|^2 \leq B\|x\|^2, \ \forall x \in \mathcal{H}.$$

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It is remarkable that the above inequalities imply the existence of a dual frame $\{\tilde{f_i}\}_{i\in I}$, such that the following reconstruction formula holds for every $x \in \mathcal{H}$: $\sum_{i\in I}\langle x, \tilde{f_i}\rangle f_i$. In particular, any orthonormal basis for $\mathcal{H}$ is a frame. However, in general, a frame need not be a basis and, in fact, most useful frames are over-complete. The redundancy that frames carry is what makes them very useful in many applications.

Today, frame theory is an exciting, dynamic and fast paced subject with applications to a wide variety of areas in mathematics and engineering, including sampling theory, operator theory, harmonic analysis, nonlinear sparse approximation, pseudodifferential operators, wavelet theory, wireless communication, data transmission with erasures, filter banks, signal processing, image processing, geophysics, quantum computing, sensor networks, and more. The last decades have seen tremendous activity in the development of frame theory and many generalizations of frames have come into existence, for more detailed information, readers are recommended to consult: [2,8,10,11,14,16,17,19–28].

Motivated by a problem regarding distributed signal processing, Bemrose et al. [3] introduced a new concept of weaving frames in separable Hilbert spaces. The fundamental properties of weaving frames were examined by Casazza and Lynch in [5]. Weaving frames were further studied by Casazza, Freeman, and Lynch [4].

The remainder of this paper is organized as follows. This section concludes with a collection of essential definitions. Section 2 studies the concept of continuous woven g-frames within the context of Hilbert C^*-modules, explores their key properties, and finishes by investigating the perturbation of continuous woven g-frames.

Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers.

Let's now review the definition of a Hilbert C^*-module, the basic properties and some facts concerning operators on Hilbert C^*-module.

Definition 1.1. *[12] Let $\mathcal{A}$ be a unital C^*- algebra and χ be a left $\mathcal{A}-$ module, such that the linear structures of $\mathcal{A}$ and χ are compatible. χ is a pre-Hilbert $\mathcal{A}$ module if χ is equipped with an $\mathcal{A}-$valued inner product $\langle\cdot,\cdot\rangle_{\mathcal{A}} : \chi\times\chi \to \mathcal{A}$ such that is sesquilinear, positive definite and respects the module action. In the other words,*

1 - $\langle y, y\rangle_{\mathcal{A}} \geq 0$, $\forall y \in \chi$ *and* $\langle y, y\rangle_{\mathcal{A}} = 0$ *if and only if* $y = 0$.

2 - $\langle az + y, x\rangle_{\mathcal{A}} = a\langle z, x\rangle_{\mathcal{A}} + \langle y, x\rangle_{\mathcal{A}}$ *for all* $a \in \mathcal{A}$ *and* $x, y, z \in \chi$

3 - $\langle x, y\rangle_{\mathcal{A}} = \langle y, x\rangle^*_{\mathcal{A}}$ *for all* $x, y \in \chi$.

For $y \in \chi$, we define $\|y\| = \|\langle y, y\rangle_{\mathcal{A}}\|^{\frac{1}{2}}$. If χ is complete with $\|.\|$, it is called a Hilbert $\mathcal{A}-$module or a Hilbert C^*-module over $\mathcal{A}$. For every x in C^*-algebra $\mathcal{A}$, we have $|x| = (x^*x)^{\frac{1}{2}}$ and the $\mathcal{A}-$valued norm on χ is defined by $|x| = \langle x, x\rangle^{\frac{1}{2}}_{\mathcal{A}}$ for $x \in \chi$.

Definition 1.2. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y\rangle_{\mathcal{A}} = \langle x, T^*y\rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.*

Remark 1.1. *It follows from the definition that if T is adjointable, T^* is adjointable and $\langle T^*x, y\rangle_{\mathcal{A}} = \langle x, Ty\rangle_{\mathcal{A}}$. That is $(T^*)^* = T$*

We reserve the notation $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H})$ is abbreviated to $End^*_{\mathcal{A}}(\mathcal{H})$.

Proposition 1.1. *[15] Let T be an adjointable map. Then T is a bounded linear module map.*

Proposition 1.2. *[15] For $T \in End^*_{\mathcal{A}}(\mathcal{H})$, we have $\langle Tx, Tx\rangle \leq \|T\|^2\langle x, x\rangle$, $\forall x \in \mathcal{H}$.*

The following proposition is given by Ljiljana Arambašić in [1].

Proposition 1.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$, and $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. The following statements are equivalent:*

(1) *T is surjective.*
(2) *T^* is bounded below with respect to the norm, i.e., there is $m > 0$ such that*
$$\|T^*x\| \geq m\|x\|, \quad \forall x \in \mathcal{K}.$$
(3) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that*
$$\langle T^*x, T^*x\rangle \geq m'\langle x, x\rangle, \quad \forall x \in \mathcal{K}.$$

Let X be a Banach space, (Ω, μ) be a measure space, and $f : \Omega \to X$ be a measurable function. Integral of the Banach-valued function f has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module are Banach spaces, we can use this integral and its properties.

Let (Ω, μ) be a measure space, U and V be two Hilbert C^*-modules, and $\{V_k : k \in \Omega\}$ be a sequence of subspaces of V, and $End^*_{\mathcal{A}}(U, V_k)$ is the collection of all adjointable $\mathcal{A}$-linear maps from U into V_k. We define
$$\bigoplus_{k\in\Omega} V_k = \left\{ G = \{G_k\}_{k\in\Omega} : G_k \in V_k, \left\| \int_{\Omega} |G_k|^2 d\mu(k) \right\| < \infty \right\}.$$

For any $F = \{F_k\}_{k\in\Omega}$ and $G = \{G_k\}_{k\in\Omega}$, the $\mathcal{A}$-valued inner product is defined by $\langle F, G\rangle = \int_{\Omega} \langle F_k, G_k\rangle d\mu(k)$ and the norm $\|G\| = \|\langle G, G\rangle\|^{\frac{1}{2}}$. In this case $\bigoplus_{k\in\Omega} V_k$ is a Hilbert C^*-module.

Definition 1.3. *[13] We say that $\Gamma := \{\Gamma_k \in End^*_{\mathcal{A}}(U, V_k)\}_{k\in\Omega}$ a continuous g-frame for Hilbert C^*-module U with respect to $\{V_k : k \in \Omega\}$ if*

- *for any $g \in U$, the function $\tilde{g} : \Omega \to V_k$ defined by $\tilde{g}(k) = \Gamma_k g$ is measurable;*
- *there is a pair of constants $0 < A, B$ such that*
$$A\langle g, g\rangle_{\mathcal{A}} \leq \int_{\Omega} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \leq B\langle g, g\rangle_{\mathcal{A}}, \quad g \in U. \tag{1.1}$$

The constants A and B are called continuous g-frame bounds. If $A = B$ we call this continuous g-frame a continuous tight g-frame. If $A = B = 1$ it is called a continuous Parseval g-frame. If only the right-hand inequality of (1.1) *is satisfied, we call* $\Gamma := \{\Gamma_k \in End^*_{\mathcal{A}}(U, V_k)\}_{k\in\Omega}$ *the continuous g-Bessel for U with respect to* $\{V_k : k \in \Omega\}$ *with Bessel bound B.*

For each $n > 1$ where $n \in N$, we define $[n] := \{1, 2, ..., n\}$.

Definition 1.4. *A family of c-frames* $\Gamma := \{\Gamma_{kj}\}_{k\in\Omega, j\in[n]}$ *in Hilbert* C^**-module U with respect to* $\{V_k : k \in \Omega\}$ *is said to be woven continuous frames if there exist universal positive constants* $0 < A \leq B$ *such that for each partition* $\{\sigma_j\}_{j\in[n]}$ *of* Ω*, the family* $\{\Gamma_{kj}\}_{k\in\sigma_j, j\in[n]}$ *is a c-frame in Hilbert* C^**-module U with bounds A and B. Each family* $\{\Gamma_{kj}\}_{k\in\sigma_j, j\in[n]}$ *is called a weaving.*

2. Main Result

Now we define woven continuous g-frames in Hilbert C^*-modules.
Let $\Gamma := \{\Gamma_{kj} \in End^*_{\mathcal{A}}(U, V_k)\}_{k\in\Omega, j\in[n]}$ and $\Phi := \{\Phi_{kj} \in End^*_{\mathcal{A}}(U, V_k)\}_{k\in\Omega, j\in[n]}$ two c-g-frames in Hilbert $\mathcal{A}$-module U.

Definition 2.1. Γ *and* Φ *are said to be woven continous g-frames if there exist universal constants* $0 < A \leq B < \infty$ *such that for every partition* $\{\sigma_j\}_{j\in[n]}$ *of* Ω*, the family* $\Gamma \cup \Phi$ *is a c-g-frame in Hilbert* $\mathcal{A}$*-module U with bounds A and B respectively,that is:*

$$A\langle g, g\rangle_{\mathcal{A}} \leq \int_{\sigma} \langle \Gamma_{kj} g, \Gamma_{kj} g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Phi_{kj} g, \Phi_{kj} g\rangle_{\mathcal{A}} d\mu(k) \leq B\langle g, g\rangle_{\mathcal{A}} \textit{ for all } g \in U.$$

Definition 2.2. *A family of c-g-frames* $\Gamma := \{\Gamma_{kj}\}_{k\in\Omega, j\in N}$ *in Hilbert* C^**-module U with respect to* $\{V_k : k \in \Omega\}$ *is said to be woven continuous g-frames if there exist universal positive constants* $0 < A \leq B$ *such that for any partition* $\{\sigma_j\}_{j\in N}$ *of* Ω*,the family* $\cup_{j\in N}\{\Gamma_{kj}\}_{k\in\sigma_j}$ *is a c-g-frames in Hilbert* C^**-module U with lower and upper c-g-frames bounds A and B,respectively.*

Let $\Gamma := \{\Gamma_{kj}\}_{k\in\Omega, j\in N}$ be a family of woven continous g-frames. The Operator $T : \bigoplus_{k\in\Omega} V_k \longrightarrow U$ defined by:

$$\langle T_\Gamma G, g\rangle_{\mathcal{A}} = \int_{\Omega} \langle \Gamma^*_k G(k), g\rangle_{\mathcal{A}} d\mu(k) \quad G \in \bigoplus_{k\in\Omega} V_k, g \in U$$

is called the synthesis operator. Therefore for each $g \in U$ and $k \in \Omega$

$$T^*_\Gamma(g)(k) = \Gamma_k(g).$$

is called the analysis operator.
By composing T_Γ and T^*_Γ, we obtain the frame operator $S_\Gamma : U \longrightarrow U$ defined by

$$S_\Gamma(g) = \int_{\Omega} \Gamma^*_k \Gamma_k g d\mu(k) \quad g \in U$$

Proposition 2.1. *Let* $\Gamma := \{\Gamma_{kj}\}_{k\in\Omega, j\in N}$ *be a family of woven continous g-frames in Hilbert* C^**-modules U with bounds* A_Γ *and* B_Γ*. Then the frame operator S is self adjoint,bounded, and positive in U.*

Proof. Now we show that S is self adjoint operator for all $g, f \in U$ we have:

$$\begin{aligned}
\langle S_\Gamma(g), f\rangle &= \langle \int_\Omega \Gamma_k^*\Gamma_k g d\mu(k), f\rangle \\
&= \int_\Omega \langle \Gamma_k^*\Gamma_k g, f\rangle d\mu(k) \\
&= \int_\Omega \langle g, \Gamma_k^*\Gamma_k f\rangle d\mu(k) \\
&= \langle g, \int_\Omega \Gamma_k^*\Gamma_k f d\mu(k)\rangle \\
&= \langle g, S_\Gamma(f)\rangle.
\end{aligned}$$

Hence S is self adjoint. We have for all $g \in U$:

$$\begin{aligned}
\|S_\Gamma\| &= \sup_{\|f\|=1} \left\|\langle S_\Gamma(f), f\rangle_{\mathcal{A}}\right\|. \\
&= \sup_{\|f\|=1} \left\|\langle \int_\Omega \Gamma_k^*\Gamma_k f d\mu(k), f\rangle_{\mathcal{A}}\right\|. \\
&= \sup_{\|f\|=1} \left\|\int_\Omega \langle \Gamma_k^*\Gamma_k f, f\rangle_{\mathcal{A}} d\mu(k)\right\|. \\
&= \sup_{\|f\|=1} \left\|\int_\Omega \langle \Gamma_k f, \Gamma_k f\rangle_{\mathcal{A}} d\mu(k)\right\|. \\
&\leq B_\Gamma
\end{aligned}$$

Thus S_Γ is bounded operator. And for for all $g \in U$ we have

$$A\langle g, g\rangle \leq \langle S_\Gamma(g), g\rangle \leq B\langle g, g\rangle$$

So

$$A_\Gamma I \leq S_\Gamma \leq B_\Gamma I$$

and S_Γ is a positive operator. □

Theorem 2.1. *Let $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ be c-g-Bessel frames in Hilbert C^*-module U with bounds B_j for each $j \in [n]$. Then every weaving is a c-g-bessel frames with bounds $\sum_{j\in[n]} B_j$.*

Proof. Let $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ be a c-g-Bessel frames in Hilbert C^*-module U with bounds B_j for each $j \in [n]$ So:

$$\int_\Omega \langle \Gamma_{kj} g, \Gamma_{kj} g\rangle_{\mathcal{A}} d\mu(k) \leq B_j \langle g, g\rangle_{\mathcal{A}}.$$

Since we have for any partition $\{\sigma_j\}_{j\in[n]}$ of Ω and $g \in U$,:

$$\int\limits_{\sigma_j} \langle \Gamma_{kj} g, \Gamma_{kj} g\rangle_{\mathcal{A}} d\mu(k) \leq \int\limits_{\Omega} \langle \Gamma_{kj} g, \Gamma_{kj} g\rangle_{\mathcal{A}} d\mu(k)$$

Hence

$$\sum_{j\in[n]} \int\limits_{\sigma_j} \langle \Gamma_{kj} g, \Gamma_{kj} g\rangle_{\mathcal{A}} d\mu(k) \leq \sum_{j\in[n]} \int\limits_{\Omega} \langle \Gamma_{kj} g, \Gamma_{kj} g\rangle_{\mathcal{A}} d\mu(k) \leq \sum_{j\in[n]} B_j \langle g, g\rangle_{\mathcal{A}}.$$

yielding the desired bound. □

Proposition 2.2. *Let $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ and $\Phi := \{\Phi_k\}_{k\in\Omega}$two c-g-bessel frames in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ and with c-g-bessel bounds B_Γ,B_Φ respectively.If $Y \subset \Omega$ be measurable subsets such that $\Gamma_Y := \{\Gamma_k\}_{k\in Y}$ and $\Phi_Y := \{\Phi_k\}_{k\in Y}$ are woven c-g-frames .Then Γ and Φ are woven c-g-frames in U.*

Proof. Let $A_{\Gamma\Phi}$ be the universal lower c-g-frame bounds for the woven c-g-frames Φ_Y and Γ_Y, and let $\sigma \subset \Omega$ be an arbitrary subset and $g \in U$. Then

$$\begin{aligned} A_{\Gamma\Phi} \langle g, g\rangle_{\mathcal{A}} &\leq \int\limits_{\sigma\cap Y} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int\limits_{\sigma^c\cap Y} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) \\ &\leq \int\limits_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int\limits_{\sigma^c} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) \\ &\leq (B_\Gamma + B_\Phi)\langle g, g\rangle_{\mathcal{A}}. \end{aligned}$$

Hence, Γ and Φ are woven c-g-frames in U. □

Theorem 2.2. *Let $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ and $\Phi := \{\Phi_k\}_{k\in\Omega}$ be woven c- g-frames in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ with universal c-g-frames bounds A and B. If: $Y \subset \Omega$ and:*

$$\int\limits_{Y} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \leq D\langle g, g\rangle_{\mathcal{A}}$$

for all $g \in U$ and for some $0 \leq D \leq A$.Then $\Gamma_0 := \{\Gamma_k\}_{k\in\Omega\setminus Y}$ and $\Phi_0 := \{\Phi_k\}_{k\in\Omega\setminus Y}$ are woven c-g-frames in U with universal c-g-frames bounds (A-D) and B.

Proof. Let σ be any subset of $\Omega \setminus Y$ we have:

$$\begin{aligned} \Xi &= \int\limits_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int\limits_{(\Omega\setminus Y)\setminus\sigma} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) \\ &= \int\limits_{\sigma\cup Y} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) - \int\limits_{Y} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) + \int\limits_{(\Omega\setminus Y)\setminus\sigma} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) \\ &= \int\limits_{\sigma\cup Y} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int\limits_{(\Omega\setminus Y)\setminus\sigma} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) - \int\limits_{Y} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k) \\ &\geq A\langle g, g\rangle_{\mathcal{A}} - D\langle g, g\rangle_{\mathcal{A}} \\ &= (A - D)\langle g, g\rangle_{\mathcal{A}} \qquad \forall g \in U \end{aligned}$$

On the other hand, for all $g \in U$ we have:

$$\begin{aligned}\Xi &= \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{(\Omega\backslash Y)\backslash\sigma} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k)\\ &\leq \int_{\sigma\cup Y} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{(\Omega\backslash Y)\backslash\sigma} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k)\\ &\leq B\langle g, g\rangle_{\mathcal{A}}.\end{aligned}$$

Then $\Gamma_0 := \{\Gamma_k\}_{k\in\Omega\backslash Y}$ and $\Phi_0 := \{\Phi_k\}_{k\in\Omega\backslash Y}$ are woven c-g-frames in U with universal c-g-frames bounds $(A - D)$ and B. □

Theorem 2.3. *Let* $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ *and* $\Phi := \{\Phi_k\}_{k\in\Omega}$ *be a pair of c-g-frames in Hilbert* $\mathcal{A}$*-module U with respect to* $\{V_k : k \in \Omega\}$*. Then for every partition* $\sigma \subset \Omega$*,*Γ *and* Φ *are woven c-g-frame with universal c-g-frames bounds A and B,if and only if:*

$$A\left\|\langle g, g\rangle_{\mathcal{A}}\right\| \leq \left\|\int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k)\right\| \leq B\left\|\langle g, g\rangle_{\mathcal{A}}\right\| \text{ for all } g \in U.$$

Proof. $(\Rightarrow)$ Obvious.
Now assume that there exist constants $0 < A \leq B < \infty$ such that for all $g \in U$

$$A\left\|\langle g, g\rangle_{\mathcal{A}}\right\| \leq \left\|\int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k)\right\| \leq B\left\|\langle g, g\rangle_{\mathcal{A}}\right\|. \tag{2.1}$$

We prove that Γ and Φ are woven c-g-frames in Hilbert $\mathcal{A}$-module U with the universal lower and upper c-g-frame bounds A and B, respectively.
Since S is positive, self adjoint operator. We have
$\langle S^{\frac{1}{2}} g, S^{\frac{1}{2}} g\rangle_{\mathcal{A}} = \langle Sg, g\rangle_{\mathcal{A}} = \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Phi_k g, \Phi_k g\rangle_{\mathcal{A}} d\mu(k)$. From (2.1), we have

$$\sqrt{A}\left\|g\right\| \leq \left\|S^{\frac{1}{2}} g\right\| \leq \sqrt{B}\left\|g\right\| \text{ for all } g \in U.$$

By using proposition 1.3, we have

$$\langle S^{\frac{1}{2}} g, S^{\frac{1}{2}} g\rangle_{\mathcal{A}} = \langle Sg, g\rangle_{\mathcal{A}} \geq A\langle g, g\rangle_{\mathcal{A}}.$$

Since $S^{\frac{1}{2}}$ is bounded and A-linear, then we have:

$$\langle S^{\frac{1}{2}} g, S^{\frac{1}{2}} g\rangle_{\mathcal{A}} = \langle Sg, g\rangle_{\mathcal{A}} \leq B\langle g, g\rangle_{\mathcal{A}}.$$

□

Theorem 2.4. *Let* $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ *and* $\Phi := \{\Phi_k\}_{k\in\Omega}$ *be a pair of c-g-frames in Hilbert* $\mathcal{A}$*-module U with respect to* $\{V_k : k \in \Omega\}$ *with c-g-frames bounds* A_Γ, B_Γ *and* A_Φ, B_Φ *respectively. Assume that there are constants* $0 < \lambda_1, \lambda_2, \lambda_3 < 1$ *such that:*

$$\lambda_1\sqrt{B_\Gamma} + \lambda_2\sqrt{B_\Phi} + \lambda_3 \leq \frac{A_\Gamma}{2(\sqrt{B_\Gamma} + \sqrt{B_\Phi})}.$$

And

$$\left\| \int_{\Omega} \langle (\Gamma_k^* - \Phi_k^*)G, f \rangle_{\mathcal{A}} d\mu(k) \right\| \leq \lambda_1 \left\| \int_{\Omega} \langle \Gamma_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_2 \left\| \int_{\Omega} \langle \Phi_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_3 \left\| \langle G \rangle_{\mathcal{A}} \right\| \quad (2.2)$$

for all $G \in (\bigoplus_{k\in\Omega} V_k)$ and $f \in U$. Then, Γ and Φ are woven c- g-frames with universal lower and upper frame bounds $\frac{A_\Gamma}{2}$ and $B_\Gamma + B_\Phi$, respectively.

Proof. Let T_Γ and T_Φ be the synthesis operator for the c-g-frames $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ and $\Phi := \{\Phi_k\}_{k\in\Omega}$ respectively. $T_\Gamma : \bigoplus_{k\in\Omega} V_k \longrightarrow U$ defined by:

$$\langle T_\Gamma G, f \rangle_{\mathcal{A}} = \int_{\Omega} \langle \Gamma_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \qquad f \in U$$

And $T_\Phi : \bigoplus_{k\in\Omega} V_k \longrightarrow U$ defined by:

$$\langle T_\Phi G, f \rangle_{\mathcal{A}} = \int_{\Omega} \langle \Phi_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \qquad f \in U.$$

For each $\sigma \subset \Omega$, define the bounded operators: $T_\Gamma^\sigma : \bigoplus_{k\in\Omega} V_k \longrightarrow U$ defined by:

$$\langle T_\Gamma^\sigma G, f \rangle_{\mathcal{A}} = \int_{\sigma} \langle \Gamma_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \qquad f \in U.$$

And $T_\Phi^\sigma : \bigoplus_{k\in\Omega} V_k \longrightarrow U$ defined by:

$$\langle T_\Phi^\sigma G, f \rangle_{\mathcal{A}} = \int_{\sigma} \langle \Phi_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \qquad f \in U.$$

We note that $\|T_\Gamma^\sigma\| \leq \|T_\Gamma\|$, $\|T_\Phi^\sigma\| \leq \|T_\Phi\|$ and $\|T_\Gamma^\sigma - T_\Phi^\sigma\| \leq \|T_\Gamma - T_\Phi\|$. By Equation (2.2) and $\|g\| = \|\langle g, g \rangle\|_{\mathcal{A}}^{\frac{1}{2}}$ for all $g \in U$, we have

$$\begin{aligned}
\|(T_\Gamma - T_\Phi)(G)\| &= \sup_{\|f\|=1} \left\| \langle (T_\Gamma - T_\Phi)(G), f \rangle_{\mathcal{A}} \right\|. \\
&= \sup_{\|f\|=1} \left\| \langle (T_\Gamma(G) - T_\Phi(G)), f \rangle_{\mathcal{A}} \right\|. \\
&= \sup_{\|f\|=1} \left\| \langle (T_\Gamma(G), f \rangle_{\mathcal{A}} - \langle T_\Phi(G)), f \rangle_{\mathcal{A}} \right\|. \\
&= \sup_{\|f\|=1} \left\| \int_{\Omega} \langle \Gamma_k^* G, f \rangle_{\mathcal{A}} d\mu(k) - \int_{\Omega} \langle \Phi_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \right\|. \\
&= \sup_{\|f\|=1} \left\| \int_{\Omega} \langle (\Gamma_k^* - \Phi_k^*)G, f \rangle_{\mathcal{A}} d\mu(k) \right\|. \\
&\leq \sup_{\|f\|=1} \left\{ \lambda_1 \left\| \int_{\Omega} \langle \Gamma_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_2 \left\| \int_{\Omega} \langle \Phi_k^* G, f \rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_3 \left\| \langle G \rangle_{\mathcal{A}} \right\| \right\}.
\end{aligned}$$

$$\begin{aligned}
&\leq \lambda_1 \sup_{\|f\|=1} \left\| \int_{\Omega} \langle \Gamma_k^* G, f\rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_2 \sup_{\|f\|=1} \left\| \int_{\Omega} \langle \Phi_k^* G, f\rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_3 \|\langle G\rangle_{\mathcal{A}}\| . \\
&\leq \lambda_1 \left\| T_\Gamma(G) \right\| + \lambda_2 \left\| T_\Phi(G) \right\| + \lambda_3 \|G\| . \\
&\leq (\lambda_1 \|T_\Gamma\| + \lambda_2 \|T_\Phi\| + \lambda_3) \|G\| .
\end{aligned}$$

Then: $\|T_\Gamma - T_\Phi\| \leq \lambda_1 \|T_\Gamma\| + \lambda_2 \|T_\Phi\| + \lambda_3$.
So for every partition $\sigma \subset \Omega$, we have

$$\begin{aligned}
M &= \left\| \int_{\sigma} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma} \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \\
&\leq \sup_{\|f\|=1} \left\| \int_{\sigma} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma} \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \\
&\leq \sup_{\|f\|=1} \left\| \langle (T_\Gamma^\sigma \{\Gamma_k g\}_{k\in\sigma} - T_\Phi^\sigma \{\Phi_k g\}_{k\in\sigma}), f\rangle_{\mathcal{A}} \right\| \\
&\leq \sup_{\|f\|=1} \left\| \langle (T_\Gamma^\sigma T_\Gamma^{\sigma *} g - T_\Gamma^\sigma T_\Phi^{\sigma\,*} g + T_\Gamma^\sigma T_\Phi^{\sigma\,*} g - T_\Phi^\sigma T_\Phi^{\sigma\,*} g), f\rangle_{\mathcal{A}} \right\| \\
&\leq \left\| T_\Gamma^\sigma \right\| \left\| T_\Phi^{\sigma\,*} - T_\Phi^{\sigma\,*} \right\| \left\| g \right\| + \left\| T_\Gamma^\sigma - T_\Phi^\sigma \right\| \left\| T_\Phi^{\sigma\,*} \right\| \left\| g \right\| \\
&\leq (\|T_\Gamma\| \|T_\Gamma - T_\Phi\| \left\| g \right\| + \|T_\Phi\| \|T_\Gamma - T_\Phi\| \left\| g \right\| \\
&\leq (\lambda_1 \|T_\Gamma\| + \lambda_2 \|T_\Phi\| + \lambda_3)(\|T_\Gamma\| + \|T_\Phi\|)\|g\| \\
&\leq \frac{A_\Gamma}{2(\sqrt{B_\Gamma} + \sqrt{B_\Phi})} (\sqrt{B_\Gamma} + \sqrt{B_\Phi})\|g\| \\
&\leq \frac{A_\Gamma}{2}\|g\|.
\end{aligned}$$

Hence for every $g \in U$, we put

$$R = \left\| \int_{\sigma^c} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma} \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| .$$

Hence

$$\begin{aligned}
R &= \left\| \int_{\sigma^c} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma} \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \\
&\geq \left\| \int_{\Omega} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma} \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \\
&\geq \left\| \int_{\Omega} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| - \left\| \int_{\sigma} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma} \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\|
\end{aligned}$$

$$
\begin{aligned}
&\geq \quad A_\Gamma \left\|g\right\| - \left\| \int_\sigma \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_\sigma \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \\
&\geq \quad (A_\Gamma - \frac{A_\Gamma}{2}) \left\|g\right\| \\
&= \quad \frac{A_\Gamma}{2} \left\|g\right\|.
\end{aligned}
$$

So the lower c-g-frames bounds is $\frac{A_\phi}{2}$ and the upper bounds is $B_\Gamma + B_\Phi$.
Thus Γ and Φ are woven c-g-frames. □

Theorem 2.5. *Let $\Gamma := \{\Gamma_k\}_{k\in\Omega}$ and $\Phi := \{\Phi_k\}_{k\in\Omega}$ be a pair of c-g-frames in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ with c-g-frames bounds A_Γ, B_Γ and A_Φ, B_Φ respectively. Assume that there are constants $0 < \lambda_1, \lambda_2, \lambda_3 < 1$ such that:*

$$\lambda_1 B_\Gamma + \lambda_2 B_\Phi + \lambda_3 \leq A_\Gamma.$$

And for all $f, g \in U$ and for every $\sigma \subset \Omega$,

$$
\begin{aligned}
\left\| \int_\sigma \langle (\Gamma_k^* \Gamma_k - \Phi_k^* \Phi_k) g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \leq \lambda_1 \left\| \int_\sigma \langle (\Gamma_k^* \Gamma_k) g, f\rangle_{\mathcal{A}} d\mu(k) \right\| + \\
\lambda_2 \left\| \int_\sigma \langle (\Phi_k^* \Phi_k) g, f\rangle_{\mathcal{A}} d\mu(k) \right\| + \lambda_3 (\int_\sigma \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k))^{\frac{1}{2}}. \quad (2.3)
\end{aligned}
$$

Then, Γ and Φ are woven c-g-frames with universal c-g-frame bounds $(A_\Gamma - \lambda_1 B_\Gamma - \lambda_2 B_\Phi - \lambda_3 \sqrt{B_\Gamma})$ and $(B_\Gamma + \lambda_1 B_\Gamma - \lambda_2 B_\Phi - \lambda_3 \sqrt{B_\Gamma})$.

Proof. For every $\sigma \subset \Omega$ we use the fact that for $g, f \in U$ where $\left\|f\right\| = 1$, we have

$$\left\| \int_\sigma \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \leq B_\Gamma \|g\| \text{ and } \left\| \int_\sigma \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\| \leq B_\Phi \|g\|$$

and as we know that $\|f\|^2 = \|\langle f, f\rangle\|, \forall f \in U$. Equation (2.3) implies

$$\left\| \int_\sigma (\Gamma_k^* \Gamma_k - \Phi_k^* \Phi_k) g d\mu(k) \right\| \leq \lambda_1 \left\| \int_\sigma (\Gamma_k^* \Gamma_k) g d\mu(k) \right\| + \lambda_2 \left\| \int_\sigma (\Phi_k^* \Phi_k) g d\mu(k) \right\| + \lambda_3 (\int_\sigma \|\Gamma_k g\|^2 d\mu(k))^{\frac{1}{2}}.$$

Hence for every $g \in U$, we put:

$$L = \left\| \int_{\sigma^c} \langle \Gamma_k^* \Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_\sigma \langle \Phi_k^* \Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) \right\|.$$

Hence:

$$\begin{aligned}
L &= \left\|\int_{\sigma^c}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&= \left\|\int_{\Omega}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&\geq \left\|\int_{\Omega}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| - \left\|\int_{\sigma}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&\geq A_\Gamma\|g\| - \left\|\int_{\sigma}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&\geq A_\Gamma\|g\| - \lambda_1\left\|\int_{\sigma}(\Gamma_k^*\Gamma_k) g d\mu(k)\right\| - \lambda_2\left\|\int_{\sigma}(\Phi_k^*\Phi_k) g d\mu(k)\right\| - \lambda_3(\int_{\sigma}\|\Gamma_k g\|^2 d\mu(k))^{\frac{1}{2}}) \\
&\geq (A_\Gamma - \lambda_1 B_\Gamma - \lambda_2 B_\Phi - \lambda_3\sqrt{B_\Gamma})\|g\|.
\end{aligned}$$

And:

$$\begin{aligned}
L &= \left\|\int_{\sigma^c}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&= \left\|\int_{\Omega}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&\leq \left\|\int_{\Omega}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| + \left\|\int_{\sigma}\langle\Gamma_k^*\Gamma_k g, f\rangle_{\mathcal{A}} d\mu(k) - \int_{\sigma}\langle\Phi_k^*\Phi_k g, f\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&\leq (B_\Gamma\left\|g\right\| + \lambda_1\left\|\int_{\sigma}(\Gamma_k^*\Gamma_k) g d\mu(k)\right\| + \lambda_2\left\|\int_{\sigma}(\Phi_k^*\Phi_k) g d\mu(k)\right\| + \lambda_3(\int_{\sigma}\left\|\Gamma_k g\right\|^2 d\mu(k))^{\frac{1}{2}})) \\
&\leq (B_\Gamma + \lambda_1 B_\Gamma + \lambda_2 B_\Phi + \lambda_3\sqrt{B_\Gamma})\left\|g\right\|.
\end{aligned}$$

Hence, Γ and Φ are c-woven g-frames with universal c-g-frame bounds $(A_\Gamma - \lambda_1 B_\Gamma - \lambda_2 B_\Phi - \lambda_3\sqrt{B_\Gamma})$ and $(B_\Gamma + \lambda_1 B_\Gamma + \lambda_2 B_\Phi + \lambda_3\sqrt{B_\Gamma})$. □

Theorem 2.6. *For $i \in N$, let $\Gamma_i = \{\Gamma_{ik}\}_{k\in\Omega}$ be a family of c-g-frames for a Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ and c-g-frames bounds A_i and B_i. for any $\sigma \subset \Omega$ and a fix $r \in N$, let*

$$P_i^\sigma(g) = \int_{\sigma}\Gamma_{ik}^*\Gamma_{ik} g d\mu(k) - \int_{\sigma}\Gamma_{rk}^*\Gamma_{rk} g d\mu(k)$$

for $r \neq i$. If P_i^σ is a positive linear operator, then the family $\{\Gamma_i\}_{i\in N}$ is woven c-g-frames.

Proof. Let $\{\sigma_i\}_{i\in[n]}$ be any partition of Ω. Then, for every $g \in U$, a fix $r \in N$ and $k \in \sigma_i$, we have

$$\int_{\sigma_i} \langle \Gamma^*_{rk}\Gamma_{rk}g, g\rangle_{\mathcal{A}} d\mu(k) = \int_{\sigma_i} \langle \Gamma^*_{ik}\Gamma_{ik}g - P^{\sigma}_i(g), g\rangle_{\mathcal{A}} d\mu(k) \leq \int_{\sigma_i} \langle \Gamma^*_{ik}\Gamma_{ik}g, g\rangle_{\mathcal{A}} d\mu(k) \quad (2.4)$$

since P^{σ}_i is a positive linear operator. Now, using (2.4) we have

$$\begin{aligned}
A_r\langle g, g\rangle &\leq \int_{\Omega} \langle \Gamma^*_{rk}\Gamma_{rk}g, g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \int_{\sigma_1} \langle \Gamma^*_{rk}\Gamma_{rk}g, g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma_2} \langle \Gamma^*_{rk}\Gamma_{rk}g, g\rangle_{\mathcal{A}} d\mu(k) + ... + \int_{\sigma_n} \langle \Gamma^*_{rk}\Gamma_{rk}g, g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \int_{\sigma_1} \langle \Gamma^*_{1k}\Gamma_{1k}g, g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma_2} \langle \Gamma^*_{2k}\Gamma_{2k}g, g\rangle_{\mathcal{A}} d\mu(k) + ... + \int_{\sigma_n} \langle \Gamma^*_{nk}\Gamma_{nk}g, g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq (B_1 + B_2 + ... + B_n)\langle g, g\rangle \\
&= (\sum_{i=1}^{n} B_i)\langle g, g\rangle. \\
&\leq (\sum_{i\in I} B_i)\langle g, g\rangle.
\end{aligned}$$

This implies that

$$A_r\langle g, g\rangle \leq \int_{\Omega} \langle \Gamma^*_{ik}\Gamma_{ik}g, g\rangle_{\mathcal{A}} d\mu(k) \leq (\sum_{i\in I} B_i)\langle g, g\rangle.$$

□

Theorem 2.7. *Let $\Gamma_i = \{\Gamma_{ik}\}_{k\in\Omega}$ be a family of c-g-frames in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ with c-g-frames bounds A_i and B_i For each $i \in [n]$. Suppose there exists D such that*

$$\left\|\int_{Y} \langle (\Gamma_{ik} - \Gamma_{jk})g, (\Gamma_{ik} - \Gamma_{jk})g\rangle_{\mathcal{A}} d\mu(k)\right\| \leq D \min\left\{\left\|\int_{Y} \langle \Gamma_{ik}g, \Gamma_{ik}g\rangle_{\mathcal{A}} d\mu(k)\right\|, \left\|\int_{Y} \langle \Gamma_{jk}g, \Gamma_{jk}g\rangle_{\mathcal{A}} d\mu(k)\right\|\right\}.$$

for all $g \in U, i \neq j \in [n]$ and $Y \subset \Omega$. Then, the family $\{\Gamma_i\}_{i\in[n]}$ is woven c-g-frames with universal bounds $\dfrac{\sum_{i\in[n]} A_i}{2(n-1)(D+1)+1}$ and $\sum_{j\in[n]} B_j$.

Proof. Suppose that $\{\sigma_i\}_{i\in[n]}$ is a partition of Ω and $g \in U$. Therefore.

$$\begin{aligned}
\left\|\sum_{i\in[n]} A_i\langle g, g\rangle\right\| &\leq \left\|\sum_{i\in[n]} \int_{\Omega} \langle \Gamma_{ik}g, \Gamma_{ik}g\rangle_{\mathcal{A}} d\mu(k)\right\| \\
&= \left\|\sum_{i\in[n]} \sum_{j\in[n]} \int_{\sigma_j} \langle \Gamma_{ik}g, \Gamma_{ik}g\rangle_{\mathcal{A}} d\mu(k)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \sum_{i\in[n]} \Big(\int_{\sigma_i} \langle \Gamma_{ik}g, \Gamma_{ik}g\rangle_{\mathcal{A}} d\mu(k) \right. \\
&+ \left. \sum_{j\in[n]} \int_{\sigma_j} \{ \langle \Gamma_{ik}g - \Gamma_{jk}g, \Gamma_{ik}g - \Gamma_{jk}g\rangle_{\mathcal{A}} d\mu(k) + \langle \Gamma_{jk}g, \Gamma_{jk}g\rangle_{\mathcal{A}} d\mu(k)\} \Big) \right\| \\
&\leq \left\| \sum_{i\in[n]} \Big(\int_{\sigma_i} \langle \Gamma_{ik}g, \Gamma_{ik}g\rangle_{\mathcal{A}} d\mu(k) + \sum_{j\in[n], j\neq i} \int_{\sigma_j} (D+1) \langle \Gamma_{jk}g, \Gamma_{jk}g\rangle_{\mathcal{A}} d\mu(k) \Big) \right\| \\
&= \{(n-1)(D+1)+1\} \left\| \sum_{i\in[n]} \int_{\sigma_i} \langle \Gamma_{ik}g, \Gamma_{ik}g\rangle_{\mathcal{A}} d\mu(k) \right\|.
\end{aligned}
$$

For The upper bound is evident. □

Proposition 2.3. *Let $\Gamma := \{\Gamma_{kj}\}_{k\in\Omega, j\in[n]}$ be a family of woven continous g-Bessel sequences in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ and with c-g-Bessel bound B. Then $\{\Gamma_{kj}P\}_{k\in\Omega, j\in[n]}$ is woven continous g-Bessel sequences with bound $B\|P\|^2$ for each $P \in End^*_{\mathcal{A}}(U)$.*

Proof. Let $\{\Gamma_{kj}\}_{k\in\Omega, j\in[n]}$ be a family of woven continous g-Bessel sequences in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ and with c-g-Bessel bound B. So for any partition $\{\sigma_j\}_{j\in[n]}$ of Ω, we have

$$
\sum_{j\in[n]} \int_{\sigma_j} \langle \Gamma_{jk}g, \Gamma_{jk}g\rangle_{\mathcal{A}} d\mu(k) \leq B\langle g, g\rangle_{\mathcal{A}}.
$$

Since $Pg \in U$ then for every $g \in U$:

$$
\sum_{j\in[n]} \int_{\sigma_j} \langle \Gamma_{jk}Pg, \Gamma_{jk}Pg\rangle_{\mathcal{A}} d\mu(k) \leq B\langle Pg, Pg\rangle_{\mathcal{A}} \leq B\|P\|^2 \langle g, g\rangle_{\mathcal{A}}.
$$

□

Theorem 2.8. *Let $\{\Gamma_k\}_{k\in\Omega}$ be a family of c-g-frame in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ and with upper and lower c-g-frame bounds A and B, Suppose S is the c-g-frame operator of $\{\Gamma_k\}_{k\in\Omega}$ such that $S^{-1}\Gamma_k$ is self adjoint for all $k \in \Omega$. Hence $\{\Gamma_k\}_{k\in\Omega}$ and $\{\Gamma_k^* S^{-1}\}_{k\in\Omega}$ are woven c-g-frame in U.*

Proof. Let $\{\Gamma_k\}_{k\in\Omega}$ be a family of c-g-frame in Hilbert $\mathcal{A}$-module U with respect to $\{V_k : k \in \Omega\}$ and with upper and lower c-g-frame bounds A and B and S is the c-g-frame operator of $\{\Gamma_k\}_{k\in\Omega}$ such that $S^{-1}\Gamma_k$ is self adjoint,and σ be any partition of Ω so we have:

$$
\begin{aligned}
A\langle g, g\rangle &\leq \int_{\Omega} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \\
&= \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k)
\end{aligned}
$$

$$\begin{aligned}
&= \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle SS^{-1}\Gamma_k g, SS^{-1}\Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \|S\|^2 \langle S^{-1}\Gamma_k g, S^{-1}\Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + B^2 \int_{\sigma^c} \langle (S^{-1}\Gamma_k)^* g, (S^{-1}\Gamma_k)^* g\rangle_{\mathcal{A}} d\mu(k) \\
&= \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + B^2 \int_{\sigma^c} \langle \Gamma_k^* S^{-1} g, \Gamma_k^* S^{-1} g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \max\{1, B^2\}\Big(\int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Gamma_k^* S^{-1} g, \Gamma_k^* S^{-1} g\rangle_{\mathcal{A}} d\mu(k)\Big).
\end{aligned}$$

Then

$$\frac{A}{\max\{1, B^2\}} \langle g, g\rangle \leq \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Gamma_k^* S^{-1} g, \Gamma_k^* S^{-1} g\rangle_{\mathcal{A}} d\mu(k)$$

So $\frac{A}{\max\{1,B^2\}}$ is the universal lower c-g-frame bound.

And we note $L := \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Gamma_k^* S^{-1} g, \Gamma_k^* S^{-1} g\rangle_{\mathcal{A}} d\mu(k)$

$$\begin{aligned}
L &= \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle \Gamma_k^* S^{-1} g, \Gamma_k^* S^{-1} g\rangle_{\mathcal{A}} d\mu(k) \\
&= \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \langle (S^{-1}\Gamma_k)^* g, (S^{-1}\Gamma_k)^* g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \int_{\sigma^c} \|S^{-1}\|^2 \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \int_{\sigma} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) + \frac{1}{A^2} \int_{\sigma^c} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq \max\{1, \frac{1}{A^2}\} \int_{\Omega} \langle \Gamma_k g, \Gamma_k g\rangle_{\mathcal{A}} d\mu(k) \\
&\leq B \max\{1, \frac{1}{A^2}\} \langle g, g\rangle.
\end{aligned}$$

Thus $\{\Gamma_k\}_{k\in\Omega}$ and $\{\Gamma_k^* S^{-1}\}_{k\in\Omega}$ are c-woven g-frame in U with universal bounds $\frac{A}{max\{1,B^2\}}$ and $Bmax\{1, \frac{1}{A^2}\}$. □

Conclusion

In this note, we have explored the concept of woven continuous generalized frames in Hilbert C^*-modules, and provided several proposions. We also present new results related to generalized frames in these modules and discuss several properties of woven c-g-frames in Hilbert C^*-modules.

The results extend known results in Hilbert spaces to Hilbert $\mathcal{A}-$ module by allowing the inner product to take values in a C^*-algebra rather than in the field of complex numbers. Finally, we have explored the perturbation theory related to woven c-g frames.

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