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Lower Characteristic and Essential Spectra

Aref Jeribi*

Department of Mathematics and statistics, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia

**Corresponding author:* aref.jeribi@fss.rnu.tn, abjeribi@imamu.edu.sa

Abstract. In this paper, we show that a lower characteristic linear operator T acting on a Banach space, can be characterized by closed subspace. Some results concerning the essential spectra of the sum of the two bounded linear operators and the essential spectra of each of these operators, where their products are compacts operators on a Banach space X, are given.

1. Introduction

Let X and Y be two Banach spaces. By $\mathcal{L}(X,Y)$ the Banach space of all bounded linear operators from X into Y and by $\mathcal{K}(X,Y)$ the subspace of all compact operators of $\mathcal{L}(X,Y)$. If $T \in \mathcal{L}(X,Y)$ then $\alpha(T)$ denotes the dimension of the kernel $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in Y. The classes of upper semi-Fredholm from X into Y are defined respectively by

$$\Phi_+(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } R(T) \text{ closed in } Y\},$$

and

$$\Phi_-(X,Y) := \{T \in \mathcal{L}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } R(T) \text{ closed in } Y\}.$$

$\Phi(X,Y) := \Phi_+(X,Y) \bigcap \Phi_-(X,Y)$ is the set of Fredholm operators from X into Y. If $X = Y$, the sets $\mathcal{L}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, and $\Phi_-(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, and $\Phi_-(X)$, respectively. The index of an operator $T \in \Phi(X)$ is $i(T) := \alpha(T) - \beta(T)$ (see [10–13]).

Proposition 1.1. *[18] Let X be a Banach space. Then the following hold true*

(i) The sets $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi(X)$ are open.
(ii) The index is constant on every component of each of the sets $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi(X)$.

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If $A \in \mathcal{L}(X)$, we define the ascent of A, $\operatorname{asc}(A)$, and the descent of A, $\operatorname{desc}(A)$, by

$$\operatorname{asc}(A) := \min\left\{n \in \mathbb{N} \text{ such that } N(A^n) = N(A^{n+1})\right\},$$

and

$$\operatorname{desc}(A) := \min\left\{n \in \mathbb{N} \text{ such that } R(A^n) = R(A^{n+1})\right\}.$$

Definition 1.1. *An operator $A \in \mathcal{L}(X)$ is called a Riesz operator, if it satisfies the following conditions*
(i) for all $\lambda \in \mathbb{C}^*$, $[\lambda - A]_a > 0$ and $[\overline{\lambda} - A^*]_a > 0$,
(ii) for all $\lambda \in \mathbb{C}^*$, $\lambda - A$ has a finite ascent and a finite descent, and
(iii) all $\lambda \in \sigma(A)\backslash\{0\}$ are eigenvalues of a finite multiplicity, and have no accumulation points, except possibly zero.

Let $\mathcal{R}(X)$ be denote the class of all Riesz operators.

Lemma 1.1. *Let $A \in \mathcal{L}(X)$ and let $E \in \mathcal{R}(X)$. If $[A]_a > 0$ and $AE - EA \in \mathcal{K}(X)$, then $[A + E]_a > 0$ and $i(A + E) = i(A)$.*

Theorem 1.1. *([16, Theorem 6, p. 157]) Let X, Y, and Z be three Banach spaces, $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$. If $BA \in \Phi^b_+(X, Z)$, then $A \in \Phi^b_+(X, Y)$.*

Theorem 1.2. *Let X, Y, and Z be three Banach spaces, $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$. If $[BA]_a > 0$, then $[A]_a > 0$.*

We recall that T is quasi-Fredholm if there exists $d \geq 0$ such that $R(T^{d+1})$ is closed and

$$R(T) + N(T) = R(T) + N^\infty(T). \tag{1.1}$$

Equivalently, $N(T) \bigcap R(T^d) = N(T) \bigcap R^\infty(T)$, where $N^\infty(T) = \bigcup_{n=0}^\infty N(T^n)$ and $R^\infty(T) = \bigcap_{n=0}^\infty (R(T^n))$. The degree of stable iteration $\operatorname{dis}(T)$ of T is defined as the smallest integer d such that the equality (1.1) is satisfied (with $\operatorname{dis}(T) = \infty$ if no such integer exists). For more details, the reader can refer back to [1,3–5].
For $T \in \mathcal{L}(X)$, we define the resolvent set of T by $\rho(T) := \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ has a bounded inverse}\}$, and the spectrum of T by $\sigma(T) = \mathbb{C}\backslash\rho(T)$.

Definition 1.2. *[14] Let D be a bounded subset of X. We define $\gamma(D)$, the Kuratowski measure of noncompactness of D, to be* $\inf\{d > 0$ *such that D can be covered by a finite number of sets of diameter less than or equal to d*$\}$.

The following proposition gives somes properties of the Kuratowski measure of noncompactness which are frequently used.

Proposition 1.2. *Let D and D' be two bounded subsets of X, then we have the following properties. Then,*

(i) $\gamma(D) = 0$ if and only if D is relatively compact.
(ii) If $D \subseteq D'$, then $\gamma(D) \leq \gamma(D')$.
(iii) $\gamma(D + D') \leq \gamma(D) + \gamma(D')$.
(iv) For every $\alpha \in \mathbb{C}$, $\gamma(\alpha D) = |\alpha|\gamma(D)$.

Definition 1.3. *[2,7] Let $T \in \mathcal{L}(X,Y)$, $\gamma(.)$ is the Kuratowski measure of noncompactness in X. Let $k \geq 0$, T is said to be k-set-contraction if, for any bounded subset B of X, $T(B)$ is a bounded subset of X and $\gamma(T(B)) \leq k\gamma(B)$. T is said to be condensing if, for any bounded subset B of X such that $\gamma(B) > 0$, $T(B)$ is a bounded subset of X and $\gamma(T(B)) < \gamma(B)$.*

Definition 1.4. *[6,15] Let X be a Banach space and let $T : X \longrightarrow X$ be a bounded linear operator. The operator T is said to be demicompact (or relative demicompact), if for every bounded sequence $(x_n)_n \in X$ such that $x_n - Tx_n \to x \in X$, then there exists a convergent subsequence of $(x_n)_n$.*

Remark 1.1. *It is well known that*

(i) Every k-set-contraction operator such that $k < 1$ is condensing.
(ii) Every condensing operator is 1-set-contraction.
(iii) Every condensing operator is demicompact.

Definition 1.5. *Let $T \in \mathcal{L}(X)$. We define $\bar{\gamma}(T)$ by*

$$\bar{\gamma}(T) := \inf\{k \text{ such that } T \text{ is } k\text{-set-contraction}\}.$$

In the following proposition, we give some properties of $\bar{\gamma}(.)$ that we will need in the sequel.

Proposition 1.3. *[8,9] Let X be a Banach space and $T \in \mathcal{L}(X)$, then we have the following properties*

(i) $\bar{\gamma}(T) = 0$ if and only if T is compact.
(ii) If $T, S \in \mathcal{L}(X)$, then $\bar{\gamma}(ST) \leq \bar{\gamma}(S)\bar{\gamma}(T)$.
(iii) If $K \in \mathcal{K}(X)$, then $\bar{\gamma}(T+K) = \bar{\gamma}(T)$.
(iv) If B is a bounded subset of X, then $\gamma(T(B)) \leq \bar{\gamma}(T)\gamma(B)$.

The paper is organized in the following way. In Section 2, we present the main results of this paper. We prove some results concerning the lower characteristic. In Section 3, we shows the relation between the essential spectra of the sum of the two bounded linear operators and the essential spectra of each of these operators, where their products are compacts operators on a Banach space X.

2. Lower characteristic operators under restrictions

Definition 2.1. *For $T \in \mathcal{L}(X,Y)$, we define the "lower" characteristic*

$$[T]_a = \sup\{k \;:\; k > 0,\; \gamma(T(M)) \geq k\gamma(M) \text{ for all bounded } M \subset X\} \tag{2.1}$$

as elements of $[0, \infty]$.

Note that in finite dimensional spaces we have $[T]_a = \infty$. In infinite dimensional spaces, where this characteristic is of more use, we get

$$[T]_a = \inf_{0<\gamma(M)<\infty} \frac{\gamma(T(M))}{\gamma(M)}.$$

Sets with $\gamma(M) = 0$ can be left out here, since the continuity of T assures that also $\gamma(T(M)) = 0$. This can be seen by considering $\gamma(T(M)) \leq \gamma(T(\overline{M}))$.

Theorem 2.1. Let X be a Banach space and let $T \in \mathcal{L}(X)$. Then, T is demicompact if, and only if, $[I-T]_a > 0$.

Proof. We first show that $N(I-T)$ is finite dimensional. Let $S := \{x \in X$ such that $(I-T)x = 0$ and $\|x\| = 1\}$ and $(x_n)_n$ be a bounded sequence of S. Since T is demicompact, there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ which converges to $x \in X$. Thus it follows from the continuity of the norm and the boundness of T that $x \in X$, $x - Tx = 0$ and $\|x\| = 1$. Hence $\alpha(I-T)$ is finite. Now, we claim that $R(I-T)$ is closed. We can write $X = N(I-T) \oplus X_0$, where X_0 is a closed subspace of X, then it is a Banach space. In view of Theorem 3.12 in [18], it suffices to prove that there is a constant $\lambda > 0$ such that for every $x \in X_0$, $\|Tx\| \geq \lambda\|x\|$. If not, there exists a sequence $(x_n)_n$ of X_0 such that $\|x_n\| = 1$ and $\|(I-T)x_n\| \to 0$. Since T is demicompact, there exists a subsequence $(x_{n_i})_i$ of $(x_n)_n$ which converges to $x \in X$. Moreover, $I - T$ is closed thus $(I-T)x = 0$, hence $x = 0$ which contradicts the continuity of the norm. Since $\dim N(I-T) < \infty$, we may find a closed subspace X_0 of X with $X = X_0 \oplus N(I-T)$. The projection $P : X \longrightarrow X_0$ satisfies $[P]_a = 1$, since $I - P$ is compact. Consider the canonical isomorphism $\widetilde{L} : X_0 \longrightarrow R(I-T)$. Since $I - T = \widetilde{L}P$ and $[\widetilde{L}]_a > 0$, we conclude that also

$$[I-T]_a \geq [\widetilde{L}]_a[P]_a > 0.$$

Inversely, suppose that $[I-T]_a > 0$ and fix $k \in (0, [I-T]_a)$. Since the set $M = N(I-L) \bigcap B_X$ is mapped into $(I-T)(M) = \{0\}$, we get

$$\gamma(M) \leq \frac{1}{k}\gamma((I-T)(M)) = 0,$$

which show that $\overline{M}$ is compact, and hence $N(I-T)$ is finite dimensional. We prove now that the range $R(I-T)$ of $I-T$ is closed. Since $\dim N(I-T) < \infty$, there exists a closed subspace $X_0 \subset X$ such that $X = X_0 \oplus N(I-T)$. Let $(y_n)_n$ be a sequence in $R(I-T)$ converging to some $y \in Y$, and choose $(x_n)_n$ in X with $(I-T)x_n = y_n$. Now, we distinguish two cases. First, suppose that $(x_n)_n$ is bounded. With $k > 0$ as before we get then

$$\gamma(\{x_1, x_2, \cdots, x_n, \cdots\}) \leq \frac{1}{k}\gamma(\{y_1, y_2, \cdots, y_n, \cdots\}) = 0,$$

and hence $x_{n_k} \to x$ for some subsequence $(x_{n_k})_k$ of $(x_n)_n$ and suitable $x \in X$. By continuity we see that $(I-T)x = y$, and so $y \in R(I-T)$. On the other hand, suppose that $\|x_n\| \to \infty$. Set $e_n = \frac{x_n}{\|x_n\|}$ and $E = \{e_1, e_2, \cdots, e_n, \cdots\}$. Then, clearly $E \subset \{x \in X \ : \ \|x\| = 1\}$ and

$$(I-T)e_n = \frac{(I-T)x_n}{\|x_n\|} = \frac{y_n}{\|x_n\|} \to 0 \text{ as } n \to \infty.$$

Hence, $\gamma((I-T)(E)) = 0$. On the other hand, $\gamma((I-T)(E)) \geq k\gamma(E)$, by (2.1), and thus $\gamma(E) = 0$. Whithout loss of generality we may assume that the sequence $(e_n)_n$ converge to some element

$e \in \{x \in X_0 \ : \ \|x\| = 1\}$. So, $(I-T)e = 0$, contradicting the fact that $X_0 \bigcap N(I-T) = \{0\}$. Thus, $I - T \in \Phi_+(X)$. By $\alpha(I-T) < \infty$, we deduce that there exists a closed subspace C of X such that

$$N(I-T) \oplus C = X.$$

We deduce that

$$(I-T)_{|C} : (C, \|\cdot\|) \longrightarrow (R(I-T), \|\cdot\|)$$

is invertible with bounded inverse on $R(I-T)$. Now, take a bounded sequence $(x_n)_n$ of X such that $((I-T)x_n)_n$ converges to y. Obviously, $y \in R(I-T)$. Using the boundedness of $((I-T)_{|C})^{-1}$ on $R(I-T)$, we deduce that $(x_n)_n$ converges to $((I-T)_{|C})^{-1}(y) = z$. Hence, $(x_n)_n$ converges to z. So, T is demicompact and the proof is achieved. □

Corollary 2.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then, T is demicompact if, and only if, $I - T \in \Phi_+(X)$.*

Corollary 2.2. *[1,19] Let X and Y be two Banach spaces and $T \in \mathcal{L}(X,Y)$. Then,*

(i) $T \in \Phi_+(X,Y)$ if, and only if, $[T]_a > 0$.
(i) $T \in \Phi_-(X,Y)$ if, and only if, $[T^*]_a > 0$.

Theorem 2.2. *([16]) Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$. If $AB \in \Phi^b(X)$ and $BA \in \Phi^b(X)$, then $A \in \Phi^b(X)$ and $B \in \Phi^b(X)$.*

A consequence of Corollary 2.2, we have the following:

Theorem 2.3. *Let $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(X)$. If $[AB]_a > 0$, $[BA]_a > 0$, $[(AB)^*]_a > 0$, and $[(BA)^*]_a > 0$, then $[A]_a > 0$, $[A^*]_a > 0$, $[B]_a > 0$, and $[B^*]_a > 0$.*

Proposition 2.1. *[19] Let X, Y, Z be three Banach spaces, T, $S \in \mathcal{L}(X,Y)$ and $R \in \mathcal{L}(Y,Z)$. Then,*

(i) $[R]_a[T]_a \leq [RT]_a$.
(ii) $[T+S]_a = [T]_a$ if S is compact.

Theorem 2.4. *([18, Theorem 7.3, p. 157]) Let X and Y be Banach spaces. If $A \in \Phi(X,Y)$ and $B \in \Phi(Y,Z)$, then $BA \in \Phi(X,Z)$ and $i(BA) = i(B) + i(A)$.*

Lemma 2.1. *Let $A \in \mathcal{L}(X,Y)$ and let $J : X \longrightarrow Y$ be a linear operator. Assume that $J \in \mathcal{K}(X,Y)$. Then, if $A \in \Phi(X,Y)$, then $A + J \in \Phi(X,Y)$ and $i(A+J) = i(A)$.*

Theorem 2.5. *Let X be a Banach space and $T \in \mathcal{L}(X)$. If $\lim\limits_{n\to\infty}[\overline{\gamma}(T^n)]^{\frac{1}{n}} < 1$, then $[I - \mu T]_a > 0$ for every $\mu \in [0,1[$.*

Proof. Let $(x_k)_k$ be a bounded sequence of X such that $y_k = x_k - \mu T x_k \to y$. Since $\lim\limits_{n\to\infty}[\overline{\gamma}(T^n)]^{\frac{1}{n}} < 1$, then there exists a positive integer n_0 such that $\overline{\gamma}(T^{n_0}) < 1$. Obviously, there exists a bounded linear operator S such that

$$I - (\mu T)^{n_0} = S(I - \mu T).$$

Hence,

$$z_k := x_k - (\mu T)^{n_0} x_k \to Sy.$$

Since

$$\{x_k,\ k \in \mathbb{N}\} \subset \{z_k,\ k \in \mathbb{N}\} + \{(\mu T)^{n_0} x_k,\ k \in \mathbb{N}\},$$

it follows that

$$\begin{aligned} \gamma(\{x_k,\ k \in \mathbb{N}\}) &\leq \gamma(\{z_k,\ k \in \mathbb{N}\}) + \mu^{n_0}\overline{\gamma}(T^{n_0})\gamma(\{x_k,\ k \in \mathbb{N}\}) \\ &< \overline{\gamma}(T^{n_0})\gamma(\{x_k,\ k \in \mathbb{N}\}). \end{aligned}$$

Thus,

$$(1 - \overline{\gamma}(T^{n_0}))\gamma(\{x_k,\ k \in \mathbb{N}\}) < 0.$$

Since $\overline{\gamma}(T^{n_0}) < 1$, then $\gamma(\{x_k,\ k \in \mathbb{N}\}) = 0$. Therefore, $\{x_k,\ k \in \mathbb{N}\}$ is relatively compact. Thus, there exists a convergent subsequence $(x_{k_i})_i$ of $(x_k)_k$. By applying Theorem 2.1, we infer that $[I - \mu T]_a > 0$ for every $\mu \in [0, 1[$. □

In this section, we focus on the lower characteristic of the restriction T_n of an operator T to $R(T^n)$ viewed as an operator from $R(T^n)$ into $R(T^n)$.

Theorem 2.6. *Let X be a Banach space and $T \in \mathcal{L}(X)$ be two commuting operators, $T \neq 0$, $n \in \mathbb{N}$ an integer such that $I - T$ and T^n are relatively prime. Then, the following conditions are equivalent*

(i) $R(T^n)$ is closed and $[I - T_n]_a > 0$,
(ii) $R((T^n - T^{n+1})$ is closed and $\alpha(I - T) < \infty$.

Proof. $(i) \Longrightarrow (ii)$ Assume $[I - T_n]_a > 0$. Then, by using Corollary 2.2, we infer that $I - T_n \in \Phi_+(X)$. As $R((I - T)T^n) = R(I - T_n)$, then $R((I - T)T^n)$ is closed. Moreover, as $I - T$ and T^n are relatively prime, there exists $U, V \in \mathcal{L}(X)$, commuting with T, such that

$$U(I - T) + VT^n = I.$$

Then, if $x \in N(I - T)$, we have $x = T^n(V(x))$. Thus, $N(I - T) \subset R(T^n)$ and therefore $\alpha(I - T) = \dim N(I - T_n)$. Hence, $\alpha(I - T) < \infty$.
$(ii) \Longrightarrow (i)$ Assume that there exists $n \in \mathbb{N}$, such that $R((I - T)T^n)$ is closed and $\alpha(I - T) < \infty$. Since T^n and $I - T$ are relatively prime, it follows that $R(I - T)$ and $R(T^n)$ are closed. Since T^n and $I - T$ are relatively prime, then $N(I - T_n) = N(I - T)$. As $\alpha(I - T) < \infty$, then $\alpha(I - T_n) < \infty$. Since $R((I - T)T^n)$ is closed, then T_n is an demicompact operator. So, by using Theorem 2.1, we deduce that $[I - T_n]_a > 0$. This completes the proof. □

Theorem 2.7. *Let X be a Banach space, $T \in \mathcal{L}(X)$ and $T \neq 0$. Let m, n be two integers such that $m > n$ and the operator $I - T$ is prime with T^n and T^m. Assume also that $R(T^m)$ and $R(T^n)$ are closed. Then, the following conditions are equivalent*

(i) $[I - T_n]_a > 0$.
(ii) $[I - T_m]_a > 0$.

Proof. $(i) \Longrightarrow (ii)$ Assume that $[I - T_n]_a > 0$. As $N(I - T) \subset R(T^n)$ and $N(I - T) \subset R(T^m)$, then $N(I - T_n) = N(I - T) = N(I - T_m)$ and $\alpha(I - T_m) < \infty$. Since $[I - T_n]_a > 0$, then by using Theorem 2.6, we infer that $R((I - T)T^n) = R(I - T_n)$ is closed. As $I - T$ and T^n are relatively prime, then $R(I - T)$ is closed. Again, as $I - T$ and T^m are relatively prime, $R(T^m)$ and $R((I - T)$ are closed, then $R((I - T)T^m) = R(I - T_m)$ is closed. Thus, $I - T_m$ is upper semi-Fredholm. By using Corollary 2.2, we deduce that $[I - T_m]_a > 0$.
$(ii) \Longrightarrow (i)$ Exactly in the same way as the proof of $(i) \Longrightarrow (ii)$, by interchanging m and n. □

Corollary 2.3. *Let X be a Banach space and let T be a quasi-Fredholm operator such that $dis(T) = d$. Let m, n be two integers such that $m > n \geq d$. Then, the following conditions are equivalent*

(i) $[I - T_n]_a > 0$.
(ii) $[I - T_m]_a > 0$.

Theorem 2.8. *Let X be a Banach space, $T \in \mathcal{L}(X)$ and $n \in \mathbb{N}^*$. If $[I - T]_a > 0$ and $R(T^n)$ is closed, then $[I - T_n]_a > 0$.*

Proof. From the closedness of $R(T^n)$ and T, it follows that T_n is a closed operator from $R(T^n)$ into itself. As $[I - T]_a > 0$, then based on Corollary 2.2 it follows that $\alpha(I - T_n) \leq \alpha(I - T) < \infty$. Now, let $y \in R(T^n)$ be such that $y \in \overline{R(I - T_n)}$, the closure of $R(I - T_n)$ in $R(T^n)$. So, there exists a sequence $(y_p)_p \in R(T^n)$ such that $(I - T)(y_p)$ converges to y. As $R(I - T)$ is closed in X, then $y = (I - T)(v)$. Hence $v = y + T(v)$. Since there exists $u \in X$ such that $y = T^n(u)$, then $v = y + T(v) = T^n(u) + T(v) = T(T^{n-1}(u) + v)$. Thus, $v \in R(T)$ and by induction, we infer that $v \in R(T^n)$ and then $y \in R(I - T_n)$. Hence, the range of $R(I - T_n)$ is closed in $R(T^n)$ and $I - T_n$ is an upper semi-Fredholm. Thus, by using Corollary 2.2, we infer that $[I - T_n]_a > 0$. □

Theorem 2.9. *Let X be a Banach space, $T \in \mathcal{L}(X)$, $T \neq I$ and $q \in \mathbb{N}$. Assume that $\Omega \neq \emptyset$ is a connected open subset of $\mathbb{C}$ such that $\sigma(T) \subset \Omega$ and let $f : \Omega \longrightarrow \Omega$ be an analyticf unction such that $f(0) = 0$. If $R(T^q)$ is closed and $[f(I - T_q)]_a > 0$, then $[I - T]_a > 0$.*

Proof. If $[f(I - T_q)]_a > 0$, then by using Corollary 2.2, we have $f(I - T_q)$ is an upper semi-Fredholm. As f is analytic in the compact $\sigma(T_q)$, then they are in finite number $\{\mu_1, \cdots, \mu_n\}$. As a matter of fact, we can state

$$f(z) = \prod_{i=1}^{n} (z - \mu_i)^{m_i} g(z),$$

where m_i stands for the multiplicity of μ_i and g corresponds to an analytic function on neighborhood of $\sigma(T_n)$ such that $g(z)$ has no zero in $\sigma(T_q)$. Let $h(z) = \frac{1}{g(z)}$, with h being an analytic function on neighborhood of $\sigma(T_q)$. Note

$$\prod_{i=1}^{n} (z - \mu_i)^{m_i} = f(z)h(z).$$

Thus, by [18, Lemma 6.15], we have

$$\prod_{i=1}^{n}(I-T_q-\mu_i)^{m_i}=f(T_q)h(T_q).$$

As $f(0)=0$, then there exists a complex polynomial Q such that

$$Q(T_q)(I-T_q)=f(T_q)h(T_q).$$

Since $[h(T_q)]_a>0$ (because $g(T_q)$ is invertible) and $[f(T_q)]_a>0$, and using Proposition 2.1 (i), we conclude that $[Q(T_q)(I-T_q)]_a>0$. As a matter of fact, according to Theorem 1.2, the operator $[I-T_q]_a>0$. Then, $R((I-T)T^q)=R(I-T)_{|R(T^q)}$ is closed. Again, as $I-T$ and T^q are prime, then $R(I-T)$ is closed. Since $N(I-T)\subset R(T^q)$, then $\alpha(I-T)<\infty$. Thus, by using Corollary 2.2, we infer that $[I-T]_a>0$. □

The authors asserted in [17] that if T is a bounded linear operator acting on a Banach space X for which there exists a nonzero complex polynomial $P(z)=\sum_{r=0}^{p}a_rz^r$ satisfying $P(1)\neq 0, P(1)-a_0\neq 0$ and $P(T)\in\mathcal{K}(X)$, then $[I-T]_a>0$. In order to draw our results, we introduce the following set, denoted by $\mathcal{PDC}_n(X)$, $n\in\mathbb{N}$ and which is defined by

$$\mathcal{PDC}_n(X):=\bigcup_{p\in\mathbb{C}(z)\backslash\{0\},\ P(1)\neq 0}\mathcal{H}_P,$$

where

$$\mathcal{H}_P=\left\{T\in\mathcal{L}(X)\text{ such that }R(T^n)\text{ is closed and }\left[I-\frac{1}{P(1)}P(T_n)\right]_a>0\right\}.$$

Theorem 2.10. *Let $T\in\mathcal{L}(X)$ and $n\in\mathbb{N}$. The following statements are equivalent*

(i) $T\in\mathcal{PDC}_n(X)$.

(ii) $R(T^n-T^{n+1})$ *is closed and* $[I-T_n]_a>0$.

Proof. $(i)\implies(ii)$ Suppose that $T\in\mathcal{PDC}_n(X)$, then there exists a nonzero complex polynomial $P(z)=\sum_{j=0}^{p}a_jz^j$ such that $R(T^n)$ is closed and $\left[I-\frac{1}{P(1)}P(T_n)\right]_a>0$. As $I-T$ commutes with I, Newton's binomial formula allows us to write the following relation

$$T_n^j=I+\sum_{i=1}^{j}(-1)^iC_j^i(I-T_n)^i,\ \text{ for all } j\geq 1.$$

Performing a simple calculation, we find

$$P(T_n)=P(1)I+\sum_{j=1}^{p}a_j\left(\sum_{i=1}^{j}(-1)^iC_j^i(I-T_n)^i\right).$$

Since $P(1\neq 0$, we get

$$I-\frac{1}{P(1)}P(T_n)=-\frac{1}{P(1)}\sum_{j=1}^{p}a_j\left(\sum_{i=1}^{j}(-1)^iC_j^i(I-T_n)^i\right). \tag{2.2}$$

Now take $(x_k)_k$ a bounded sequence in $R(T^n)$ such that $(I-T_n)x_k$ converges in X. Using the boundedness of T together with the relation (2.2), we deduce that there exists $x \in R(T^n)$ such that

$$\left(I - \frac{1}{P(1)}P(T_n)\right)x_k \to x.$$

Now, owing to the demicompactness of $\frac{1}{P(1)}P(T_n)$, we conclude that $(x_k)_k$ admits a convergent subsequence in $R(T^n)$. Thus $[I-T_n]_a > 0$ and $R(T^n - T^{n+1}) = R(I-T_n)$ is then closed.

$(ii) \Longrightarrow (i)$ Since $R(T^n - T^{n+1}) = R((I-T)T^n)$ is closed, T^n and $I-T$ are prime, it follows that $R(T^n)$ are closed. Now, take $P(z) = z$, then $\left[I - \frac{1}{P(1)}P(T_n)\right]_a > 0$. Therefore, $T \in \mathcal{PDC}_n(X)$. □

3. The Jeribi essential spectra

We define the Jeribi essential spectra by

$$\sigma_{e1}(T) := \{\lambda \in \mathbb{C} \ : \ [\lambda - T]_a = 0\} = \mathbb{C}\backslash\{\lambda \in \mathbb{C} \ : \ [\lambda - T]_a > 0\}$$

$$\sigma_{e4}(T) := \{\lambda \in \mathbb{C} \ : \ [\lambda - T]_a = 0 \text{ or } [\overline{\lambda} - T^*]_a = 0\} = \mathbb{C}\backslash\{\lambda \in \mathbb{C} \ : \ [\lambda - T]_a > 0 \text{ and } [\overline{\lambda} - T^*]_a > 0\}$$

and the Schechter essential spectrum by

$$\sigma_{e5}(T) := \bigcap_{K \in \mathcal{K}(X)} \sigma(A+K).$$

Remark 3.1. Let $A \in \mathcal{L}(X)$. *If* $\mathbb{C}\backslash\sigma_{e4}(A)$ *is connected, then* $\sigma_{e4}(A) = \sigma_{e5}(A)$.

Theorem 3.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then, $\lambda \notin \sigma_{e5}(T)$ if and only if $[\lambda - T]_a > 0$, $[\overline{\lambda} - T^*]_a > 0$, and $i(\lambda - T) = 0$.*

Proof. If $\lambda \notin \sigma_{e5}(T)$, then there is $K \in \mathcal{K}(X)$ such that $\lambda \in \rho(T+K)$. In particular, $\lambda \in \Phi_{T+K}$ and $i(T+K-\lambda) = 0$. Since $K \in \mathcal{K}(X)$, then $\lambda \in \Phi_T$ and $i(T-\lambda) = 0$. Hence, by using Corollary 2.2, we have $[\lambda - T]_a > 0$, $[\overline{\lambda} - T^*]_a > 0$, and $i(\lambda - T) = 0$. To prove the converse, suppose that $[\lambda - T]_a > 0$, $[\overline{\lambda} - T^*]_a > 0$, and $i(\lambda - T) = 0$. By using Corollary 2.2, we have $\lambda \in \Phi_T$ and $i(T-\lambda) = 0$. Now the rest of the proof may be sketched in a similar way to that in [18, Theorem 7.27]. The details are therefore omitted. □

Corollary 3.1. *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then,*

$$\sigma_{e5}(T) := \mathbb{C}\backslash\{\lambda \in \mathbb{C} \ : \ [\lambda - T]_a > 0 \textit{ and } [\overline{\lambda} - T^*]_a > 0 \textit{ with } i(\lambda - T) = 0\}.$$

The following theorem shows the relation between the essential spectra of the sum of the two bounded linear operators and the essential spectra of each of these operators, where their products are compacts operators on a Banach space X.

Theorem 3.2. *Let A and B be two bounded linear operators on a Banach space X such that AB is a Riesz operator. If $AB - BA \in \mathcal{K}(X)$, then*

$$\sigma_{e1}(A+B)\backslash\{0\} = \left[\sigma_{e1}(A) \bigcup \sigma_{e1}(B)\right]\backslash\{0\} \tag{3.1}$$

Proof. Let $\lambda \in \mathbb{C}$. Using Eqs (3.6) and (3.7), we have

$$AB\,(A+B-\lambda) - (A+B-\lambda)\,AB = A\,(BA-AB) + (AB-BA)\,B \tag{3.2}$$

and

$$BA\,(A+B-\lambda) - (A+B-\lambda)\,BA = (BA-AB)\,A + B\,(AB-BA)\,. \tag{3.3}$$

Let $\lambda \notin \sigma_{e1}(A)\bigcup\sigma_{e1}(B)\bigcup\{0\}$, then $[A-\lambda]_a > 0$ and $[B-\lambda]_a > 0$. Using Proposition 2.1 (i), we have $[(A-\lambda)\,(B-\lambda)]_a > 0$. Since $AB-BA \in \mathcal{K}(X)$, we can apply Eq. (3.2), we infer that $\lambda AB\,(A+B-\lambda) - \lambda\,(A+B-\lambda)\,AB \in \mathcal{K}(X)$. Also, since $AB \in \mathcal{R}(X)$, then by Lemma 1.1 and Eq. (3.6), $[A+B-\lambda]_a > 0$. So, $\lambda \notin \sigma_{e1}(A+B)$. Therefore

$$\sigma_{e1}(A+B)\backslash\{0\} \subset \left[\sigma_{e1}(A)\bigcup\sigma_{e1}(B)\right]\backslash\{0\}. \tag{3.4}$$

Suppose $\lambda \notin \sigma_{e1}(A+B)\bigcup\{0\}$ then $[A+B-\lambda]_a > 0$. Since $AB-BA \in \mathcal{K}(X)$, then by Eqs (3.2) and (3.3), we have $\lambda AB\,(A+B-\lambda) - \lambda\,(A+B-\lambda)\,AB \in \mathcal{K}(X)$ and $\lambda BA\,(A+B-\lambda) - \lambda\,(A+B-\lambda)\,BA \in \mathcal{K}(X)$. Also, since $AB \in \mathcal{R}(X)$ and $BA \in \mathcal{R}(X)$, then by Eqs (3.6), (3.7) and Lemma 1.1 (i), we have $[(A-\lambda)\,(B-\lambda)]_a > 0$ and $[(B-\lambda)\,(A-\lambda)]_a > 0$. Again, using Theorem 1.2, we have $[A-\lambda]_a > 0$ and $[B-\lambda]_a > 0$. Hence $\lambda \notin \sigma_{e1}(A)\bigcup\sigma_{e1}(B)$. Therefore $[\sigma_{e1}(A)\bigcup\sigma_{e1}(B)]\,\backslash\{0\} \subset \sigma_{e1}(A+B)\backslash\{0\}$. This proves that Eq. (3.1). □

Theorem 3.3. *Let A and B be two bounded linear operators on a Banach space X.*

(i) *If* $AB \in \mathcal{K}(X)$, *then* $\sigma_{ei}(A+B)\backslash\{0\} \subset [\sigma_{ei}(A)\bigcup\sigma_{ei}(B)]\,\backslash\{0\}$, $i = 4,\ 5$. *Furthermore, if* $BA \in \mathcal{K}(X)$, *then* $\sigma_{e4}(A+B)\backslash\{0\} = [\sigma_{e4}(A)\bigcup\sigma_{e4}(B)]\,\backslash\{0\}$. *Moreover, if* $\mathbb{C}\backslash\sigma_{e4}(A)$ *is connected, then*

$$\sigma_{e5}(A+B)\backslash\{0\} = \left[\sigma_{e5}(A)\bigcup\sigma_{e5}(B)\right]\backslash\{0\}. \tag{3.5}$$

Proof. For $\lambda \in \mathbb{C}$, we can write

$$(A-\lambda)(B-\lambda) = AB - \lambda(A+B-\lambda), \tag{3.6}$$

and

$$(B-\lambda)(A-\lambda) = BA - \lambda(A+B-\lambda). \tag{3.7}$$

(i) Let $\lambda \notin \sigma_{e4}(A)\bigcup\sigma_{e4}(B)\bigcup\{0\}$. Then, $[A-\lambda]_a > 0$ and $[B-\lambda]_a > 0$. Proposition 2.1 (i) ensures that $[(A-\lambda)(B-\lambda)]_a > 0$. Since $AB \in \mathcal{K}(X)$, and applying Eq. (3.6), we have $[A+B-\lambda]_a > 0$. Hence, $\lambda \notin \sigma_{e4}(A+B)$, and we obtain

$$\sigma_{e4}(A+B)\backslash\{0\} \subset \left[\sigma_{e4}(A)\bigcup\sigma_{e4}(B)\right]\backslash\{0\}. \tag{3.8}$$

Let $\lambda \notin \sigma_{e5}(A)\bigcup\sigma_{e5}(B)\bigcup\{0\}$. Then, by using Theorem 3.1, we get $[A-\lambda]_a > 0$, $[A^*-\overline{\lambda}]_a > 0$, $i(A-\lambda) = 0$, $[B-\lambda]_a > 0$, $[B^*-\overline{\lambda}]_a > 0$ and $i(B-\lambda) = 0$ and therefore, Proposition 2.1 (i) gives $[(A-\lambda)(B-\lambda)]_a > 0$, $[(A^*-\overline{\lambda})(B^*-\overline{\lambda})]_a > 0$ and $i((A-\lambda)(B-\lambda)) = 0$. Moreover, since $AB \in \mathcal{K}(X)$, we can apply both Eq. (3.6) and Proposition 2.1 (ii), hence ensuring that

$[A+B-\lambda]_a > 0$, $[A^*+B^*-\overline{\lambda}]_a > 0$ and $i(A+B-\lambda)=0$. Again, by applying Theorem 3.1, we infer that $\lambda \notin \sigma_{e5}(A+B)$ and, then

$$\sigma_{e5}(A+B)\backslash\{0\} \subset \Big[\sigma_{e5}(A) \bigcup \sigma_{e5}(B)\Big] \backslash\{0\}. \tag{3.9}$$

In order to prove the inverse inclusions of Eqs (3.8) and (3.9), let us suppose that $\lambda \notin \sigma_{e4}(A+B) \bigcup\{0\}$. Then, $[A+B-\lambda]_a > 0$ and $[A^*+B^*-\overline{\lambda}]_a > 0$. Since $AB \in \mathcal{K}(X)$ and $BA \in \mathcal{K}(X)$, then by using Eqs (3.6) and (3.7), we have

$$[(A-\lambda)(B-\lambda)]_a > 0,\ \ [(A^*-\overline{\lambda})(B^*-\overline{\lambda})]_a > 0,\ \ [(B-\lambda)(A-\lambda)]_a > 0 \text{ and } [(B^*-\overline{\lambda})(A^*-\overline{\lambda})]_a > 0. \tag{3.10}$$

Eq. (3.10) and Theorem 2.3 show clearly that $[A-\lambda]_a > 0$, $[A^*-\overline{\lambda}]_a > 0$, $[B-\lambda]_a > 0$, and $[B^*-\overline{\lambda}]_a > 0$. Therefore, $\lambda \notin \sigma_{e4}(A) \bigcup \sigma_{e4}(B)$. This proves that $[\sigma_{e4}(A) \bigcup \sigma_{e4}(B)] \backslash\{0\} \subset \sigma_{e4}(A+B)\backslash\{0\}$. Hence, $\sigma_{e4}(A+B)\backslash\{0\} = [\sigma_{e4}(A) \bigcup \sigma_{e4}(B)] \backslash\{0\}$. It remains to prove the following $[\sigma_{e5}(A) \bigcup \sigma_{e5}(B)] \backslash\{0\} \subset \sigma_{e5}(A+B)\backslash\{0\}$. Let $\lambda \notin \sigma_{e5}(A+B) \bigcup\{0\}$. Then, by using Theorem 3.1, we have $[A+B-\lambda]_a > 0$, $[A^*+B^*-\overline{\lambda}]_a > 0$, and $i(A+B-\lambda)=0$. Since $AB \in \mathcal{K}(X)$ and $BA \in \mathcal{K}(X)$, it is easy to deduce that $[A-\lambda]_a > 0$, $[A^*-\overline{\lambda}]_a > 0$, $[B-\lambda]_a > 0$, and $[B^*-\overline{\lambda}]_a > 0$. Again, the use of Eqs (3.6), (3.10), Theorem 2.4 and Lemma 2.1 (i) allows us to have

$$i[(A-\lambda)(B-\lambda)] = i(A-\lambda)+i(B-\lambda) = i(A+B-\lambda) = 0. \tag{3.11}$$

Since A is a bounded linear operator, we get $\rho(A) \neq \emptyset$. As $\mathbb{C}\backslash\sigma_{e4}(A)$ is a connected set, and from Remark 3.1, we deduce that $\sigma_{e4}(A) = \sigma_{e5}(A)$. Using the last equality and the fact that $[A-\lambda]_a > 0$ and $[A^*-\overline{\lambda}]_a > 0$, we deduce that $i(A-\lambda)=0$. It follows, from Eq. (3.11), that $i(B-\lambda)=0$. We conclude that $\lambda \notin \sigma_{e5}(A) \bigcup \sigma_{e5}(B)$ and hence, we have $[\sigma_{e5}(A) \bigcup \sigma_{e5}(B)] \backslash\{0\} \subset \sigma_{e5}(A+B)\backslash\{0\}$. So, we prove Eq. (3.5). □

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