Int. J. Anal. Appl. (2025), 23:114

International Journal of Analysis and Applications

A Study on W_6 and W_8-Curvature Tensors on $(LPK)_n$-Manifold with a Quarter-Symmetric Metric Connection

Rajendra Prasad[1], Abdul Haseeb[2,*], Sushmita Sen[1], Mohd Bilal[3]

[1]*Department of Mathematics and Astronomy, University of Lucknow, Lucknow-226007, India*
[2]*Department of Mathematics, College of Science, Jazan University, P.O. Box. 114, Jazan 45142, Kingdom of Saudi Arabia*
[3]*Department of Mathematical Sciences, Faculty of Applied Sciences, Umm Al-Qura University, Makkah 21955, Saudi Arabia*

[*]*Corresponding author:* haseeb@jazanu.edu.sa, malikhaseeb80@gmail.com

Abstract. The present paper deals with the study of W_6 and W_8-curvature tensors in an n-dimensional Lorentzian para-Kenmotsu manifold (briefly, $(LPK)_n$-manifold) with a quarter-symmetric metric connection.

1. Introduction

In 1971, a class of contact Riemannian manifolds satisfying some special conditions was proposed by Kenmotsu [1], we call it Kenmotsu manifold. After that Kenmotsu manifolds have been studied by many authors, such as Yoldas and Yasar [2]; Jun, De and Pathak [3]; Prasad, Haseeb and Pooja [4] and many others. In 1976, the concept of almost paracontact manifolds was proposed by Sato [5]. An almost paracontact structure on a semi-Riemannian manifold was established by Kaneyuki and Kozai in [6]. They constructed almost paracomplex structure on $M \times R$. According to Kaneyuki et al. [7], the key variation among almost paracontact manifolds is the signature of the metric. In 1995, the authors Sinha and Prasad studied para-Kenmotsu as well as special para-Kenmotsu manifolds and found their significant properties [8]. Afterwards, para-Kenmotsu manifolds drew huge attention of geometers, and significant characteristics of such manifolds have been obtained. Semi-Riemannian geometry, used in the relativity theory, was studied by Neill [9]. About four decades ago, Kaigorodov has explored the curvature structure of the spacetime [10].

Received: Feb. 24, 2025.

2020 *Mathematics Subject Classification.* 53C15, 53C25.

Key words and phrases. W_6-curvature tensor; W_8-curvature tensor; Lorentzian para-Kenmotsu manifolds; Riemann curvature tensor; Ricci tensor.

https://doi.org/10.28924/2291-8639-23-2025-114
ISSN: 2291-8639

Raychaudhuri et al. [11] extended the concept of general theory of spacetime. Recently, Haseeb and Prasad introduced and studied Lorentzian para-Kenmotsu manifolds of dimension n (briefly, $(LPK)_n$-manifold) [12, 13].

In 1975, the notion of quarter-symmetric connection in a differentiable manifold was defined and studied by Golab [14]. A quarter-symmetric metric connection has been studied by many geometers in many ways to a different extent as Mandal and De [15], Ahmad et al. [16], Prasad and Haseeb [17] and others. In [18], Pokhariyal have defined W_6 and W_8-curvature tensors, and it is shown that if the divergence of W_6-curvature tensor in an electromagnetic field vanishes then it is a purely electric field.

This paper has been organized in the following way: Section 2 contains preliminaries, where some fundamental results are given. In Section 3, curvature tensor of $(LPK)_n$-manifolds with a quarter-symmetric metric connection is described. In Section 4, we express W_6 and W_8 curvature tensors of $(LPK)_n$-manifolds with a quarter-symmetric metric connection. In Section 5, non-flatness of W_6 and W_8-curvature tensors in $(LPK)_n$-manifold with a quarter-symmetric metric connection are discussed. In Section 6, the relation between W_6 and $\bar{W}_8$-curvature tensors is established. In Section 7, we study $(LPK)_n$ manifolds with a quarter-symmetric metric connection satisfying the conditions $\bar{W}_6 \cdot \bar{R}=0$ and $\bar{W}_8 \cdot \bar{R}=0$.

2. Preliminaries

We begin this section with the following definition:

Definition 2.1. *A differentiable manifold M is said to be a Lorentzian manifold, if M has a Lorentzian metric g which is a symmetric non-degenerate (0, 2) tensor field of index 1. Since the Lorentzian metric g is of index 1, therefore, Lorentzian manifolds M has not only spacelike vector fields but also lightlike and timelike vector fields.*

Definition 2.2. *Let M be an n-dimensional Lorentzian metric manifold. If it is endowed with a structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is a Lorentzian metric, satisfying*

$$\phi^2 \mathscr{X}_1 = \mathscr{X}_1 + \eta(\mathscr{X}_1)\xi, \qquad g(\phi\mathscr{X}_1, \phi\mathscr{X}_2) = g(\mathscr{X}_1, \mathscr{X}_2) + \eta(\mathscr{X}_1)\eta(\mathscr{X}_2), \qquad \eta(\xi) = -1, \tag{2.1}$$

$$g(\mathscr{X}_1, \xi) = \eta(\mathscr{X}_1), \qquad \phi\xi = 0, \qquad \eta(\phi\mathscr{X}_1) = 0, \qquad \Phi(\mathscr{X}_1, \mathscr{X}_2) = \Phi(\mathscr{X}_2, \mathscr{X}_1), \tag{2.2}$$

for any vector fields $\mathscr{X}_1, \mathscr{X}_2 \in \chi(M)$: the set of all differentiable vector fields on M, where $\Phi(\mathscr{X}_1, \mathscr{X}_2) = g(\mathscr{X}_1, \phi\mathscr{X}_2)$, then it is called Lorentzian almost paracontact manifold.

Definition 2.3. *A Lorentzian almost paracontact manifold M is called $(LPK)_n$-manifold, if [12]*

$$(\nabla_{\mathscr{X}_1}\phi)\mathscr{X}_2 = -g(\phi\mathscr{X}_1, \mathscr{X}_2)\xi - \eta(\mathscr{X}_2)\phi\mathscr{X}_1,$$

where $\mathscr{X}_1, \mathscr{X}_2 \in \chi(M)$.

In an $(LPK)_n$-manifold, we have [12]

$$\nabla_{\mathscr{X}_1}\xi = -\mathscr{X}_1 - \eta(\mathscr{X}_1)\xi,$$

$$(\nabla_{\mathscr{X}_1}\eta)\mathscr{X}_2 = -g(\mathscr{X}_1, \mathscr{X}_2) - \eta(\mathscr{X}_1)\eta(\mathscr{X}_2),$$

where ∇ denotes the operator of the covariant differentiation with respect to the Lorentzian metric g.

Further, in an $(LPK)_n$-manifold, the following relations hold [12,19]:

$$\eta(R(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3) = g(\mathscr{X}_2, \mathscr{X}_3)\eta(\mathscr{X}_1) - g(\mathscr{X}_1, \mathscr{X}_3)\eta(\mathscr{X}_2), \tag{2.3}$$

$$R(\xi, \mathscr{X}_1)\mathscr{X}_2 = g(\mathscr{X}_1, \mathscr{X}_2)\xi - \eta(\mathscr{X}_2)\mathscr{X}_1, \tag{2.4}$$

$$R(\mathscr{X}_1, \mathscr{X}_2)\xi = \eta(\mathscr{X}_2)\mathscr{X}_1 - \eta(\mathscr{X}_1)\mathscr{X}_2, \tag{2.5}$$

$$R(\xi, \mathscr{X}_1)\xi = \mathscr{X}_1 + \eta(\mathscr{X}_1)\xi, \tag{2.6}$$

$$S(\mathscr{X}_1, \xi) = (n-1)\eta(\mathscr{X}_1), \;\; S(\xi, \xi) = -(n-1), \tag{2.7}$$

$$Q\xi = (n-1)\xi, \tag{2.8}$$

$$S(\phi\mathscr{X}_1, \phi\mathscr{X}_2) = S(\mathscr{X}_1, \mathscr{X}_2) + (n-1)\eta(\mathscr{X}_1)\eta(\mathscr{X}_2), \tag{2.9}$$

for any $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \in \chi(M)$, here R, S and Q are the curvature tensor, the Ricci tensor, and the Ricci operator of $M(\phi, \xi, \eta, g)$, respectively.

Definition 2.4. *A linear connection $\bar{\nabla}$ defined on (M, g) is said to be a quarter-symmetric connection [14], if its torsion tensor T*

$$T(\mathscr{X}_1, \mathscr{X}_2) = \bar{\nabla}_{\mathscr{X}_1}\mathscr{X}_2 - \bar{\nabla}_{\mathscr{X}_2}\mathscr{X}_1 - [\mathscr{X}_1, \mathscr{X}_2]$$

satisfies

$$T(\mathscr{X}_1, \mathscr{X}_2) = \eta(\mathscr{X}_2)\phi\mathscr{X}_1 - \eta(\mathscr{X}_1)\phi\mathscr{X}_2,$$

where η is a 1-form and ϕ is a $(1,1)$-tensor field. Moreover, if a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_{\mathscr{X}_1}g)(\mathscr{X}_2, \mathscr{X}_3) = 0,$$

where $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \in \chi(M)$, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection.

A relation between a quarter-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in an $(LPK)_n$-manifold M is given by

$$\bar{\nabla}_{\mathscr{X}_1}\mathscr{X}_2 = \nabla_{\mathscr{X}_1}\mathscr{X}_2 + \eta(\mathscr{X}_2)\phi\mathscr{X}_1 - g(\phi\mathscr{X}_1, \mathscr{X}_2)\xi. \tag{2.10}$$

3. Curvature tensor of $(LPK)_n$-manifold with a quarter-symmetric metric connection

The curvature tensor $\bar{R}$ with a quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 = \bar{\nabla}_{\mathscr{X}_1}\bar{\nabla}_{\mathscr{X}_2}\mathscr{X}_3 - \bar{\nabla}_{\mathscr{X}_2}\bar{\nabla}_{\mathscr{X}_1}\mathscr{X}_3 - \bar{\nabla}_{[\mathscr{X}_1,\mathscr{X}_2]}\mathscr{X}_3. \tag{3.1}$$

Using the relation (2.10) in (3.1), we have

$$\begin{aligned}\bar{R}(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 &= R(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 + g(\mathscr{X}_2, \mathscr{X}_3)\phi\mathscr{X}_1 - g(\mathscr{X}_1, \mathscr{X}_3)\phi\mathscr{X}_2 + g(\phi\mathscr{X}_2, \mathscr{X}_3)\mathscr{X}_1 \\ &\quad - g(\phi\mathscr{X}_1, \mathscr{X}_3)\mathscr{X}_2 + g(\phi\mathscr{X}_2, \mathscr{X}_3)\phi\mathscr{X}_1 - g(\phi\mathscr{X}_1, \mathscr{X}_3)\phi\mathscr{X}_2,\end{aligned} \tag{3.2}$$

where

$$R(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 = \nabla_{\mathscr{X}_1}\nabla_{\mathscr{X}_2}\mathscr{X}_3 - \nabla_{\mathscr{X}_2}\nabla_{\mathscr{X}_1}\mathscr{X}_3 - \nabla_{[\mathscr{X}_1,\mathscr{X}_2]}\mathscr{X}_3,$$

is the Riemannian curvature tensor of the connection ∇. Taking the inner product of (3.2) with $\mathscr{U}_3$, we have

$$\begin{aligned}\bar{R}(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_3) &= R(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_3) + g(\mathscr{X}_2, \mathscr{X}_3)g(\phi\mathscr{X}_1, \mathscr{U}_3) - g(\mathscr{X}_1, \mathscr{X}_3)g(\phi\mathscr{X}_2, \mathscr{U}_3) \\ &+ g(\phi\mathscr{X}_2, \mathscr{X}_3)g(\mathscr{X}_1, \mathscr{U}_3) - g(\phi\mathscr{X}_1, \mathscr{X}_3)g(\mathscr{X}_2, \mathscr{U}_3) + g(\phi\mathscr{X}_2, \mathscr{X}_3)g(\phi\mathscr{X}_1, \mathscr{U}_3) \\ &- g(\phi\mathscr{X}_1, \mathscr{X}_3)g(\phi\mathscr{X}_2, \mathscr{U}_3),\end{aligned} \tag{3.3}$$

where $\bar{R}(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_3) = g(\bar{R}(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3, \mathscr{U}_3)$, and $R(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_3) = g(R(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3, \mathscr{U}_3)$.

Contracting (3.3) over $\mathscr{X}_1$ and $\mathscr{U}_3$, we obtain

$$\bar{S}(\mathscr{X}_2, \mathscr{X}_3) = S(\mathscr{X}_2, \mathscr{X}_3) + (n + \psi - 2)g(\phi\mathscr{X}_2, \mathscr{X}_3) + (\psi - 1)g(\mathscr{X}_2, \mathscr{X}_3) - \eta(\mathscr{X}_2)\eta(\mathscr{X}_3), \tag{3.4}$$

where, S and $\bar{S}$ are the Ricci tensors of the connections ∇ and $\bar{\nabla}$, respectively on M, and ψ = trace ϕ.

Replacing $\mathscr{X}_3 = \xi$ in (3.4), and using (2.2) and (2.7), we have

$$\bar{S}(\mathscr{X}_2, \xi) = (n + \psi - 1)\eta(\mathscr{X}_2). \tag{3.5}$$

From (3.4), we have

$$\bar{Q}\mathscr{X}_2 = Q\mathscr{X}_2 + (n + \psi - 2)\phi\mathscr{X}_2 + (\psi - 1)\mathscr{X}_2 - \eta(\mathscr{X}_2)\xi, \tag{3.6}$$

where Q and $\bar{Q}$ are the Ricci operators of the connections ∇ and $\bar{\nabla}$, respectively on M.

Replacing $\mathscr{X}_3 = \xi$ in (3.6), we have

$$\bar{Q}\xi = (n + \psi - 1)\xi. \tag{3.7}$$

Contracting (3.4) over $\mathscr{X}_2$ and $\mathscr{X}_3$, we find

$$\bar{r} = r + (2\psi - 1)(n - 1) + \psi^2, \tag{3.8}$$

where r and $\bar{r}$ are the scalar curvatures of the connections ∇ and $\bar{\nabla}$, respectively on M.

From (3.2), we deduce

$$\bar{R}(\xi, \mathscr{X}_2)\mathscr{X}_3 = g(\mathscr{X}_2, \mathscr{X}_3)\xi - \eta(\mathscr{X}_3)\mathscr{X}_2 - \eta(\mathscr{X}_3)\phi\mathscr{X}_2 + g(\phi\mathscr{X}_2, \mathscr{X}_3)\xi, \tag{3.9}$$

$$\bar{R}(\mathscr{X}_1,\xi)\mathscr{X}_3=\eta(\mathscr{X}_3)\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_3)\xi+\eta(\mathscr{X}_3)\phi\mathscr{X}_1-g(\phi\mathscr{X}_1,\mathscr{X}_3)\xi, \tag{3.10}$$

$$\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\xi=\eta(\mathscr{X}_2)\mathscr{X}_1-\eta(\mathscr{X}_1)\mathscr{X}_2+\eta(\mathscr{X}_2)\phi\mathscr{X}_1-\eta(\mathscr{X}_1)\phi\mathscr{X}_2, \tag{3.11}$$

for any $\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3\in\chi(M)$.

4. Expression of W_6 and W_8-curvature tensors on $(LPK)_n$-manifolds with a quarter-symmetric metric connection

The W_6-curvature tensor field in a Riemannian manifold is defined as [20,21]

$$W_6(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3=R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{1}{n-1}[g(\mathscr{X}_1,\mathscr{X}_2)Q\mathscr{X}_3-S(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1], \tag{4.1}$$

for any $\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3\in\chi(M)$.

Analogous to (4.1), the W_6-curvature tensor in an $(LPK)_n$-manifold with a quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{W}_6(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3=\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{1}{n-1}[g(\mathscr{X}_1,\mathscr{X}_2)\bar{Q}\mathscr{X}_3-\bar{S}(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1]. \tag{4.2}$$

By using (3.2) and (3.6) in (4.2), we have

$$\begin{aligned}\bar{W}_6(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 &= R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{1}{n-1}[g(\mathscr{X}_1,\mathscr{X}_2)Q\mathscr{X}_3-S(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1]\\ &+ \frac{n+\psi-2}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\phi\mathscr{X}_3+g(\mathscr{X}_2,\mathscr{X}_3)\phi\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_3)\phi\mathscr{X}_2\\ &+ \frac{\psi-1}{n-1}[g(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3-g(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-g(\phi\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1]-g(\phi\mathscr{X}_1,\mathscr{X}_3)\mathscr{X}_2\\ &+ g(\phi\mathscr{X}_2,\mathscr{X}_3)\phi\mathscr{X}_1-g(\phi\mathscr{X}_1,\mathscr{X}_3)\phi\mathscr{X}_2\\ &+ \frac{1}{n-1}[\eta(\mathscr{X}_2)\eta(\mathscr{X}_3)\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{X}_3)\xi].\end{aligned} \tag{4.3}$$

Taking $\mathscr{X}_1=\xi$, $\mathscr{X}_2=\xi$ and $\mathscr{X}_3=\xi$ in (4.3), we respectively have

$$\begin{aligned}\bar{W}_6(\xi,\mathscr{X}_2)\mathscr{X}_3 &= \frac{n-\psi}{n-1}g(\mathscr{X}_2,\mathscr{X}_3)\xi-\frac{\psi-1}{n-1}g(\phi\mathscr{X}_2,\mathscr{X}_3)\xi+\frac{\psi-1}{n-1}\eta(\mathscr{X}_2)\mathscr{X}_3-\eta(\mathscr{X}_3)\mathscr{X}_2\\ &+ \frac{n+\psi-2}{n-1}\eta(\mathscr{X}_2)\phi\mathscr{X}_3-\eta(\mathscr{X}_3)\phi\mathscr{X}_2+\frac{1}{n-1}[\eta(\mathscr{X}_2)Q\mathscr{X}_3-S(\mathscr{X}_2,\mathscr{X}_3)\xi],\end{aligned} \tag{4.4}$$

$$\begin{aligned}\bar{W}_6(\mathscr{X}_1,\xi)\mathscr{X}_3 &= \frac{\psi-1}{n-1}\eta(\mathscr{X}_1)\mathscr{X}_3-\frac{\psi}{n-1}\eta(\mathscr{X}_3)\mathscr{X}_1+\frac{n+\psi-2}{n-1}\eta(\mathscr{X}_1)\phi\mathscr{X}_3+\eta(\mathscr{X}_3)\phi\mathscr{X}_1\\ &- g(\mathscr{X}_1,\mathscr{X}_3)\xi-g(\phi\mathscr{X}_1,\mathscr{X}_3)\xi-\frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_3)\xi+\frac{1}{n-1}\eta(\mathscr{X}_1)Q\mathscr{X}_3,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\bar{W}_6(\mathscr{X}_1,\mathscr{X}_2)\xi &= -\eta(\mathscr{X}_1)\mathscr{X}_2-\frac{\psi}{n-1}\eta(\mathscr{X}_2)\mathscr{X}_1+\eta(\mathscr{X}_2)\phi\mathscr{X}_1\\ &\quad -\eta(\mathscr{X}_1)\phi\mathscr{X}_2+\frac{n+\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\xi.\end{aligned} \tag{4.6}$$

Taking the inner product of (4.3) with ξ, we get

$$\begin{aligned}\eta(\bar{W}_6(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3) &= \frac{n+\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{X}_3)+\frac{n-\psi}{n-1}g(\mathscr{X}_2,\mathscr{X}_3)\eta(\mathscr{X}_1) \\ &- g(\mathscr{X}_1,\mathscr{X}_3)\eta(\mathscr{X}_2)-\frac{\psi-1}{n-1}g(\phi\mathscr{X}_2,\mathscr{X}_3)\eta(\mathscr{X}_1)-g(\phi\mathscr{X}_1,\mathscr{X}_3)\eta(\mathscr{X}_2) \\ &+ \frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\eta(\mathscr{X}_3)-\frac{1}{n-1}S(\mathscr{X}_2,\mathscr{X}_3)\eta(\mathscr{X}_1).\end{aligned} \tag{4.7}$$

Now, the W_8-curvature tensor in Kenmotsu manifolds with the Levi-Civita connection ∇ is given by

$$W_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 = R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3-\frac{1}{n-1}[S(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-S(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3]. \tag{4.8}$$

Analogous to (4.8), the W_8-curvature tensor in $(LPK)_n$-manifold with a quarter-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 = \bar{R}(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3-\frac{1}{n-1}[\bar{S}(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-\bar{S}(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3]. \tag{4.9}$$

By using (3.2) and (3.4) in (4.9), we have

$$\begin{aligned}&\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 \\ =\ & R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3-\frac{1}{n-1}[S(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-S(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3] \\ -\ & \frac{\psi-1}{n-1}[g(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3]+g(\mathscr{X}_2,\mathscr{X}_3)\phi\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_3)\phi\mathscr{X}_2 \\ +\ & \frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3-\frac{\psi-1}{n-1}g(\phi\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-g(\phi\mathscr{X}_1,\mathscr{X}_3)\mathscr{X}_2 \\ +\ & g(\phi\mathscr{X}_2,\mathscr{X}_3)\phi\mathscr{X}_1-g(\phi\mathscr{X}_1,\mathscr{X}_3)\phi\mathscr{X}_2+\frac{1}{n-1}[\eta(\mathscr{X}_2)\eta(\mathscr{X}_3)\mathscr{X}_1-\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\mathscr{X}_3].\end{aligned} \tag{4.10}$$

Taking $\mathscr{X}_1=\xi$, $\mathscr{X}_2=\xi$ and $\mathscr{X}_3=\xi$ in (4.10), we respectively have

$$\begin{aligned}\bar{W}_8(\xi,\mathscr{X}_2)\mathscr{X}_3 &= \frac{n-\psi}{n-1}g(\mathscr{X}_2,\mathscr{X}_3)\xi-\eta(\mathscr{X}_3)\mathscr{X}_2+\frac{n+\psi-1}{n-1}\eta(\mathscr{X}_2)\mathscr{X}_3-\eta(\mathscr{X}_3)\phi\mathscr{X}_2 \\ &- \frac{\psi-1}{n-1}g(\phi\mathscr{X}_2,\mathscr{X}_3)\xi+\frac{1}{n-1}\eta(\mathscr{X}_2)\eta(\mathscr{X}_3)\xi-\frac{1}{n-1}S(\mathscr{X}_2,\mathscr{X}_3)\xi.\end{aligned} \tag{4.11}$$

$$\begin{aligned}\bar{W}_8(\mathscr{X}_1,\xi)\mathscr{X}_3 &= \frac{n+\psi-1}{n-1}\eta(\mathscr{X}_1)\mathscr{X}_3-\frac{\psi}{n-1}\eta(\mathscr{X}_3)\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_3)\xi \\ &\quad +\eta(\mathscr{X}_3)\phi\mathscr{X}_1-g(\phi\mathscr{X}_1,\mathscr{X}_3)\xi.\end{aligned} \tag{4.12}$$

$$\begin{aligned}\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\xi &= \frac{\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\xi+\frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\xi \\ &\quad -\eta(\mathscr{X}_1)\mathscr{X}_2-\frac{\psi}{n-1}\eta(\mathscr{X}_2)\mathscr{X}_1+\eta(\mathscr{X}_2)\phi\mathscr{X}_1-\eta(\mathscr{X}_1)\phi\mathscr{X}_2 \\ &\quad -\frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\xi+\frac{1}{n-1}S(\mathscr{X}_1,\mathscr{X}_2)\xi.\end{aligned} \tag{4.13}$$

Taking the inner product of equation (4.10) with ξ, we have

$$\begin{aligned} & \eta(\bar{W}_8(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3) \\ = & \frac{\psi-1}{n-1} g(\mathscr{X}_1, \mathscr{X}_2)\eta(\mathscr{X}_3) + \frac{n-\psi}{n-1} g(\mathscr{X}_2, \mathscr{X}_3)\eta(\mathscr{X}_1) \\ - & g(\mathscr{X}_1, \mathscr{X}_3)\eta(\mathscr{X}_2) + \frac{n+\psi-2}{n-1} g(\phi\mathscr{X}_1, \mathscr{X}_2)\eta(\mathscr{X}_3) - \frac{\psi-1}{n-1} g(\phi\mathscr{X}_2, \mathscr{X}_3)\eta(\mathscr{X}_1) \\ - & g(\phi\mathscr{X}_1, \mathscr{X}_3)\eta(\mathscr{X}_2) - \frac{1}{n-1}[S(\mathscr{X}_2, \mathscr{X}_3)\eta(\mathscr{X}_1) - S(\mathscr{X}_1, \mathscr{X}_2)\eta(\mathscr{X}_3)]. \end{aligned} \tag{4.14}$$

5. Non-flatness of W_6 and W_8-curvature tensor in $(LPK)_n$-manifolds with a quarter-symmetric metric connection

First, we consider non-flatness of W_6-curvature tensor in $(LPK)_n$-manifolds with a quarter-symmetric metric connection. We prove our fact by contradiction, i.e., we assume that an $(LPK)_n$-manifold with a quarter-symmetric metric connection is W_6-flat, i.e.,

$$\bar{W}_6(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 = 0, \tag{5.1}$$

for any $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \in \chi(M)$.

Taking the inner product of (4.2) with $\mathscr{U}_2$, we have

$$\begin{aligned} & \bar{W}_6(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_2) \\ & = \bar{R}(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_2) + \frac{1}{n-1}[g(\mathscr{X}_1, \mathscr{X}_2)g(\bar{Q}\mathscr{X}_3, \mathscr{U}_2) - \bar{S}(\mathscr{X}_2, \mathscr{X}_3)g(\mathscr{X}_1, \mathscr{U}_2)]. \end{aligned} \tag{5.2}$$

By using equation (5.1), (5.2) gives

$$\bar{R}(\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3, \mathscr{U}_2) = \frac{1}{n-1}[\bar{S}(\mathscr{X}_2, \mathscr{X}_3)g(\mathscr{X}_1, \mathscr{U}_2) - g(\mathscr{X}_1, \mathscr{X}_2)g(\bar{Q}\mathscr{X}_3, \mathscr{U}_2)].$$

Taking $\mathscr{X}_3 = \xi$ in the above equation, we have

$$\bar{R}(\mathscr{X}_1, \mathscr{X}_2, \xi, \mathscr{U}_2) = \frac{1}{n-1}[\bar{S}(\mathscr{X}_2, \xi)g(\mathscr{X}_1, \mathscr{U}_2) - g(\mathscr{X}_1, \mathscr{X}_2)g(\bar{Q}\xi, \mathscr{U}_2)]. \tag{5.3}$$

Using equations (3.5) and (3.7) in (5.3), we find

$$\bar{R}(\mathscr{X}_1, \mathscr{X}_2, \xi, \mathscr{U}_2) = \frac{n+\psi-1}{n-1}[g(\mathscr{X}_1, \mathscr{U}_2)\eta(\mathscr{X}_2) - g(\mathscr{X}_1, \mathscr{X}_2)\eta(\mathscr{U}_2)] \neq 0. \tag{5.4}$$

Hence, the above relation leads to the following theorem:

Theorem 5.1. *In an $(LPK)_n$-manifold with a quarter-symmetric metric connection, $\bar{W}_6$-curvature tensor is not flat.*

Next, we consider non-flatness of W_8-curvature tensor in $(LPK)_n$-manifolds with a quarter-symmetric metric connection. We assume that an $(LPK)_n$-manifold with a quarter-symmetric metric connection is W_8-flat, i.e.,

$$\bar{W}_8(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 = 0, \tag{5.5}$$

for all $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3 \in \chi(M)$.

By taking the inner product of (4.9) with $\mathscr{U}_2$, we have

$$\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)=\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)-\frac{1}{n-1}[\bar{S}(\mathscr{X}_2,\mathscr{X}_3)g(\mathscr{X}_1,\mathscr{U}_2)-\bar{S}(\mathscr{X}_1,\mathscr{X}_2)g(\mathscr{X}_3,\mathscr{U}_2)]. \tag{5.6}$$

By using (5.5), (5.6) gives

$$\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)=\frac{1}{n-1}[\bar{S}(\mathscr{X}_2,\mathscr{X}_3)g(\mathscr{X}_1,\mathscr{U}_2)-\bar{S}(\mathscr{X}_1,\mathscr{X}_2)g(\mathscr{X}_3,\mathscr{U}_2)]. \tag{5.7}$$

Taking $\mathscr{X}_3=\xi$ in (5.7), we have

$$\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)=\frac{1}{n-1}[\bar{S}(\mathscr{X}_2,\xi)g(\mathscr{X}_1,\mathscr{U}_2)-\bar{S}(\mathscr{X}_1,\mathscr{X}_2)g(\xi,\mathscr{U}_2)].$$

In view of (3.4) and (3.5), the last equation tuns to

$$\begin{aligned}\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2) &= \frac{1}{n-1}[(n+\psi-1)\eta(\mathscr{X}_2)g(\mathscr{X}_1,\mathscr{U}_2)-(\psi-1)g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2) \\ &- S(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)-(n+\psi-2)g(\phi\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)+\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\eta(\mathscr{U}_2)]\neq 0.\end{aligned} \tag{5.8}$$

Hence, the above relation leads to the following theorem:

Theorem 5.2. *In an $(LPK)_n$-manifold with a quarter-symmetric metric connection, $\bar{W}_8$-curvature tensor is not flat.*

6. Relationship between W_6 and $\bar{W}_8$-curvature tensors in $(LPK)_n$-manifolds

Taking the inner product of (4.9) with $\mathscr{U}_2$, we have

$$\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)=\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)-\frac{1}{n-1}[\bar{S}(\mathscr{X}_2,\mathscr{X}_3)g(\mathscr{X}_1,\mathscr{U}_2)-\bar{S}(\mathscr{X}_1,\mathscr{X}_2)g(\mathscr{X}_3,\mathscr{U}_2)], \tag{6.1}$$

where $\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)=g(\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3,\mathscr{U}_2)$.

Putting $\mathscr{X}_3=\xi$ in (6.1), we have

$$\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2) = \bar{R}(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)-\frac{1}{n-1}[\bar{S}(\mathscr{X}_2,\xi)g(\mathscr{X}_1,\mathscr{U}_2)-\bar{S}(\mathscr{X}_1,\mathscr{X}_2)g(\xi,\mathscr{U}_2)]. \tag{6.2}$$

From (3.3), we also have

$$\begin{aligned}\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2) &= R(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)+g(\mathscr{X}_2,\xi)g(\phi\mathscr{X}_1,\mathscr{U}_2)-g(\mathscr{X}_1,\xi)g(\phi\mathscr{X}_2,\mathscr{U}_2) \\ &+ g(\phi\mathscr{X}_2,\xi)g(\mathscr{X}_1,\mathscr{U}_2)-g(\phi\mathscr{X}_1,\xi)g(\mathscr{X}_2,\mathscr{U}_2) \\ &+ g(\phi\mathscr{X}_2,\xi)g(\phi\mathscr{X}_1,\mathscr{U}_2)-g(\phi\mathscr{X}_2,\mathscr{U}_2).\end{aligned} \tag{6.3}$$

By using (2.2), (6.3) reduces to

$$\bar{R}(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)=R(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)+\eta(\mathscr{X}_2)g(\phi\mathscr{X}_1,\mathscr{U}_2)-\eta(\mathscr{X}_1)g(\phi\mathscr{X}_2,\mathscr{U}_2). \tag{6.4}$$

Now, using (2.2), (3.5) and (6.4) in (6.2), we have

$$\begin{aligned}\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2) &= R(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)+\frac{\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)-\frac{n+\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{U}_2)\eta(\mathscr{X}_2)\\ &- g(\phi\mathscr{X}_2,\mathscr{U}_2)\eta(\mathscr{X}_1)+g(\phi\mathscr{X}_1,\mathscr{U}_2)\eta(\mathscr{X}_2)+\frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)\\ &- \frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\eta(\mathscr{U}_2)+\frac{1}{n-1}S(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2).\end{aligned}$$

Interchanging $\mathscr{U}_2$ and ξ in the foregoing equation, we have

$$\begin{aligned}&\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2,\mathscr{U}_2,\xi)\\ =\;& R(\mathscr{X}_1,\mathscr{X}_2,\mathscr{U}_2,\xi)+\frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\eta(\mathscr{U}_2)-\frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)\\ -\;& \frac{\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)+\frac{n+\psi-1}{n-1}g(\mathscr{X}_2,\mathscr{U}_2)\eta(\mathscr{X}_1)-\frac{1}{n-1}S(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2). \qquad (6.5)\end{aligned}$$

Equation (6.5) can be written as

$$\begin{aligned}\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2 &= R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2+\frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\mathscr{U}_2-\frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2 \qquad (6.6)\\ &-\frac{\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2+\frac{n+\psi-1}{n-1}g(\mathscr{X}_2,\mathscr{U}_2)\mathscr{X}_1-\frac{1}{n-1}S(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2.\end{aligned}$$

Replacing $\mathscr{U}_2$ by $\mathscr{X}_3$ in (6.6), we have

$$\begin{aligned}\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 &= R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\mathscr{X}_3-\frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 \qquad (6.7)\\ &-\frac{\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{n+\psi-1}{n-1}g(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-\frac{1}{n-1}S(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3.\end{aligned}$$

Using equation (2.3) in (6.7) and simplifying, we get

$$\begin{aligned}\bar{W}_8(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 &= -\frac{\psi-1}{n-1}g(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{2n+\psi-2}{n-1}g(\mathscr{X}_2,\mathscr{X}_3)\mathscr{X}_1-g(\mathscr{X}_1,\mathscr{X}_3)\mathscr{X}_2 \qquad (6.8)\\ &-\frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3+\frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\mathscr{X}_3-\frac{1}{n-1}S(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3.\end{aligned}$$

Taking the inner product of (4.1) with $\mathscr{U}_2$, we have

$$W_6(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)=R(\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{U}_2)+\frac{1}{n-1}[g(\mathscr{X}_1,\mathscr{X}_2)S(\mathscr{X}_3,\mathscr{U}_2)-S(\mathscr{X}_2,\mathscr{X}_3)g(\mathscr{X}_1,\mathscr{U}_2)],$$

which by putting $\mathscr{X}_3=\xi$ turns to

$$W_6(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)=R(\mathscr{X}_1,\mathscr{X}_2,\xi,\mathscr{U}_2)+[g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)-g(\mathscr{X}_1,\mathscr{U}_2)\eta(\mathscr{X}_2)]. \qquad (6.9)$$

Interchanging $\mathscr{U}_2$ and ξ in the above equation, we have

$$-W_6(\mathscr{X}_1,\mathscr{X}_2,\mathscr{U}_2,\xi)=-R(\mathscr{X}_1,\mathscr{X}_2,\mathscr{U}_2,\xi)+[g(\mathscr{X}_1,\mathscr{X}_2)\eta(\mathscr{U}_2)-g(\mathscr{X}_2,\mathscr{U}_2)\eta(\mathscr{X}_1)],$$

from which we write

$$W_6(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2=R(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2-[g(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_2-g(\mathscr{X}_2,\mathscr{U}_2)\mathscr{X}_1]. \qquad (6.10)$$

Replacing $\mathscr{U}_2 = \mathscr{X}_3$ in (6.10), we have

$$W_6(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 = R(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 - [g(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 - g(\mathscr{X}_2, \mathscr{X}_3)\mathscr{X}_1]. \tag{6.11}$$

Simplifying above, we have

$$W_6(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 = -g(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 + 2g(\mathscr{X}_2, \mathscr{X}_3)\mathscr{X}_1 - g(\mathscr{X}_1, \mathscr{X}_3)\mathscr{X}_2. \tag{6.12}$$

Now, from (6.8) and (6.12), we get

$$\begin{aligned} \bar{W}_8(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 &= W_6(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 + \frac{n-\psi}{n-1}g(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 + \frac{\psi}{n-1}g(\mathscr{X}_2, \mathscr{X}_3)\mathscr{X}_1 \\ &- \frac{n+\psi-2}{n-1}g(\phi\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 + \frac{1}{n-1}\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\mathscr{X}_3 - \frac{1}{n-1}S(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3. \end{aligned} \tag{6.13}$$

Thus, we conclude:

Theorem 6.1. *A W_8-curvature tensor with a quarter-symmetric metric connection is related to the W_6-curvature tensor with the Levi-Civita connection in $(LPK)_n$-manifolds by* (6.13).

7. $(LPK)_n$-manifolds satisfying $\bar{W}_6 \cdot \bar{R} = 0$ and $\bar{W}_8 \cdot \bar{R} = 0$

In this section, first we study an $(LPK)_n$-manifold satisfying $\bar{W}_6(\xi, \mathscr{U}_1) \cdot \bar{R} = 0$. Thus, we have

$$\begin{aligned} &\bar{W}_6(\xi, \mathscr{U}_1)\bar{R}(\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 - \bar{R}(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_1, \mathscr{X}_2)\mathscr{X}_3 \\ &-\bar{R}(\mathscr{X}_1, \bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_2)\mathscr{X}_3 - \bar{R}(\mathscr{X}_1, \mathscr{X}_2)\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_3 = 0. \end{aligned} \tag{7.1}$$

Replacing $\mathscr{X}_3$ by ξ in (7.1), we have

$$\begin{aligned} &\bar{W}_6(\xi, \mathscr{U}_1)\bar{R}(\mathscr{X}_1, \mathscr{X}_2)\xi - \bar{R}(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_1, \mathscr{X}_2)\xi \\ &-\bar{R}(\mathscr{X}_1, \bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_2)\xi - \bar{R}(\mathscr{X}_1, \mathscr{X}_2)\bar{W}_6(\xi, \mathscr{U}_1)\xi = 0. \end{aligned} \tag{7.2}$$

Using (3.11) in (7.2), we have

$$\begin{aligned} &\eta(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_1)\mathscr{X}_2 - \eta(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_2)\mathscr{X}_1 + \eta(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_1)\phi\mathscr{X}_2 \\ &-\eta(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_2)\phi\mathscr{X}_1 + \eta(\mathscr{X}_2)\bar{W}_6(\xi, \mathscr{U}_1)\phi\mathscr{X}_1 - \eta(\mathscr{X}_1)\bar{W}_6(\xi, \mathscr{U}_1)\phi\mathscr{X}_2 \\ &+\eta(\mathscr{X}_1)\phi(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_2) - \eta(\mathscr{X}_2)\phi(\bar{W}_6(\xi, \mathscr{U}_1)\mathscr{X}_1) - \bar{R}(\mathscr{X}_1, \mathscr{X}_2)(\mathscr{U}_1 + \phi\mathscr{U}_1 + \eta(\mathscr{U}_1)\xi) = 0. \end{aligned} \tag{7.3}$$

By using (4.4) in (7.3), we lead to

$$\begin{aligned} &\frac{1}{n-1}S(\mathscr{U}_1, \mathscr{X}_1)\mathscr{X}_2 - \frac{1}{n-1}S(\mathscr{U}_1, \mathscr{X}_2)\mathscr{X}_1 + \frac{1}{n-1}S(\mathscr{U}_1, \mathscr{X}_1)\phi\mathscr{X}_2 \\ &-\frac{1}{n-1}S(\mathscr{U}_1, \mathscr{X}_2)\phi\mathscr{X}_1 + \frac{1}{n-1}S(\mathscr{U}_1, \phi\mathscr{X}_2)\eta(\mathscr{X}_1)\xi - \frac{1}{n-1}S(\mathscr{U}_1, \phi\mathscr{X}_1)\eta(\mathscr{X}_2)\xi \\ &+\frac{n-\psi}{n-1}g(\mathscr{U}_1, \mathscr{X}_2)\mathscr{X}_1 - \frac{n-\psi}{n-1}g(\mathscr{U}_1, \mathscr{X}_1)\mathscr{X}_2 + \frac{n-\psi}{n-1}g(\mathscr{U}_1, \mathscr{X}_2)\phi\mathscr{X}_1 \\ &-\frac{n-\psi}{n-1}g(\mathscr{U}_1, \mathscr{X}_1)\phi\mathscr{X}_2 + \frac{\psi-1}{n-1}g(\phi\mathscr{U}_1, \mathscr{X}_1)\mathscr{X}_2 - \frac{\psi-1}{n-1}g(\phi\mathscr{U}_1, \mathscr{X}_2)\mathscr{X}_1 \\ &+\frac{\psi-1}{n-1}g(\phi\mathscr{U}_1, \mathscr{X}_1)\phi\mathscr{X}_2 - \frac{\psi-1}{n-1}g(\phi\mathscr{U}_1, \mathscr{X}_2)\phi\mathscr{X}_1 + \frac{\psi-1}{n-1}g(\mathscr{U}_1, \mathscr{X}_2)\eta(\mathscr{X}_1)\xi \end{aligned} \tag{7.4}$$

$$-\frac{\psi-1}{n-1}g(\mathscr{U}_1,\mathscr{X}_1)\eta(\mathscr{X}_2)\xi+\frac{n-\psi}{n-1}g(\mathscr{U}_1,\phi\mathscr{X}_1)\eta(\mathscr{X}_2)\xi-\frac{n-\psi}{n-1}g(\mathscr{U}_1,\phi\mathscr{X}_2)\eta(\mathscr{X}_1)\xi$$
$$+\frac{n+\psi-2}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_1)\mathscr{X}_2-\frac{n+\psi-2}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_2)\mathscr{X}_1+\frac{n+\psi-2}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_1)\phi\mathscr{X}_2$$
$$-\frac{n+\psi-2}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_2)\phi\mathscr{X}_1-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_1-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\phi\mathscr{U}_1=0.$$

Substituting $\mathscr{X}_2=\xi$, and using (3.10) in (7.4), we arrive at

$$\frac{1}{n-1}S(\mathscr{U}_1,\mathscr{X}_1)\xi+\frac{1}{n-1}S(\mathscr{U}_1,\phi\mathscr{X}_1)\xi+\frac{n+2\psi-3}{n-1}g(\mathscr{U}_1,\mathscr{X}_1)\xi \tag{7.5}$$
$$+\frac{n+\psi-2}{n-1}g(\phi\mathscr{U}_1,\mathscr{X}_1)\xi+\frac{\psi-1}{n-1}g(\mathscr{U}_1,\phi\mathscr{X}_1)\xi+\frac{2(n+\psi-2)}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_1)\xi=0.$$

Taking the inner product of (7.5) with ξ, we have

$$S(\mathscr{U}_1,\mathscr{X}_1)+S(\mathscr{U}_1,\phi\mathscr{X}_1) = -(n+2\psi-3)g(\mathscr{U}_1,\mathscr{X}_1)-(\psi-1)g(\mathscr{U}_1,\phi\mathscr{X}_1) \tag{7.6}$$
$$-(n+\psi-2)g(\phi\mathscr{U}_1,\mathscr{X}_1)-2(n+\psi-2)\eta(\mathscr{U}_1)\eta(\mathscr{X}_1).$$

Contracting above expression over $\mathscr{U}_1$ and $\mathscr{X}_1$, we obtain

$$r=-[n^2+(5\psi-1)n-(\psi+4)]-tr\phi Q. \tag{7.7}$$

The above relation leads to the following theorem:

Theorem 7.1. *The scalar curvature of an $(LPK)_n$-manifold satisfying $\bar{W}_6.\bar{R}=0$ is given by* (7.7).

Next, we study an $(LPK)_n$-manifold satisfying $\bar{W}_8(\xi,\mathscr{U}_1)\cdot\bar{R}=0$. Thus, we have

$$\bar{W}_8(\xi,\mathscr{U}_1)\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3-\bar{R}(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_1,\mathscr{X}_2)\mathscr{X}_3 \tag{7.8}$$
$$-\bar{R}(\mathscr{X}_1,\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_2)\mathscr{X}_3-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_3=0.$$

Replacing $\mathscr{X}_3$ by ξ in (7.8), we have

$$\bar{W}_8(\xi,\mathscr{U}_1)\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\xi-\bar{R}(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_1,\mathscr{X}_2)\xi \tag{7.9}$$
$$-\bar{R}(\mathscr{X}_1,\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_2)\xi-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\bar{W}_8(\xi,\mathscr{U}_1)\xi=0.$$

Using (3.11) in (7.9), we have

$$\eta(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_1)\mathscr{X}_2-\eta(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_2)\mathscr{X}_1+\eta(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_1)\phi\mathscr{X}_2 \tag{7.10}$$
$$-\eta(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_2)\phi\mathscr{X}_1+\eta(\mathscr{X}_2)\bar{W}_8(\xi,\mathscr{U}_1)\phi\mathscr{X}_1-\eta(\mathscr{X}_1)\bar{W}_8(\xi,\mathscr{U}_1)\phi\mathscr{X}_2$$
$$+\eta(\mathscr{X}_1)\phi(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_2)-\eta(\mathscr{X}_2)\phi(\bar{W}_8(\xi,\mathscr{U}_1)\mathscr{X}_1)-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)(\mathscr{U}_1$$
$$+\phi\mathscr{U}_1+\frac{n}{n-1}\eta(\mathscr{U}_1)\xi)=0.$$

By using (4.11), (7.10) becomes

$$
\begin{aligned}
&\frac{1}{n-1}S(\mathscr{U}_1,\mathscr{X}_1)\mathscr{X}_2-\frac{1}{n-1}S(\mathscr{U}_1,\mathscr{X}_2)\mathscr{X}_1+\frac{1}{n-1}S(\mathscr{U}_1,\mathscr{X}_1)\phi\mathscr{X}_2\\
&-\frac{1}{n-1}S(\mathscr{U}_1,\mathscr{X}_2)\phi\mathscr{X}_1+\frac{1}{n-1}S(\mathscr{U}_1,\phi\mathscr{X}_2)\eta(\mathscr{X}_1)\xi-\frac{1}{n-1}S(\mathscr{U}_1,\phi\mathscr{X}_1)\eta(\mathscr{X}_2)\xi\\
&+\frac{n-\psi}{n-1}g(\mathscr{U}_1,\mathscr{X}_2)\mathscr{X}_1-\frac{n-\psi}{n-1}g(\mathscr{U}_1,\mathscr{X}_1)\mathscr{X}_2+\frac{n-\psi}{n-1}g(\mathscr{U}_1,\mathscr{X}_2)\phi\mathscr{X}_1\\
&-\frac{n-\psi}{n-1}g(\mathscr{U}_1,\mathscr{X}_1)\phi\mathscr{X}_2+\frac{\psi-1}{n-1}g(\phi\mathscr{U}_1,\mathscr{X}_1)\mathscr{X}_2-\frac{\psi-1}{n-1}g(\phi\mathscr{U}_1,\mathscr{X}_2)\mathscr{X}_1\\
&+\frac{\psi-1}{n-1}g(\phi\mathscr{U}_1,\mathscr{X}_1)\phi\mathscr{X}_2-\frac{\psi-1}{n-1}g(\phi\mathscr{U}_1,\mathscr{X}_2)\phi\mathscr{X}_1+\frac{\psi-1}{n-1}g(\mathscr{U}_1,\mathscr{X}_2)\eta(\mathscr{X}_1)\xi \qquad (7.11)\\
&-\frac{\psi-1}{n-1}g(\mathscr{U}_1,\mathscr{X}_1)\eta(\mathscr{X}_2)\xi+\frac{n-\psi}{n-1}g(\mathscr{U}_1,\phi\mathscr{X}_1)\eta(\mathscr{X}_2)\xi-\frac{n-\psi}{n-1}g(\mathscr{U}_1,\phi\mathscr{X}_2)\eta(\mathscr{X}_1)\xi\\
&+\frac{n+\psi}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_1)\mathscr{X}_2-\frac{n+\psi}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_2)\mathscr{X}_1+\frac{n+\psi}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_1)\phi\mathscr{X}_2\\
&-\frac{n+\psi}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_2)\phi\mathscr{X}_1+\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\phi^2\mathscr{U}_1-\eta(\mathscr{X}_1)\eta(\mathscr{X}_2)\phi\mathscr{U}_1\\
&-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\mathscr{U}_1-\bar{R}(\mathscr{X}_1,\mathscr{X}_2)\phi\mathscr{U}_1=0.
\end{aligned}
$$

Substituting $\mathscr{X}_2=\xi$, and using (3.10) in (7.11), we find

$$
\begin{aligned}
&\frac{1}{n-1}S(\mathscr{U}_1,\mathscr{X}_1)\xi+\frac{1}{n-1}S(\mathscr{U}_1,\phi\mathscr{X}_1)\xi+\frac{n+2\psi-3}{n-1}g(\mathscr{U}_1,\mathscr{X}_1)\xi \qquad (7.12)\\
&+\frac{n+\psi-2}{n-1}g(\phi\mathscr{U}_1,\mathscr{X}_1)\xi+\frac{\psi-1}{n-1}g(\mathscr{U}_1,\phi\mathscr{X}_1)\xi+\frac{n+2\psi-1}{n-1}\eta(\mathscr{U}_1)\eta(\mathscr{X}_1)\xi\\
&-\eta(\mathscr{X}_1)\mathscr{U}_1+\frac{2}{n-1}\eta(\mathscr{U}_1)\mathscr{X}_1+\frac{2}{n-1}\eta(\mathscr{U}_1)\phi\mathscr{X}_1+\eta(\mathscr{X}_1)\phi\mathscr{U}_1=0.
\end{aligned}
$$

Taking inner product of (7.12) with ξ, we lead to

$$
\begin{aligned}
S(\mathscr{U}_1,\mathscr{X}_1)+S(\mathscr{U}_1,\phi\mathscr{X}_1) \;=\; &-(n+2\psi-3)g(\mathscr{U}_1,\mathscr{X}_1)-(\psi-1)g(\mathscr{U}_1,\phi\mathscr{X}_1) \qquad (7.13)\\
&-(n+\psi-2)g(\phi\mathscr{U}_1,\mathscr{X}_1)-2(n+\psi-2)\eta(\mathscr{U}_1)\eta(\mathscr{X}_1).
\end{aligned}
$$

Contracting above expression over $\mathscr{U}_1$ and $\mathscr{X}_1$, we get

$$
r=-[n^2+(5\psi-1)n-(\psi+4)]-tr\phi Q. \qquad (7.14)
$$

The above relation leads to the following theorem:

Theorem 7.2. *The scalar curvature of an $(LPK)_n$-manifold satisfying $\bar{W}_8\cdot\bar{R}=0$ is given by* (7.14).

Acknowledgement: The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 25UQU4330007GSSR04.

Funding statement: This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 25UQU4330007GSSR04.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] K. Kenmotsu, A Class of Almost Contact Riemannian Manifolds, Tohoku Math. J. 24 (1972), 93–103. https://doi.org/10.2748/tmj/1178241594.

[2] H.I. Yoldaş, E. Yasar, Some Notes on Kenmotsu Manifold, Facta Univ., Ser. Math. Inform. (2021), 949–961. https://doi.org/10.22190/FUMI2004949Y.

[3] J.B. Jun, U.C. De, G. Pathak, On Kenmotsu Manifolds, J. Korean Math. Soc. 42 (2005), 435–445. https://doi.org/10.4134/JKMS.2005.42.3.435.

[4] R. Prasad, A. Haseeb, P. Gupta, Quasi Hemi-Slant Submanifolds of Kenmotsu Manifolds, J. Appl. Math. Inform. 40 (2022), 475–490. https://doi.org/10.14317/JAMI.2022.475.

[5] I. Sato, On a Structure Similar to the Almost Contact Structure I, Tensor, N. S. 30 (1976), 219–224. https://cir.nii.ac.jp/crid/1572261550129279744.

[6] S. Kaneyuki, M. Kozai, Paracomplex Structures and Affine Symmetric Spaces, Tokyo J. Math. 8 (1985), 81–98. https://doi.org/10.3836/tjm/1270151571.

[7] S. Kaneyuki, F.L. Williams, Almost Paracontact and Parahodge Structures on Manifolds, Nagoya Math. J. 99 (1985), 173–187. https://doi.org/10.1017/S0027763000021565.

[8] B.B. Sinha, K.L.S. Prasad, A Class of Almost Para Contact Metric Manifolds, Bull. Calcutta Math. Soc. 87 (1995), 307–312.

[9] B. O'Neill, Semi-Riemannian Geometry: With Applications to Relativity, Academic Press, New York, 1983.

[10] V.R. Kaigorodov, The Curvature Structure of Spacetime, Prob. Geom. 14 (1983), 177–202.

[11] A.K. Raychaudhuri, S. Banerji, A. Banerjee, General Relativity, Astrophysics and Cosmology, Springer, 1992.

[12] A. Haseeb, R. Prasad, Certain Results on Lorentzian Para-Kenmotsu Manifolds, Bol. Soc. Parana. Mat. 39 (2021), 201–220. https://doi.org/10.5269/bspm.40607.

[13] A. Haseeb, R. Prasad, Some Results on Lorentzian Para-Kenmotsu Manifolds, Bull. Transilv. Univ. Brașov Ser. III Math. Inform. Phys. 13 (2020), 185–198. https://doi.org/10.31926/but.mif.2020.13.62.1.14.

[14] S. Golab, On Semi-Symmetric and Quarter-Symmetric Linear Connections, Tensor, N. S. 29 (1975), 249–254.

[15] K. Mandal, U.C. De, Quarter-Symmetric Metric Connection in a P-Sasakian Manifold, Ann. West Univ. Timisoara - Math. Comput. Sci. 53 (2015), 137–150. https://doi.org/10.1515/awutm-2015-0007.

[16] M. Ahmad, J.B. Jun, A. Haseeb, Hypersurfaces of Almost-Para Contact Riemannian Manifold With a Quarter-Symmetric Connection, Bull. Korean Math. Soc. 46 (2009), 477–487. https://doi.org/10.4134/BKMS.2009.46.3.477.

[17] R. Prasad, A. Haseeb, Conformal Curvature Tensor on K-Contact Manifolds With Respect to the Quarter-Symmetric Metric Connection, Facta Univ. (NIS) Ser. Math. Inform. 32 (2017), 503–514.

[18] G.P. Pokhariyal, Relativistic Significance of Curvature Tensors, Int. J. Math. Math. Sci. 5 (1982), 133–139.

[19] R. Prasad, A. Haseeb, V.S. Yadav, A Study of φ-Ricci Symmetric LP-Kenmotsu Manifolds, Int. J. Maps Math. 7 (2024), 33–44.

[20] P.W. Njori, N.K. Moindi, G.P. Pokhariyal, A Study on W_6-Curvature Tensors and W_8-Curvature Tensors in Kenmotsu Manifold Admitting Semi-Symmetric Metric Connection, Int. J. Stat. Appl. Math. 6 (2021), 1–5.

[21] G.P. Pokhariyal, R.S. Mishra, Curvature Tensors and Their Relativistics Significance, Yokohama Math. J. 18 (1970), 105–108.