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Soft Union Bi-quasi Ideals of Semigroup

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ABSTRACT. Mathematicians attach importance to extending ideals in algebraic structures. The concept of bi-quasi (ƁQ) ideal was introduced as a generalization version of quasi-ideal, bi-ideal, and left (right) ideals in semigroups. This paper applies this concept to soft set theory and semigroups, introducing the notion of "Soft union (S-uni) ƁQ ideal." The aim of this paper is to explore the relationships between S-uni ƁQ ideals and other types of S-uni ideals in semigroups. It is shown that every S-uni bi-ideal, S-uni ideal, S-uni quasi-ideal, and S-uni interior ideal of an idempotent soft set are S-uni ƁQ ideals. Counterexamples demonstrate that the converses are not always true unless the semigroup is special soft simple or regular. For special soft simple semigroups, the S-uni ƁQ ideal coincides with the S-uni bi-ideal, S-uni left (right) ideal, and S-uni quasi-ideal. Additionally, we provide conceptual definitions and analyses of the new concept in the context of soft set operations, supporting our claims with clear examples.

1. Introduction

Semigroups are crucial in various areas of mathematics as they provide the abstract algebraic foundation for "memoryless" systems, which reset after every iteration. Initially studied in the early 1900s, semigroups serve as key models for linear time-invariant systems in applied mathematics. Their connection to finite automata makes the study of finite semigroups particularly important in theoretical computer science. In probability theory, semigroups are also linked to Markov processes. The concept of ideals is vital for understanding the structure and

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applications of mathematical systems, and thus, many mathematicians have focused on extending the theory of ideals in algebraic structures. In fact, advancing the study of algebraic systems requires a broader understanding of ideals. By utilizing the concept and properties of generalized ideals, mathematicians have made significant contributions to the characterization of algebraic structures. Dedekind introduced ideals in the context of algebraic number theory, and Noether expanded this concept to include associative rings. The notion of a one-sided ideal extends the idea of an ideal, and the theory of one-sided and two-sided ideals remains central to ring theory.

In 1952, Good and Hughes [1] introduced the concept of bi-ideals for semigroups. Steinfeld [2] was the first to present the idea of quasi-ideals for semigroups, later extending it to rings. Quasi-ideals are a generalization of right and left ideals, while bi-ideals are a further generalization of quasi-ideals. The concept of interior ideals was initially introduced by Lajos [3] and later explored by Szasz [4,5]. Interior ideals represent a generalization of the traditional ideal concept. Rao [6-9] developed several novel types of semigroup ideals that generalize existing ones, such as bi-interior ideals, bi-quasi ideals, quasi-ideal, interior ideals, weak-interior ideals and bi-quasi-interior ideals. Moreover, Baupradist et al. [10] introduced the concept of essential ideals in semigroups. The concept of "almost" ideals was proposed as a broader form of various ideal types, and a comprehensive study was conducted on their properties and interrelationships. In this regard, the idea of almost ideals was first presented in [11]. Additionally, various types of almost ideals for semigroups were introduced in [12-18]. Furthermore, in [13, 15–20], several fuzzy types of almost ideals for semigroups were explored.

In 1999, Molodtsov [21] introduced the "Soft Set Theory" to address and provide solutions for problems involving uncertainty. Since its inception, extensive research has been conducted on the concepts of soft sets, particularly focusing on operations involving soft sets. Maji et al. [22] defined specific operations on soft sets and introduced related concepts. Pei and Miao [23], as well as Ali et al. [24], proposed various operations on soft sets. Sezgin and Atagün [25] conducted studies on soft set operations. For more information on soft set operations, which have gained significant attention since their introduction, we refer to [26–36]. Çağman and Enginoğlu [37] revisited the notions and operations of soft sets. In addition, Çağman et al. [38] developed the concept of soft intersection groups, which led to the exploration of different soft algebraic systems. Sezgin [39], using soft sets in the context of semigroup theory, defined soft union (S-uni) semigroups, left (right/two-sided) ideals, and bi-ideals of semigroups. Sezgin et al. [40] defined S-uni interior ideals, quasi-ideals, and generalized bi-ideals of semigroups, thoroughly analyzing their fundamental properties. Regarding the S-uni substructures of semigroups, Sezer et al. [41] defined and classified several types of semigroups. In [42], various types of regularities in semigroups were characterized using soft union quasi-ideals, soft union (generalized) bi-ideals, and soft union semiprime ideals of a semigroup. Several types of soft intersection almost ideals were introduced and studied in [43–54] as a generalization of soft intersection ideals. Finally, in [55–72], the soft versions of various algebraic structures were explored.

As a generalization of bi-ideals and interior ideals of semigroups, The first of these is the study by Rao [6] on the bi-quasi ideals of Γ-semigroups and the fuzzy bi-quasi ideals of these semigroups. Additionally, the bi-quasi ideals of Γ-semirings were examined by Rao, Venkateswarlu, and Rafi [73]. Rao [74,75] provided an extensive study on the bi-quasi ideals of semirings. Similarly, Rao [8] made significant contributions to the study of bi-quasi ideals of semigroups. In this study, we extend this idea to semigroups and soft set theory by introducing "S-uni bi-quasi ideals of semigroups." We analyze the relationships between S-uni bi-quasi ideals and various types of S-uni ideals of semigroups. Under certain necessary conditions, it is demonstrated that an S-uni ideal (bi-ideal, quasi-ideal, interior ideal) is indeed an S-uni ƁQ ideal of a semigroup. Counterexamples are provided to show that the reverse of these statements does not always hold. However, the converse statements are not true, as demonstrated by counterexamples. We show that for the converses to hold, the semigroup must be a special soft simple semigroup. Additionally, we provide conceptual characterizations and analyses of this new concept in the context of soft set operations, supporting our claims with specific, illustrative examples. This study is structured into four sections: Section 1 offers an introduction to the topic, Section 2 presents the basic concepts of semigroups and soft set ideals along with relevant definitions and implications, Section 3 introduces the idea of S-uni bi-quasi ideals of semigroups and uses specific examples to examine their characteristics and their connections to other S-uni ideals, and Section 4 outlines our findings and suggests directions for future research.

2. Preliminaries

In this study, S is used to represent a semigroup. A nonempty subset Ƙ of S is called a subsemigroup of S if ƘƘ ⊆ Ƙ, is called a left (right) ideal of S if SƘ ⊆ Ƙ (ƘS ⊆ Ƙ), is called a bi-ideal of S if ƘƘ ⊆ Ƙ and ƘSƘ ⊆ Ƙ, is called an interior ideal of S if SƘS ⊆ Ƙ, and is called a quasi-ideal of S if ƘS ∩ SƘ ⊆ Ƙ.

A subsemigroup Ƙ of S is called a left (L-) ƁQ ideal of S if SƘ ∩ ƘSƘ ⊆ Ƙ , is called a right (R-) ƁQ ideal of S if ƘS ∩ ƘSƘ ⊆ Ƙ, and is called a ƁQ ideal of S if it is both L-ƁQ ideal of S and R-ƁQ ideal [8].

Definition 2.1. [21, 37] Let E be the parameter set, U be the universal set, $P(U)$ be the power set of U, and Ɗ ⊇ E. The soft set (SS) $ϱ_{Ɗ}$ over U is a function such that $ϱ_{Ɗ}: E \to P(U)$, where for all ⱴ ∉ Ɗ, $ϱ_{Ɗ}$(ⱴ) = ∅. That is,

$$ϱ_{Ɗ} = \{(ⱴ, ϱ_{Ɗ}(ⱴ)): ⱴ \in E, ϱ_{Ɗ}(ⱴ) \in P(U)\}$$

The set of all SSs over U is designated by $S_E(U)$ throughout this paper.

Definition 2.2. [37] Let $\varrho_{\mathrm{D}} \in S_E(U)$. If $\varrho_{\mathrm{D}}(t) = \emptyset$ for all $t \in E$, then ϱ_{D} is called a null SS and indicated by $\emptyset_{\mathrm{E}}$.

Definition 2.3. [37] Let $\varrho_{\mathcal{M}}, \varrho_{\mathrm{N}} \in S_E(U)$. If $\varrho_{\mathcal{M}}(\varpi) \subseteq \varrho_{\mathrm{N}}(\varpi)$, for all $\varpi \in E$, then $\varrho_{\mathcal{M}}$ is a soft subset of ϱ_{N} and indicated by $\varrho_{\mathcal{M}} \tilde{\subseteq} \varrho_{\mathrm{N}}$. If $\varrho_{\mathcal{M}}(\varpi) \supseteq \varrho_{\mathrm{N}}(\varpi)$, for all $\varpi \in E$, then $\varrho_{\mathcal{M}}$ is a soft superset of ϱ_{N} and indicated by $\varrho_{\mathcal{M}} \tilde{\supseteq} \varrho_{\mathrm{N}}$, and if $\varrho_{\mathcal{M}}(\varsigma) = \varrho_{\mathrm{N}}(\varsigma)$, for all $\varsigma \in E$, then $\varrho_{\mathcal{M}}$ is called soft equal to ϱ_{N} and denoted by $\varrho_{\mathcal{M}} = \varrho_{\mathrm{N}}$.

Definition 2.4. [37] Let $\text{ꝗ}_{\mathcal{M}}, \text{ꝗ}_{\text{И}} \in S_E(U)$. The union (intersection) of $\text{ꝗ}_{\mathcal{M}}$ and $\text{ꝗ}_{\text{И}}$ is the SS $\text{ꝗ}_{\mathcal{M}} \tilde{\cup} \text{ꝗ}_{\text{И}}$ $(\text{ꝗ}_{\mathcal{M}} \tilde{\cap} \text{ꝗ}_{\text{И}})$, where $(\text{ꝗ}_{\mathcal{M}} \tilde{\cup} \text{ꝗ}_{\text{И}})(\upsilon) = \text{ꝗ}_{\mathcal{M}}(\upsilon) \cup \text{ꝗ}_{\text{И}}(\upsilon)$ $\left((\text{ꝗ}_{\mathcal{M}} \tilde{\cap} \text{ꝗ}_{\text{И}})(\upsilon) = \text{ꝗ}_{\mathcal{M}}(\upsilon) \cap \text{ꝗ}_{\text{И}}(\upsilon)\right)$, for all $\upsilon \in E$, respectively.

Definition 2.5. [39] Let $\text{f}_{\text{Ƙ}} \in S_E(U)$ and $\alpha \subseteq U$. Then, lower α-inclusion of $\text{f}_{\text{Ƙ}}$, denoted by $\mathcal{L}(\text{f}_{\text{Ƙ}}; \alpha)$, is defined as $\mathcal{L}(\text{f}_{\text{Ƙ}}; \alpha) = \{x \in \text{Ƙ} \mid \text{f}_{\text{Ƙ}}(x) \subseteq \alpha\}$.

Definition 2.6. [39] Let $\text{ɲ}_S, g_S \in S_S(U)$. S-uni product $\text{ɲ}_S * g_S$ is defined by

$$(\text{ɲ}_S * g_S)(\text{ɤ}) = \begin{cases} \bigcap_{\text{ɤ}=yz} \{\text{ɲ}_S(y) \cup g_S(z)\}, & if\ \exists y, z \in S\ such\ that\ \text{ɤ} = yz \\ U, & otherwise \end{cases}$$

Theorem 2.7. [39] Let $\text{ɦ}_S, \text{ɋ}_S, \text{n}_S \in S_S(U)$. Then,

i. $(\text{ɦ}_S * \text{ɋ}_S) * \text{n}_S = \text{ɦ}_S * (\text{ɋ}_S * \text{n}_S)$
ii. $\text{ɦ}_S * \text{ɋ}_S \neq \text{ɋ}_S * \text{ɦ}_S$
iii. $\text{ɦ}_S * (\text{ɋ}_S \tilde{\cup} \text{n}_S) = (\text{ɦ}_S * \text{ɋ}_S) \tilde{\cup} (\text{ɦ}_S * \text{n}_S)$ and $(\text{ɦ}_S \tilde{\cup} \text{ɋ}_S) * \text{n}_S = (\text{ɦ}_S * \text{n}_S) \tilde{\cup} (\text{ɋ}_S * \text{n}_S)$
iv. $\text{ɦ}_S * (\text{ɋ}_S \tilde{\cap} \text{n}_S) = (\text{ɦ}_S * \text{ɋ}_S) \tilde{\cap} (\text{ɦ}_S * \text{n}_S)$ and $(\text{ɦ}_S \tilde{\cap} \text{ɋ}_S) * \text{n}_S = (\text{ɦ}_S * \text{n}_S) \tilde{\cap} (\text{ɋ}_S * \text{n}_S)$
v. If $\text{ɦ}_S \tilde{\subseteq} \text{ɋ}_S$, then $\text{ɦ}_S * \text{n}_S \tilde{\subseteq} \text{ɋ}_S * \text{n}_S$ and $\text{n}_S * \text{ɦ}_S \tilde{\subseteq} \text{n}_S * \text{ɋ}_S$
vi. If $\text{ɉ}_S, \text{ɥ}_S \in S_S(U)$ such that $\text{ɉ}_S \tilde{\subseteq} \text{ɦ}_S$ and $\text{ɥ}_S \tilde{\subseteq} \text{ɋ}_S$, then $\text{ɉ}_S * \text{ɥ}_S \tilde{\subseteq} \text{ɦ}_S * \text{ɋ}_S$.

Definition 2.8. [39] Let $\mathcal{T} \subseteq S$. We denote by $\zeta_{\mathcal{T}^c}$ the soft characteristic function (SCF) of the complement $\mathcal{T}$ and defined as

$$\zeta_{\mathcal{T}^c}(v) = \begin{cases} U, & if\ v \in \mathcal{T} \\ \emptyset, & if\ v \in S \setminus \mathcal{T} \end{cases}$$

Definition 2.9. [39, 40] An SS $\text{ɦ}_S \in S_S(U)$ is called

i. an S-uni subsemigroup of S over U if $\text{ɦ}_S(ab) \subseteq \text{ɦ}_S(a) \cup \text{ɦ}_S(b)$ for all $a, b \in S$,
ii. an S-uni left (right) ideal of S over U if $\text{ɦ}_S(n\upsilon) \subseteq \text{ɦ}_S(\upsilon)$ $\left(\text{ɦ}_S(n\upsilon) \subseteq \text{ɦ}_S(n)\right)$ for all $n, \upsilon \in S$, and is called an S-uni two-sided ideal (S-uni ideal) of S over U if it is both S-uni left ideal of S over U and S-uni right ideal of S over U,
iii. an S-uni bi-ideal of S over U if ɦ_S is an S-uni subsemigroup of S over U and $\text{ɦ}_S(jn\rho) \subseteq \text{ɦ}_S(j) \cup \text{ɦ}_S(\rho)$ for all $j, n, \rho \in S$,
iv. an S-uni interior ideal of S over U if $\text{ɦ}_S(jn\rho) \subseteq \text{ɦ}_S(n)$ for all $j, n, \rho \in S$.

Note that in [39], the definition of "S-uni subsemigroup of S" is given as "S-uni semigroup of S"; however in this paper, without loss of generality, we prefer to use "S-uni subsemigroup of S".

If $\vartheta_S(x) = \emptyset$ for all $x \in S$, then ϑ_S is an S-uni subsemigroup (left ideal, right ideal, ideal, bi-ideal, interior ideal). Such a kind of S-uni subsemigroup (left ideal, right ideal, ideal, bi-ideal, interior ideal) is denoted by $\tilde{\theta}$. It is obvious that $\tilde{\theta} = \zeta_{S^c}$, that is, $\tilde{\theta}(x) = \emptyset$ for all $x \in S$ [39, 40].

Definition 2.10. [40] $\mathfrak{h}_S \in S_S(U)$ is called an S-uni quasi-ideal of S over U if $(\tilde{\theta} * \mathfrak{h}_S) \tilde{\cup} (\mathfrak{h}_S * \tilde{\theta}) \tilde{\supseteq} \mathfrak{h}_S$.

Theorem 2.11. [40] Let $\mathfrak{h}_S \in S_S(U)$. Then,

i. $\tilde{\theta} * \tilde{\theta} \tilde{\supseteq} \tilde{\theta}$
ii. $\tilde{\theta} * \mathfrak{h}_S \tilde{\supseteq} \tilde{\theta}$ and $\mathfrak{h}_S * \tilde{\theta} \tilde{\supseteq} \tilde{\theta}$
iii. $\mathfrak{h}_S \tilde{\cup} \tilde{\theta} = \tilde{\theta}$ and $\mathfrak{h}_S \tilde{\cap} \tilde{\theta} = \mathfrak{h}_S$.

Theorem 2.12. [39, 40] Let $\mathfrak{h}_S \in S_S(U)$. Then,

(1) $\mathfrak{h}_S$ is an S-uni subsemigroup $\Leftrightarrow (\mathfrak{h}_S * \mathfrak{h}_S) \tilde{\supseteq} \mathfrak{h}_S$,
(2) $\mathfrak{h}_S$ is an S-uni left (right) ideal $\Leftrightarrow (\tilde{\theta} * \mathfrak{h}_S) \tilde{\supseteq} \mathfrak{h}_S$ and $(\mathfrak{h}_S * \tilde{\theta}) \tilde{\supseteq} \mathfrak{h}_S$,
(3) $\mathfrak{h}_S$ is an S-uni bi-ideal $\Leftrightarrow (\mathfrak{h}_S * \mathfrak{h}_S) \tilde{\supseteq} \mathfrak{h}_S$ and $(\mathfrak{h}_S * \tilde{\theta} * \mathfrak{h}_S) \tilde{\supseteq} \mathfrak{h}_S$,
(4) $\mathfrak{h}_S$ is an S-uni interior ideal $\Leftrightarrow (\tilde{\theta} * \mathfrak{h}_S * \tilde{\theta}) \tilde{\supseteq} \mathfrak{h}_S$.

Theorem 2.13. [39,40] The following assertions hold:

(1) Every S-uni left (right/two-sided) ideal is an S-uni subsemigroup (S-uni bi-ideal/S-uni quasi-ideal),
(2) Every S-uni ideal is an S-uni bi-ideal.

Proposition 2.14. [39] Let $\mathfrak{h}_S \in S_S(U)$, α be a subset of U, $Im(f_S)$ be the image of $\mathfrak{h}_S$ such that $\alpha \in Im(\mathfrak{h}_S)$. If $\mathfrak{h}_S$ is an S-uni subsemigroup of S, then $\mathcal{L}(\mathfrak{h}_S; \alpha)$ is a subsemigroup of S.

Definition 2.15. [76] Let $\mathfrak{h}_S \in S_S(U)$. Then, S is called a special soft left (right) simple semigroup (with respect to $\mathfrak{h}_S$) if $\tilde{\theta} = \tilde{\theta} * \mathfrak{h}_S$ ($\tilde{\theta} = \mathfrak{h}_S * \tilde{\theta}$) , is called a special soft simple semigroup (with respect to $\mathfrak{h}_S$) if $\tilde{\theta} = \tilde{\theta} * \mathfrak{h}_S = \mathfrak{h}_S * \tilde{\theta}$. If S is a special soft (left/right) simple semigroup with respect to all soft sets over U, then it is called a special soft (left/right) simple semigroup.

For the sake of brevity, special soft (left/right) simple semigroup is abbreviated by special soft (L-/ R-) simple.

Corollary 2.16. [39] For a semigroup S, the following conditions are equivalent:

(1) S is regular.
(2) $\mathfrak{h}_S * \mathfrak{p}_S = \mathfrak{h}_S \tilde{\cup} \mathfrak{p}_S$ for every S-uni ideals $\mathfrak{h}_S$ and $\mathfrak{p}_S$ of S over U.

3. Soft Union Bi-quasi Ideals of Semigroups

In this section, we present the concept of soft union bi-quasi ideals of semigroups, provide its examples, thoroughly examine its relationships with other soft union ideals, and analyze the concept in terms of certain SS concepts and operations.

Definition 3.1. A soft set η_S over U is called a soft union left (right) (L-(R-)) bi-quasi ideal of S over U if $(\widetilde{\Theta} * \eta_S) \widetilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S$ $\left((\eta_S * \widetilde{\Theta}) \widetilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S) \widetilde{\supseteq} \eta_S\right)$.

An SS over U is called a soft union bi-quasi ideal of S if it is both a soft union L-bi-quasi ideal and a soft union R-bi-quasi ideal of S over U.

For the sake of brevity, soft union bi-quasi ideal of S over U is abbreviated by S-uni ƁQ ideal.

Example 3.2. Consider the semigroup $S = \{\mathfrak{f}, h, \mathfrak{r}\}$ defined by the following table:

Table 1: Cayley table of '♦' binary operation.

♦	$\mathfrak{f}$	h	$\mathfrak{r}$
$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{r}$	$\mathfrak{r}$
h	$\mathfrak{r}$	h	$\mathfrak{r}$
$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{r}$	$\mathfrak{r}$

Let η_S and $\mathcal{S}_S$ be SSs over $U = D_3 = \{< x, y >: x^3 = y^2 = e,\ xy = yx^2\} = \{e, x, x^2, y, yx, yx^2\}$ as follows:

$$\eta_S = \{(\mathfrak{f}, \{e, x, x^2\}), (h, \{e, x, x^2, y\}), (\mathfrak{r}, \{e, x\})\}$$
$$\mathcal{S}_S = \{(\mathfrak{f}, \{e, x, y\}), (h, \{e, x^2, y, yx^2\}), (\mathfrak{r}, \{e, x\})\}$$

It can be readily proven that η_S is an S-uni ƁQ ideal of S. Here, we find it appropriate to give a few concrete examples of elements for ease of illustration in order to be more understandable. In fact,

$$[(\widetilde{\Theta} * \eta_S) \widetilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S)](\mathfrak{f}) = (\widetilde{\Theta} * \eta_S)(\mathfrak{f}) \cup (\eta_S * \widetilde{\Theta} * \eta_S)(\mathfrak{f})$$
$$= [\widetilde{\Theta}(\mathfrak{f}) \cup \eta_S(\mathfrak{f})] \cup [\eta_S(\mathfrak{f}) \cup (\widetilde{\Theta} * \eta_S)(\mathfrak{f})] = \eta_S(\mathfrak{f}) \cup \eta_S(\mathfrak{f}) = \eta_S(\mathfrak{f}) \supseteq \eta_S(\mathfrak{f})$$
$$[(\widetilde{\Theta} * \eta_S) \widetilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S)](h) = (\widetilde{\Theta} * \eta_S)(h) \cup (\eta_S * \widetilde{\Theta} * \eta_S)(h)$$
$$= [\widetilde{\Theta}(h) \cup \eta_S(h)] \cup [\eta_S(h) \cup (\widetilde{\Theta} * \eta_S)(h)] = \eta_S(h) \cup \eta_S(h) = \eta_S(h) \supseteq \eta_S(h)$$
$$[(\widetilde{\Theta} * \eta_S) \widetilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S)](\mathfrak{r}) = (\widetilde{\Theta} * \eta_S)(\mathfrak{r}) \cup (\eta_S * \widetilde{\Theta} * \eta_S)(\mathfrak{r})$$
$$= \Big[[\widetilde{\Theta}(f) \cup \eta_S(h)] \cap [\widetilde{\Theta}(f) \cup \eta_S(\mathfrak{r})] \cap [\widetilde{\Theta}(h) \cup \eta_S(\mathfrak{f})] \cap [\widetilde{\Theta}(h) \cup \eta_S(\mathfrak{r})]$$
$$\cap [\widetilde{\Theta}(\mathfrak{r}) \cup \eta_S(\mathfrak{f})] \cap [\widetilde{\Theta}(\mathfrak{r}) \cup \eta_S(h)] \cap [\widetilde{\Theta}(\mathfrak{r}) \cup \eta_S(\mathfrak{r})]\Big]$$
$$\cup \Big[[\eta_S(\mathfrak{f}) \cup (\widetilde{\Theta} * \eta_S)(h)] \cap [\eta_S(\mathfrak{f}) \cup (\widetilde{\Theta} * \eta_S)(\mathfrak{r})] \cap [\eta_S(h) \cup (\widetilde{\Theta} * \eta_S)(\mathfrak{f})]$$
$$\cap [\eta_S(h) \cup (\widetilde{\Theta} * \eta_S)(\mathfrak{r})] \cap [\eta_S(\mathfrak{r}) \cup (\widetilde{\Theta} * \eta_S)(\mathfrak{f})] \cap [\eta_S(\mathfrak{r}) \cup (\widetilde{\Theta} * \eta_S)(h)]$$
$$\cap [\eta_S(\mathfrak{r}) \cup (\widetilde{\Theta} * \eta_S)(\mathfrak{r})]\Big] = [\eta_S(h) \cap \eta_S(\mathfrak{r}) \cap \eta_S(\mathfrak{f})] \cup [\eta_S(h) \cap \eta_S(\mathfrak{r}) \cap \eta_S(\mathfrak{f})]$$
$$= \eta_S(h) \cap \eta_S(\mathfrak{r}) \cap \eta_S(\mathfrak{f}) \supseteq \eta_S(\mathfrak{r})$$

It can be easily shown that the SS η_S satisfies the S-uni ʟ-ƁQ ideal condition for all other element combinations of the set S. Similarly,

$$[(\eta_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S)](\mathfrak{f}) \supseteq \eta_S(\mathfrak{f}), \quad [(\eta_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S)](h) \supseteq \eta_S(h)$$
$$[(\eta_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S)](\text{ɍ}) \supseteq \eta_S(\text{ɍ})$$

It can be easily shown that the SS η_S satisfies the S-uni ʀ-ƁQ ideal condition for all other element combinations of the set S, thus η_S is an S-uni ƁQ ideal. However, since

$$[(\widetilde{\theta} * \text{Ꞩ}_S) \,\widetilde{\cup}\, (\text{Ꞩ}_S * \widetilde{\theta} * \text{Ꞩ}_S)](\text{ɍ}) = [\text{Ꞩ}_S(h) \cap \text{Ꞩ}_S(\text{ɍ}) \cap \text{Ꞩ}_S(\mathfrak{f})] \not\supseteq \text{Ꞩ}_S(\text{ɍ})$$

Ꞩ_S is not an S-uni ƁQ ideal.

Corollary 3.3. $\widetilde{\theta}$ is an S-uni ƁQ ideals.

Proposition 3.4. Every S-uni bi-ideal is an S-uni ʀ-ƁQ ideal.

Proof: Let ҕ_S be an S-uni bi-ideal of S. Then, $\text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S \,\widetilde{\supseteq}\, \text{ҕ}_S$. Thus,

$$(\text{ҕ}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S) \,\widetilde{\supseteq}\, \text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S \,\widetilde{\supseteq}\, \text{ҕ}_S$$

Hence, ҕ_S is an S-uni ʀ-ƁQ ideal of S.

We show with a counterexample that the converse of Proposition 3.4 is not true:

Example 3.5. Consider the semigroup $S = \{\text{ɚ}, \text{ƴ}, \mathfrak{r}, \mathfrak{s}\}$ defined by the following table:

Table 2: Cayley table of '⸙' binary operation.

⸙	ɚ	ƴ	𝔯	𝔰
ɚ	ɚ	ɚ	ɚ	ɚ
ƴ	ɚ	ɚ	ɚ	ɚ
𝔯	ɚ	ɚ	ɚ	Ƴ
𝔰	ɚ	ɚ	ƴ	𝔯

Let ҕ_S be an SS over $U = N$ as follows:

$$\text{ҕ}_S = \{(\text{ɚ}, \{4\}), (\text{ƴ}, \{1,2,4\}), (\mathfrak{r}, \{4,5\}), (\mathfrak{s}, \{1,2,3,4\})\}$$

Here, ҕ_S is an S-uni ʀ-ƁQ ideal. In fact,

$$[(\text{ҕ}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S)](\text{ɚ}) = \text{ҕ}_S(\text{ɚ}) \cap \text{ҕ}_S(\text{ƴ}) \cap \text{ҕ}_S(\mathfrak{r}) \cap \text{ҕ}_S(\mathfrak{s}) \supseteq \text{ҕ}_S(\text{ɚ})$$
$$[(\text{ҕ}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S)](\text{ƴ}) = \text{ҕ}_S(\mathfrak{s}) \supseteq \text{ҕ}_S(\text{ƴ}), \quad [(\text{ҕ}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S)](\mathfrak{r}) = U \supseteq \text{ҕ}_S(\mathfrak{r})$$
$$[(\text{ҕ}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\text{ҕ}_S * \widetilde{\theta} * \text{ҕ}_S)](\mathfrak{s}) = U \supseteq \text{ҕ}_S(\mathfrak{s})$$

thus, ҕ_S is an S-uni ʀ-ƁQ ideal of S. However, since $(\text{ҕ}_S * \text{ҕ}_S)(\mathfrak{r}) = \text{ҕ}_S(\mathfrak{s}) \cup \text{ҕ}_S(\mathfrak{s}) \not\supseteq \text{ҕ}_S(\mathfrak{r})$.

ҕ_S is not an S-uni bi-ideal.

Proposition 3.6 shows that the converse of Proposition 3.4 holds for soft ʟ-simple semigroups.

Proposition 3.6. Let $\text{ҕ}_S \in S_S(U)$ and S be a special soft ʟ-simple semigroup. Then, the following conditions are equivalent:

1. ҕ_S is an S-uni bi-ideal.
2. ҕ_S is an S-uni ʀ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Proposition 3.6. Assume that $ҕ_S$ is an S-uni ʀ-ƁQ ideal. By assumption, $\widetilde{\Theta} = \widetilde{\Theta} * ҕ_S$. Thus, $ҕ_S * ҕ_S = (ҕ_S * ҕ_S) \widetilde{\cup} (ҕ_S * ҕ_S) \widetilde{\supseteq} (ҕ_S * \widetilde{\Theta}) \widetilde{\cup} (ҕ_S * \widetilde{\Theta}) = (ҕ_S * \widetilde{\Theta}) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\supseteq} ҕ_S$.

Hence, $ҕ_S$ is an S-uni subsemigroup.

$$ҕ_S * \widetilde{\Theta} * ҕ_S = (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) = (ҕ_S * \widetilde{\Theta}) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\supseteq} ҕ_S$$

Thus, $ҕ_S$ is an S-uni bi-ideal.

Proposition 3.7. Every S-uni bi-ideal is an S-uni ʟ-ƁQ ideal.

Proof: Let $ҕ_S$ be an S-uni bi-ideal of S. Then, $ҕ_S * \widetilde{\Theta} * ҕ_S \widetilde{\supseteq} ҕ_S$. Thus,

$$(\widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\supseteq} ҕ_S * \widetilde{\Theta} * ҕ_S \widetilde{\supseteq} ҕ_S$$

Hence, $ҕ_S$ is an S-uni ʟ-ƁQ ideal of S.

We show with a counterexample that the converse of Proposition 3.7 is not true:

Example 3.8. Consider the SS $ҕ_S$ in Example 3.5. The SS $ҕ_S$ is an S-uni ʟ-ƁQ ideal. Since,

$$[(\widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S)](ɚ) = ҕ_S(ɚ) \cap ҕ_S(y) \cap ҕ_S(\mathfrak{r}) \cap ҕ_S(\mathfrak{s}) \supseteq ҕ_S(ɚ)$$

$$[(\widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S)](y) = ҕ_S(\mathfrak{s}) \supseteq ҕ_S(y)$$

$$[(\widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S)](\mathfrak{r}) = \emptyset \supseteq ҕ_S(\mathfrak{r})$$

$$[(\widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S)](\mathfrak{s}) = \emptyset \supseteq ҕ_S(\mathfrak{s})$$

Hence, $ҕ_S$ is an S-uni ʟ-ƁQ ideal. However, since

$$(ҕ_S * ҕ_S)(\mathfrak{r}) = ҕ_S(\mathfrak{s}) \cup ҕ_S(\mathfrak{s}) \not\supseteq ҕ_S(\mathfrak{r})$$

$ҕ_S$ is not an S-uni bi-ideal.

Proposition 3.9 shows that the converse of Proposition 3.7 holds for special soft ʀ-simple semigroups.

Proposition 3.9. Let $ҕ_S \in S_S(U)$ and S be a special soft ʀ-simple semigroup. Then, the following conditions are equivalent:

1. $ҕ_S$ is an S-uni bi-ideal.
2. $ҕ_S$ is an S-uni ʟ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.7. Assume that $ҕ_S$ is an S-uni ʟ-ƁQ ideal. By assumption, $\widetilde{\Theta} = ҕ_S * \widetilde{\Theta}$. Thus,

$ҕ_S * ҕ_S = (ҕ_S * ҕ_S) \widetilde{\cup} (ҕ_S * ҕ_S) \widetilde{\supseteq} (ҕ_S * \widetilde{\Theta}) \widetilde{\cup} (ҕ_S * \widetilde{\Theta}) = (ҕ_S * \widetilde{\Theta}) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\supseteq} ҕ_S$.

Hence, $ҕ_S$ is an S-uni subsemigroup.

$$ҕ_S * \widetilde{\Theta} * ҕ_S = (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) = (\widetilde{\Theta} * ҕ_S) \widetilde{\cup} (ҕ_S * \widetilde{\Theta} * ҕ_S) \widetilde{\supseteq} ҕ_S$$

Thus, $ҕ_S$ is an S-uni bi-ideal.

Theorem 3.10. Every S-uni bi-ideal is an S-uni ƁQ ideal.

Proof: It is followed by Proposition 3.4 and Proposition 3.7.

Theorem 3.11 shows that the converse of Theorem 3.10 holds for special soft simple semigroup.

Theorem 3.11. Let $ҕ_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. $\mathfrak{h}_S$ is an S-uni bi-ideal.
2. $\mathfrak{h}_S$ is an S-uni ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.10. Assume that $\mathfrak{h}_S$ is an S-uni ƁQ ideal. Then, by Definition 2.15, S is both a special soft ʟ-simple and a special soft ʀ-simple semigroup. The rest of the proof follows from Proposition 3.6 and Proposition 3.9.

Proposition 3.12. Every S-uni ʀ-ideal is an S-uni ʀ-ƁQ ideal.

Proof: Let η_S be an S-uni ʀ-ideal of S. Then, $\eta_S * \widetilde{\theta} \widetilde{\supseteq} \eta_S$. Thus, $(\eta_S * \widetilde{\theta}) \widetilde{\cup} (\eta_S * \widetilde{\theta} * \eta_S) \widetilde{\supseteq} \eta_S * \widetilde{\theta} \widetilde{\supseteq} \eta_S$. Hence, η_S is an S-uni ʀ-ƁQ ideal of S.

Additionally, since η_S is an S-uni ʀ-ideal, by Theorem 2.13, it is an S-uni bi-ideal. Therefore, by Proposition 3.4, η_S is an S-uni ʀ-ƁQ ideal.

We show with a counterexample that the converse of Proposition 3.10 is not true:

Example 3.13. Consider the semigroup $S = \{ɣ, ʒ\}$ defined by the following table:

Table 3: Cayley table of '☼' binary operation.

☼	ɣ	ʒ
ɣ	ɣ	ʒ
ʒ	ɣ	ʒ

Let η_S be a SS over $U = \mathbb{Z}$ as follows:

$$\eta_S = \{(ɣ, \{1,3\}), (ʒ, \{1,2\})\}$$

Here, η_S is an S-uni ʀ-ƁQ ideal. In fact,

$$[(\eta_S * \widetilde{\theta}) \widetilde{\cup} (\eta_S * \widetilde{\theta} * \eta_S)](ɣ) = (\eta_S * \widetilde{\theta})(ɣ) \cup (\eta_S * \widetilde{\theta} * \eta_S)(ɣ) = \eta_S(ɣ) \supseteq \eta_S(ɣ)$$

$$[(\eta_S * \widetilde{\theta}) \widetilde{\cup} (\eta_S * \widetilde{\theta} * \eta_S)](ʒ) = (\eta_S * \widetilde{\theta})(ʒ) \cup (\eta_S * \widetilde{\theta} * \eta_S)(ʒ) = \eta_S(ʒ) \supseteq \eta_S(ʒ)$$

thus, η_S is an S-uni ʀ-ƁQ ideal of S. However, since

$$(\eta_S * \widetilde{\theta})(ɣ) = [\eta_S(ɣ) \cup \widetilde{\theta}(ɣ)] \cap [\eta_S(ʒ) \cup \widetilde{\theta}(ɣ)] = \eta_S(ɣ) \cap \eta_S(ʒ) \not\supseteq \eta_S(ɣ)$$

$$(\eta_S * \widetilde{\theta})(ʒ) = [\eta_S(ɣ) \cup \widetilde{\theta}(ʒ)] \cap [\eta_S(ʒ) \cup \widetilde{\theta}(ʒ)] = \eta_S(ɣ) \cap \eta_S(ʒ) \not\supseteq \eta_S(ʒ)$$

η_S is not an S-uni ʀ-ideal.

Proposition 3.14 shows that the converse of Proposition 3.12 holds for special soft ʟ-simple semigroups.

Proposition 3.14. Let $\eta_S \in S_S(U)$ and S be a special soft ʟ-simple semigroup. Then, the following conditions are equivalent:

1. η_S is an S-uni ʀ-ideal.
2. η_S is an S-uni ʀ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Proposition 3.12. Assume that η_S is an S-uni ʀ-ƁQ ideal. By assumption, $\widetilde{\theta} = \widetilde{\theta} * \eta_S$. Thus, $(\eta_S * \widetilde{\theta}) = (\eta_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta}) = (\eta_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S) \,\widetilde{\supseteq}\, \eta_S$.

Hence, η_S is an S-uni ʀ-ideal.

Proposition 3.15. Every S-uni ʀ-ideal is an S-uni ʟ-ƁQ ideal.

Proof: Let η_S be an S-uni ʀ-ideal of S. Then, $\eta_S * \widetilde{\theta} \,\widetilde{\supseteq}\, \eta_S$ and $\eta_S * \eta_S \,\widetilde{\supseteq}\, \eta_S$. Thus, $(\widetilde{\theta} * \eta_S) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S) \,\widetilde{\supseteq}\, \eta_S * \widetilde{\theta} * \eta_S \,\widetilde{\supseteq}\, \eta_S * \eta_S \,\widetilde{\supseteq}\, \eta_S$. Hence, η_S is an S-uni ʟ-ƁQ ideal of S.

Additionally, since η_S is an S-uni ʀ-ideal, by Theorem 2.13, it is an S-uni bi-ideal. Therefore, by Proposition 3.7, η_S is an S-uni ʟ-ƁQ ideal.

We show with a counterexample that the converse of Proposition 3.15 is not true:

Example 3.16. Consider the SS η_S in Example 3.13. The SS η_S is an S-uni ʟ-ƁQ ideal. Since,

$$[(\widetilde{\theta} * \eta_S) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S)](ɣ) = (\widetilde{\theta} * \eta_S)(ɣ) \cup (\eta_S * \widetilde{\theta} * \eta_S)(ɣ) = \eta_S(ɣ) \supseteq \eta_S(ɣ)$$

$$[(\widetilde{\theta} * \eta_S) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S)](ʑ) = (\widetilde{\theta} * \eta_S)(ʑ) \cup (\eta_S * \widetilde{\theta} * \eta_S)(ʑ) = \eta_S(ʑ) \supseteq \eta_S(ʑ)$$

Hence, η_S is an S-uni ʟ-ƁQ ideal. However, since

$$(\eta_S * \widetilde{\theta})(ɣ) = [\eta_S(ɣ) \cup \widetilde{\theta}(ɣ)] \cap [\eta_S(ʑ) \cup \widetilde{\theta}(ɣ)] = \eta_S(ɣ) \cap \eta_S(ʑ) \not\supseteq \eta_S(ɣ)$$

$$(\eta_S * \widetilde{\theta})(ʑ) = [\eta_S(ɣ) \cup \widetilde{\theta}(ʑ)] \cap [\eta_S(ʑ) \cup \widetilde{\theta}(ʑ)] = \eta_S(ɣ) \cap \eta_S(ʑ) \not\supseteq \eta_S(ʑ)$$

η_S is not an S-uni ʀ-ideal.

Proposition 3.17 shows that the converse of Proposition 3.15 holds for special soft simple semigroups.

Proposition 3.17. Let $\eta_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. η_S is an S-uni ʀ-ideal.
2. η_S is an S-uni ʟ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.15. Assume that η_S is an S-uni ʟ-ƁQ ideal. By assumption, $\widetilde{\theta} = \eta_S * \widetilde{\theta} = \widetilde{\theta} * \eta_S$. Thus, $(\eta_S * \widetilde{\theta}) = (\eta_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta}) = (\widetilde{\theta} * \eta_S) \,\widetilde{\cup}\, (\eta_S * \widetilde{\theta} * \eta_S) \,\widetilde{\supseteq}\, \eta_S$.

η_S is an S-uni ʀ-ideal.

Theorem 3.18. Every S-uni ʀ-ideal is an S-uni ƁQ ideal.

Proof: It is followed by Proposition 3.12 and Proposition 3.15.

Here note that the converse of Theorem 3.18 is not true follows from Example 3.13 and Example 3.16.

Theorem 3.19 shows that the converse of Theorem 3.18 holds for special soft simple semigroup.

Theorem 3.19. Let $f_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. η_S is an S-uni ʀ-ideal.
2. η_S is an S-uni ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.18. (2) implies (1) is obvious by Proposition 3.14 and Proposition 3.17.

Proposition 3.20. Every S-uni ʟ-ideal is an S-uni ʀ-ƁQ ideal.

Proof: Let f_S be an S-uni ʟ-ideal of S. Then, $\widetilde{\theta} * f_S \widetilde{\supseteq} f_S$ and $f_S * f_S \widetilde{\supseteq} f_S$. Thus, $(f_S * \widetilde{\theta}) \widetilde{\cup} (f_S * \widetilde{\theta} * f_S) \widetilde{\supseteq} f_S * \widetilde{\theta} * f_S \widetilde{\supseteq} f_S * f_S \widetilde{\supseteq} f_S$. Hence, f_S is an S-uni ʀ-ƁQ ideal of S.

Additionally, since f_S is an S-uni ʟ-ideal, by Theorem 2.13, it is an S-uni bi-ideal. Therefore, by Proposition 3.4, f_S is an S-uni ʀ-ƁQ ideal.

We show with a counterexample that the converse of Proposition 3.20 is not true:

Example 3.21. Consider the semigroup $S = \{\varrho, \mathfrak{Q}\}$ defined by the following table:

Table 4: Cayley table of '֎' binary operation.

֎	ϱ	$\mathfrak{Q}$
ϱ	ϱ	ϱ
$\mathfrak{Q}$	$\mathfrak{Q}$	$\mathfrak{Q}$

Let q_S be a SS over $U = \mathbb{Z}$ as follows:

$$q_S = \{(\varrho, \{3,6\}), (\mathfrak{Q}, \{3,9\})\}$$

Here, q_S is an S-uni ʀ-ƁQ ideal. In fact,

$$[(q_S * \widetilde{\theta}) \widetilde{\cup} (q_S * \widetilde{\theta} * q_S)](\varrho) = (q_S * \widetilde{\theta})(\varrho) \cup (q_S * \widetilde{\theta} * q_S)(\varrho) = q_S(\varrho) \supseteq q_S(\varrho)$$

$$[(q_S * \widetilde{\theta}) \widetilde{\cup} (q_S * \widetilde{\theta} * q_S)](\mathfrak{Q}) = (q_S * \widetilde{\theta})(\mathfrak{Q}) \cup (q_S * \widetilde{\theta} * q_S)(\mathfrak{Q}) = q_S(\mathfrak{Q}) \supseteq q_S(\mathfrak{Q})$$

thus, q_S is an S-uni ʀ-ƁQ ideal of S. However, since

$$(\widetilde{\theta} * q_S)(\varrho) = [\widetilde{\theta}(\varrho) \cup q_S(\varrho)] \cap [\widetilde{\theta}(\varrho) \cup q_S(\mathfrak{Q})] = q_S(\varrho) \cap q_S(\mathfrak{Q}) \not\supseteq q_S(\varrho)$$

$$(\widetilde{\theta} * q_S)(\mathfrak{Q}) = [\widetilde{\theta}(\mathfrak{Q}) \cup q_S(\varrho)] \cap [\widetilde{\theta}(\mathfrak{Q}) \cup q_S(\mathfrak{Q})] = q_S(\varrho) \cap q_S(\mathfrak{Q}) \not\supseteq q_S(\mathfrak{Q})$$

q_S is not an S-uni ʟ-ideal.

Proposition 3.22 shows that the converse of Proposition 3.20 holds for special soft simple semigroups.

Proposition 3.22. Let $q_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. q_S is an S-uni ʟ-ideal.
2. q_S is an S-uni ʀ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Proposition 3.20. Assume that q_S is an S-uni ʀ-ƁQ ideal. By assumption, $\widetilde{\theta} = q_S * \widetilde{\theta} = \widetilde{\theta} * q_S$. Thus, $\widetilde{\theta} * q_S = (\widetilde{\theta} * q_S) \widetilde{\cup} (\widetilde{\theta} * q_S) = (q_S * \widetilde{\theta}) \widetilde{\cup} (q_S * \widetilde{\theta} * q_S) \widetilde{\supseteq} q_S$. q_S is an S-uni ʟ- ideal.

Proposition 3.23. Every S-uni ʟ-ideal is an S-uni ʟ-ƁQ ideal.

Proof: Let q_S be an S-uni ʟ-ideal of S. Then, $\widetilde{\theta} * \text{q}_S \,\widetilde{\supseteq}\, \text{q}_S$. Thus, $(\widetilde{\theta} * \text{q}_S)\,\widetilde{\cup}\,(\text{q}_S * \widetilde{\theta} * \text{q}_S) \,\widetilde{\supseteq}\, \widetilde{\theta} * \text{q}_S \,\widetilde{\supseteq}\, \text{q}_S$. Hence, q_S is an S-uni ʟ-ƁQ ideal of S.

Additionally, since q_S is an S-uni ʟ-ideal, by Theorem 2.13, it is an S-uni bi-ideal. Therefore, by Proposition 3.7, q_S is an S-uni ʟ-ƁQ ideal.

We show with a counterexample that the converse of Proposition 3.23 is not true:

Example 3.24. Consider the SS q_S in Example 3.21. The SS q_S is an S-uni ʟ-ƁQ ideal. Since,

$$[(\widetilde{\theta} * \text{q}_S)\,\widetilde{\cup}\,(\text{q}_S * \widetilde{\theta} * \text{q}_S)](\varrho) = (\widetilde{\theta} * \text{q}_S)(\varrho) \cup (\text{q}_S * \widetilde{\theta} * \text{q}_S)(\varrho) = \text{q}_S(\varrho) \supseteq \text{q}_S(\varrho)$$

$$[(\widetilde{\theta} * \text{q}_S)\,\widetilde{\cup}\,(\text{q}_S * \widetilde{\theta} * \text{q}_S)](\mathfrak{Q}) = (\widetilde{\theta} * \text{q}_S)(\mathfrak{Q}) \cup (\text{q}_S * \widetilde{\theta} * \text{q}_S)(\mathfrak{Q}) = \text{q}_S(\mathfrak{Q}) \supseteq \text{q}_S(\mathfrak{Q})$$

Hence, q_S is an S-uni ʟ-ƁQ ideal. However, since

$$(\widetilde{\theta} * \text{q}_S)(\varrho) = [\widetilde{\theta}(\varrho) \cup \text{q}_S(\varrho)] \cap [\widetilde{\theta}(\varrho) \cup \text{q}_S(\mathfrak{Q})] = \text{q}_S(\varrho) \cap \text{q}_S(\mathfrak{Q}) \not\supseteq \text{q}_S(\varrho)$$

$$(\widetilde{\theta} * \text{q}_S)(\mathfrak{Q}) = [\widetilde{\theta}(\mathfrak{Q}) \cup \text{q}_S(\varrho)] \cap [\widetilde{\theta}(\mathfrak{Q}) \cup \text{q}_S(\mathfrak{Q})] = \text{q}_S(\varrho) \cap \text{q}_S(\mathfrak{Q}) \not\supseteq \text{q}_S(\mathfrak{Q})$$

q_S is not an S-uni ʟ-ideal.

Proposition 3.25 shows that the converse of Proposition 3.23 holds for special soft ʀ-simple semigroups.

Proposition 3.25. Let $\text{q}_S \in S_S(U)$ and S be a special soft ʀ-simple semigroup. Then, the following conditions are equivalent:

1. q_S is an S-uni ʟ-ideal.
2. q_S is an S-uni ʟ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.21. Assume that q_S is an S-uni ʟ-ƁQ ideal. By assumption, $\widetilde{\theta} = \text{q}_S * \widetilde{\theta}$. Thus, $\widetilde{\theta} * \text{q}_S = (\widetilde{\theta} * \text{q}_S)\,\widetilde{\cup}\,(\widetilde{\theta} * \text{q}_S) = (\widetilde{\theta} * \text{q}_S)\,\widetilde{\cup}\,(\text{q}_S * \widetilde{\theta} * \text{q}_S) \,\widetilde{\supseteq}\, \text{q}_S$. q_S is an S-uni ʟ-ideal.

Theorem 3.26. Every S-uni ʟ-ideal is an S-uni ƁQ ideal.

Proof: It is followed by Proposition 3.20 and Proposition 3.23.

Note that the converse of Theorem 3.26 is not true follows from Example 3.21 and Example 3.24.

Theorem 3.27 shows that the converse of Theorem 3.26 holds for special soft simple semigroup.

Theorem 3.27. Let $\text{q}_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. q_S is an S-uni ʟ-ideal.
2. q_S is an S-uni ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.26. (2) implies (1) is obvious by Proposition 3.22 and Proposition 3.25.

Theorem 3.28. Every S-uni ideal is an S-uni ƁQ ideal.

Proof: It follows by Theorem 3.18 and Theorem 3.26.

Theorem 3.29 shows that the converse of Theorem 3.28 holds for special soft simple semigroup.

Theorem 3.29. Let $\mathfrak{q}_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. $\mathfrak{q}_S$ is an S-uni ideal.
2. $\mathfrak{q}_S$ is an S-uni ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.28. (2) implies (1) is obvious by Proposition 3.19 and Proposition 3.26.

Proposition 3.30. Every S-uni quasi-ideal is an S-uni ʀ-ƁQ ideal.

Proof: Let f_S be an S-uni quasi-ideal of S. Then, $(\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * \mathfrak{h}_S) \widetilde{\supseteq} \mathfrak{h}_S$. Thus, $(\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\mathfrak{h}_S * \widetilde{\theta} * \mathfrak{h}_S) \widetilde{\supseteq} (\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * \widetilde{\theta} * \mathfrak{h}_S) \widetilde{\supseteq} (\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * \mathfrak{h}_S) \widetilde{\supseteq} \mathfrak{h}_S$. Hence, $\mathfrak{h}_S$ is an S-uni ʀ-ƁQ ideal of S.

We show with a counterexample that the converse of Proposition 3.30 is not true:

Example 3.31. Consider the SS $\mathfrak{h}_S$ in Example 3.5. The SS $\mathfrak{h}_S$ is an S-uni ʀ-ƁQ ideal. Since, $[(\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * \mathfrak{h}_S)](y) = \mathfrak{h}_S(\mathfrak{r}) \cap \mathfrak{h}_S(\mathfrak{s}) \not\supseteq \mathfrak{h}_S(y)$. Hence, $\mathfrak{h}_S$ is not an S-uni quasi ideal.

Proposition 3.32 shows that the converse of Proposition 3.30 holds for special soft ʀ-simple semigroups.

Proposition 3.32. Let $\mathfrak{h}_S \in S_S(U)$ and S be a special soft ʀ-simple semigroup. Then, the following conditions are equivalent:

1. $\mathfrak{h}_S$ is an S-uni quasi-ideal.
2. $\mathfrak{h}_S$ is an S-uni ʀ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.30. Assume that $\mathfrak{h}_S$ is an S-uni ʀ-ƁQ ideal. By assumption, $\widetilde{\theta} = \mathfrak{h}_S * \widetilde{\theta}$. Thus, $(\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * \mathfrak{h}_S) = (\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\mathfrak{h}_S * \widetilde{\theta} * \mathfrak{h}_S) \widetilde{\supseteq} \mathfrak{h}_S$, implying that $\mathfrak{h}_S$ is an S-uni quasi-ideal.

Proposition 3.33. Every S-uni quasi-ideal is an S-uni ʟ-ƁQ ideal.

Proof: Let g_S be an S-uni quasi-ideal of S. Then, $(g_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * g_S) \widetilde{\supseteq} g_S$. Thus, $(\widetilde{\theta} * g_S) \widetilde{\cup} (g_S * \widetilde{\theta} * g_S) \widetilde{\supseteq} (\widetilde{\theta} * g_S) \widetilde{\cup} (g_S * \widetilde{\theta} * \widetilde{\theta}) \widetilde{\supseteq} (\widetilde{\theta} * g_S) \widetilde{\cup} (g_S * \widetilde{\theta}) \widetilde{\supseteq} g_S$. Hence, g_S is an S-uni ʟ-ƁQ ideal of S.

We show with a counterexample that the converse of Proposition 3.33 is not true:

Example 3.34. Consider the SS $\mathfrak{h}_S$ in Example 3.5. The SS $\mathfrak{h}_S$ is an S-uni ʟ-ƁQ ideal. Since, $[(\mathfrak{h}_S * \widetilde{\theta}) \widetilde{\cup} (\widetilde{\theta} * \mathfrak{h}_S)](y) = \mathfrak{h}_S(\mathfrak{r}) \cap \mathfrak{h}_S(\mathfrak{s}) \not\supseteq \mathfrak{h}_S(y)$. Hence, $\mathfrak{h}_S$ is not an S-uni quasi-ideal.

Proposition 3.35 shows that the converse of Proposition 3.33 holds for special soft simple semigroups.

Proposition 3.35. Let $g_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. g_S is an S-uni quasi-ideal.
2. g_S is an S-uni ʟ-ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.33. Assume that $\mathscr{g}_S$ is an S-uni L-ƁQ ideal. By assumption, $\widetilde{\theta} = \mathscr{g}_S * \widetilde{\theta} = \widetilde{\theta} * \mathscr{g}_S$. Thus, $(\mathscr{g}_S * \widetilde{\theta}) \tilde{\cup} (\widetilde{\theta} * \mathscr{g}_S) = (\widetilde{\theta} * \mathscr{g}_S) \tilde{\cup} (\mathscr{g}_S * \widetilde{\theta} * \mathscr{g}_S) \tilde{\supseteq} \mathscr{g}_S$.

$\mathscr{g}_S$ is an S-uni quasi-ideal.

Theorem 3.36. Every S-uni quasi-ideal is an S-uni ƁQ ideal.

Proof: It follows by Theorem 3.30 and Theorem 3.33.

Here note that the converse of Theorem 3.36 is not true follows from Example 3.31 and Example 3.34.

Theorem 3.37 shows that the converse of Theorem 3.38 holds for special soft simple semigroup.

Theorem 3.37. Let $\mathscr{g}_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. $\mathscr{g}_S$ is an S-uni quasi-ideal.
2. $\mathscr{g}_S$ is an S-uni ƁQ ideal.

Proof: (1) implies (2) is obvious by Theorem 3.36. (2) implies (1) is obvious by Proposition 3.32 and Proposition 3.35.

Proposition 3.38. Let ϑ_S be an idempotent SS over U. If ϑ_S is an S-uni interior ideal, then ϑ_S is an S-uni L-ƁQ ideal.

Proof: Let ϑ_S be an idempotent S-uni interior ideal of S. Then, $\vartheta_S * \vartheta_S = \vartheta_S$ and $\widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. Thus, $(\widetilde{\theta} * \vartheta_S) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \widetilde{\theta} * \vartheta_S = \widetilde{\theta} * \vartheta_S * \vartheta_S \tilde{\supseteq} \widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. Hence, ϑ_S is an S-uni L-ƁQ ideal of S.

Proposition 3.39. Let ϑ_S be an idempotent SS over U. If ϑ_S is an S-uni interior ideal, then ϑ_S is an S-uni R-ƁQ ideal.

Proof: Let ϑ_S be an idempotent S-uni interior ideal of S. Then, $\vartheta_S * \vartheta_S = \vartheta_S$ and $\widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. Thus, $(\vartheta_S * \widetilde{\theta}) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S * \widetilde{\theta} = \vartheta_S * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. Hence, ϑ_S is an S-uni R-ƁQ ideal of S.

Theorem 3.40. Let ϑ_S be an idempotent SS over U. If ϑ_S is an S-uni interior ideal, then ϑ_S is an S-uni ƁQ ideal.

Proof: It follows by Theorem 3.38 and Theorem 3.39.

Proposition 3.41. Let $\vartheta_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. ϑ_S is an S-uni interior ideal.
2. ϑ_S is an S-uni L-ƁQ ideal.

Proof: First assume that (1) holds. Where ϑ_S is an S-uni interior ideal of S. Then, $\widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. By assumption, $\widetilde{\theta} = \vartheta_S * \widetilde{\theta} = \widetilde{\theta} * \vartheta_S$. Thus,

$$(\widetilde{\theta} * \vartheta_S) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S * \widetilde{\theta} * \vartheta_S = \widetilde{\theta} * \vartheta_S * \vartheta_S \tilde{\supseteq} \widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$$

ϑ_S is an S-uni L-ƁQ ideal.

Conversely, assume that (2) holds. Where ϑ_S is an S-uni L-ƁQ ideal of S. Then, $(\widetilde{\theta} * \vartheta_S) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S$. In order to show that ϑ_S S-uni interior ideal, we need to show that $\widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. By assumption, $\widetilde{\theta} = \vartheta_S * \widetilde{\theta} = \widetilde{\theta} * \vartheta_S$. Thus,

$$\begin{aligned}\widetilde{\theta} * \vartheta_S * \widetilde{\theta} &= (\widetilde{\theta} * \vartheta_S * \widetilde{\theta}) \tilde{\cup} (\widetilde{\theta} * \vartheta_S * \widetilde{\theta}) \\ &= (\widetilde{\theta} * \widetilde{\theta} * \vartheta_S) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \widetilde{\theta}) \tilde{\supseteq} (\widetilde{\theta} * \vartheta_S) \tilde{\cup} (\vartheta_S * \widetilde{\theta}) \\ &= (\widetilde{\theta} * \vartheta_S) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S\end{aligned}$$

ϑ_S is an S-uni interior ideal.

Proposition 3.42. Let $\vartheta_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. ϑ_S is an S-uni interior ideal.
2. ϑ_S is an S-uni R-ƁQ ideal.

Proof: First assume that (1) holds.Where ϑ_S is an S-uni interior ideal of S. Then, $\widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. By assumption, $\widetilde{\theta} = \vartheta_S * \widetilde{\theta} = \widetilde{\theta} * \vartheta_S$. Thus,

$$(\vartheta_S * \widetilde{\theta}) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S * \widetilde{\theta} * \vartheta_S = \widetilde{\theta} * \vartheta_S * \vartheta_S \tilde{\supseteq} \widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$$

ϑ_S is an S-uni R-ƁQ ideal.

Conversely, assume that (2) holds. Where f_S is an S-uni R-ƁQ ideal of S. Then, $(\vartheta_S * \widetilde{\theta}) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S$. In order to show that ϑ_S S-uni interior ideal, we need to show that $\widetilde{\theta} * \vartheta_S * \widetilde{\theta} \tilde{\supseteq} \vartheta_S$. By assumption, $\widetilde{\theta} = \vartheta_S * \widetilde{\theta} = \widetilde{\theta} * \vartheta_S$. Thus,

$$\begin{aligned}\widetilde{\theta} * \vartheta_S * \widetilde{\theta} &= (\widetilde{\theta} * \vartheta_S * \widetilde{\theta}) \tilde{\cup} (\widetilde{\theta} * \vartheta_S * \widetilde{\theta}) \\ &= (\vartheta_S * \widetilde{\theta} * \widetilde{\theta}) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \widetilde{\theta}) \tilde{\supseteq} (\vartheta_S * \widetilde{\theta}) \tilde{\cup} (\vartheta_S * \widetilde{\theta}) \\ &= (\vartheta_S * \widetilde{\theta}) \tilde{\cup} (\vartheta_S * \widetilde{\theta} * \vartheta_S) \tilde{\supseteq} \vartheta_S\end{aligned}$$

ϑ_S is an S-uni interior ideal.

Theorem 3.43. Let $\vartheta_S \in S_S(U)$ and S be a special soft simple semigroup. Then, the following conditions are equivalent:

1. ϑ_S is an S-uni interior ideal.
2. ϑ_S is an S-uni ƁQ ideal.

Proof: It follows by Theorem 3.41 and Theorem 3.42.

Proposition 3.44. Let $\mathfrak{p}_S$ and $\mathfrak{s}_S$ be S-uni L-(R-) ƁQ ideals. Then, $\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S$ is an S-uni L-(R-) ƁQ ideal.

Proof: The proof is presented only for S-uni L-ƁQ ideal, as the proof for S-uni R- ƁQ ideal can be shown similarly. Let $\mathfrak{p}_S$ and $\mathfrak{s}_S$ be S-uni L-ƁQ ideals of S. Then, $(\widetilde{\theta} * \mathfrak{p}_S) \tilde{\cup} (\mathfrak{p}_S * \widetilde{\theta} * \mathfrak{p}_S) \tilde{\supseteq} \mathfrak{p}_S$ and $(\widetilde{\theta} * \mathfrak{s}_S) \tilde{\cup} (\mathfrak{s}_S * \widetilde{\theta} * \mathfrak{s}_S) \tilde{\supseteq} \mathfrak{s}_S$. Thus,

$$[\widetilde{\theta} * (\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S)] \tilde{\cup} [(\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S)] \tilde{\supseteq} (\widetilde{\theta} * \mathfrak{p}_S) \tilde{\cup} (\mathfrak{p}_S * \widetilde{\theta} * \mathfrak{p}_S) \tilde{\supseteq} \mathfrak{p}_S$$

and

$$[\widetilde{\theta} * (\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S)] \tilde{\cup} [(\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{p}_S \tilde{\cup} \mathfrak{s}_S)] \tilde{\supseteq} (\widetilde{\theta} * \mathfrak{s}_S) \tilde{\cup} (\mathfrak{s}_S * \widetilde{\theta} * \mathfrak{s}_S) \tilde{\supseteq} \mathfrak{s}_S$$

Hence, $\left[\widetilde{\theta} * (\mathfrak{p}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\cup} \left[(\mathfrak{p}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{p}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\supseteq} \mathfrak{p}_S \widetilde{\cup} \mathfrak{s}_S$. Thus, $\mathfrak{p}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni L-ƁQ ideals.

Theorem 3.45. Let $\mathfrak{p}_S$ and $\mathfrak{s}_S$ be S-uni ƁQ ideals. Then, $\mathfrak{p}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni ƁQ ideals.

Corollary 3.46. The finite union of S-uni ƁQ ideals is an S-uni ƁQ ideal.

Proposition 3.47. Let $\mathfrak{q}_S$ and $\mathfrak{s}_S$ be S-uni L-(R-) ideals. Then, $\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni L-(R-) ƁQ ideal.

Proof: The proof is presented only for S-uni L-ƁQ ideal, as the proof for S-uni R- ƁQ ideal can be shown similarly. Let $\mathfrak{q}_S$ and $\mathfrak{s}_S$ be S-uni L-ideals of S. Then, $\widetilde{\theta} * \mathfrak{q}_S \widetilde{\supseteq} \mathfrak{q}_S$ and $\widetilde{\theta} * \mathfrak{s}_S \widetilde{\supseteq} \mathfrak{s}_S$. Thus,

$$\left[\widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\cap} \left[(\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\supseteq} (\widetilde{\theta} * \mathfrak{q}_S) \widetilde{\cup} (\mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S) \widetilde{\supseteq} \widetilde{\theta} * \mathfrak{q}_S \widetilde{\supseteq} \mathfrak{q}_S$$

and

$$\left[\widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\cup} \left[(\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\supseteq} (\widetilde{\theta} * \mathfrak{s}_S) \widetilde{\cup} (\mathfrak{s}_S * \widetilde{\theta} * \mathfrak{s}_S) \widetilde{\supseteq} \widetilde{\theta} * \mathfrak{s}_S \widetilde{\supseteq} \mathfrak{s}_S$$

Hence, $\left[\widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\cap} \left[(\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\supseteq} \mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$. Thus, $\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni L-ƁQ ideals.

Theorem 3.48. Let $\mathfrak{q}_S$ and $\mathfrak{s}_S$ be S-uni ideals. Then, $\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni ƁQ ideals.

Theorem 3.49. Let $\mathfrak{q}_S$ be an S-uni R-ideal and $\mathfrak{s}_S$ be an S-uni L- ideal. Then, $\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni ƁQ ideal.

Proof: Let $\mathfrak{q}_S$ be an S-uni R-ideal and $\mathfrak{s}_S$ be an S-uni L-ideal. Then, $\mathfrak{q}_S * \widetilde{\theta} \widetilde{\supseteq} \mathfrak{q}_S$, $\widetilde{\theta} * \mathfrak{s}_S \widetilde{\supseteq} \mathfrak{s}_S$, and $\mathfrak{q}_S * \mathfrak{q}_S \widetilde{\supseteq} \mathfrak{q}_S$, $\mathfrak{s}_S * \mathfrak{s}_S \widetilde{\supseteq} \mathfrak{s}_S$.Thus,

$$\left[\widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\cup} \left[(\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\supseteq} \left(\widetilde{\theta} * \mathfrak{s}_S\right) \widetilde{\cup} \left(\mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S\right) \widetilde{\supseteq} \mathfrak{s}_S \widetilde{\cup} (\mathfrak{q}_S * \mathfrak{q}_S) \widetilde{\supseteq} \mathfrak{s}_S \widetilde{\cup} \mathfrak{q}_S$$

Hence, $\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni L-ƁQ ideal. Similarly, since

$$\left[(\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta}\right] \widetilde{\cup} \left[(\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S) * \widetilde{\theta} * (\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S)\right] \widetilde{\supseteq} (\mathfrak{q}_S * \widetilde{\theta}) \widetilde{\cup} (\mathfrak{s}_S * \widetilde{\theta} * \mathfrak{s}_S) \widetilde{\supseteq} \mathfrak{q}_S \widetilde{\cup} (\mathfrak{s}_S * \mathfrak{s}_S) \widetilde{\supseteq} \mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$$

$\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni R-ƁQ ideal. Therefore, $\mathfrak{q}_S \widetilde{\cup} \mathfrak{s}_S$ is an S-uni ƁQ ideal.

Theorem 3.50. Let ϑ_S be an S-uni L-ƁQ ideal and t_S be an S-uni L-ideal. Then, $\vartheta_S \widetilde{\cup} \mathrm{t}_S$ is an S-uni ƁQ ideal.

Proof: Let ϑ_S be an S-uni L-ƁQ ideal and t_S be an S-uni L-ideal. Then, $(\widetilde{\theta} * \vartheta_S) \widetilde{\cup} \left(\vartheta_S * \widetilde{\theta} * \vartheta_S\right) \widetilde{\supseteq} \vartheta_S$ and $\widetilde{\theta} * \mathrm{t}_S \widetilde{\supseteq} \mathrm{t}_S$. Thus,

$$\left[\widetilde{\theta} * (\vartheta_S \widetilde{\cup} \mathrm{t}_S)\right] \widetilde{\cup} \left[(\vartheta_S \widetilde{\cup} \mathrm{t}_S) * \widetilde{\theta} * (\vartheta_S \widetilde{\cup} \mathrm{t}_S)\right] \widetilde{\supseteq} (\widetilde{\theta} * \vartheta_S) \widetilde{\cup} \left(\vartheta_S * \widetilde{\theta} * \vartheta_S\right) \widetilde{\supseteq} \vartheta_S$$

$$\left[\widetilde{\theta} * (\vartheta_S \widetilde{\cup} \mathrm{t}_S)\right] \widetilde{\cup} \left[(\vartheta_S \widetilde{\cup} \mathrm{t}_S) * \widetilde{\theta} * (\vartheta_S \widetilde{\cup} \mathrm{t}_S)\right] \widetilde{\supseteq} (\widetilde{\theta} * \mathrm{t}_S) \widetilde{\cup} \left(\mathrm{t}_S * \widetilde{\theta} * \mathrm{t}_S\right) \widetilde{\supseteq} \widetilde{\theta} * \mathrm{t}_S \widetilde{\supseteq} \mathrm{t}_S$$

Hence, $\left[\widetilde{\theta} * (\vartheta_S \widetilde{\cup} \mathrm{t}_S)\right] \widetilde{\cup} \left[(\vartheta_S \widetilde{\cup} \mathrm{t}_S) * \widetilde{\theta} * (\vartheta_S \widetilde{\cup} \mathrm{t}_S)\right] \widetilde{\supseteq} \vartheta_S \widetilde{\cup} \mathrm{t}_S$. Thus, $\vartheta_S \widetilde{\cup} \mathrm{t}_S$ is an S-uni L-ƁQ ideal.

Theorem 3.51. Let $\mathfrak{s}_S$ be an S-uni L-ideal and $\mathfrak{p}_S$ be a SS over U. Then, $\mathfrak{s}_S * \mathfrak{p}_S$ is an S-uni L-ƁQ ideal.

Proof: Let $\mathfrak{s}_S$ be an S-uni L- ideal. Then, $\widetilde{\theta} * \mathfrak{s}_S \widetilde{\supseteq} \mathfrak{s}_S$. Thus,

$$\left[\widetilde{\theta} * (\mathfrak{s}_S * \mathfrak{p}_S)\right] \widetilde{\cup} \left[(\mathfrak{s}_S * \mathfrak{p}_S) * \widetilde{\theta} * (\mathfrak{s}_S * \mathfrak{p}_S)\right] \widetilde{\supseteq} \widetilde{\theta} * (\mathfrak{s}_S * \mathfrak{p}_S) = \left(\widetilde{\theta} * \mathfrak{s}_S\right) * \mathfrak{p}_S \widetilde{\supseteq} \mathfrak{s}_S * \mathfrak{p}_S$$

Hence, $\mathfrak{s}_S * \mathfrak{p}_S$ is an S-uni L-ƁQ ideal.

Theorem 3.52. Let $\mathfrak{s}_S$ be an S-uni ʀ-ideal and $\mathfrak{p}_S$ be a SS over U. Then, $\mathfrak{p}_S * \mathfrak{s}_S$ is an S-uni ʀ-ƁQ ideal.

Proof: Let $\mathfrak{s}_S$ be an S-uni ʀ-ideal. Then, $\mathfrak{s}_S * \widetilde{\Theta} \widetilde{\supseteq} \mathfrak{s}_S$. Thus,

$$\left[(\mathfrak{p}_S * \mathfrak{s}_S) * \widetilde{\Theta}\right] \widetilde{\cup} \left[(\mathfrak{p}_S * \mathfrak{s}_S) * \widetilde{\Theta} * (\mathfrak{p}_S * \mathfrak{s}_S)\right] \widetilde{\supseteq} (\mathfrak{p}_S * \mathfrak{s}_S) * \widetilde{\Theta} = \mathfrak{p}_S * (\mathfrak{s}_S * \widetilde{\Theta}) \widetilde{\supseteq} \mathfrak{p}_S * \mathfrak{s}_S$$

Hence, $\mathfrak{p}_S * \mathfrak{s}_S$ is an S-uni ʀ-ƁQ ideal.

Theorem 3.53. Let $\mathfrak{h}_S$ be a nonempty SS over U. Then, every SS containing $\mathfrak{h}_S$ which is the soft superset of $(\widetilde{\Theta} * \mathfrak{h}_S) \widetilde{\cap} (\mathfrak{h}_S * \widetilde{\Theta})$ is an S-uni ƁQ ideal.

Proof: Let $\mathfrak{p}_S \widetilde{\supseteq} \mathfrak{h}_S$ and $\mathfrak{p}_S \widetilde{\supseteq} (\widetilde{\Theta} * \mathfrak{h}_S) \widetilde{\cap} (\mathfrak{h}_S * \widetilde{\Theta})$. Since,

$$\widetilde{\Theta} * \mathfrak{p}_S \widetilde{\supseteq} \widetilde{\Theta} * \mathfrak{h}_S \widetilde{\supseteq} (\widetilde{\Theta} * \mathfrak{h}_S) \widetilde{\cap} (\mathfrak{h}_S * \widetilde{\Theta}) \widetilde{\supseteq} \mathfrak{p}_S$$

Thus, $\widetilde{\Theta} * \mathfrak{p}_S \widetilde{\supseteq} \mathfrak{p}_S$, implying that $\mathfrak{p}_S$ is an S-uni ʟ-ideal. Similarly,

$$\mathfrak{p}_S * \widetilde{\Theta} \widetilde{\supseteq} \mathfrak{h}_S * \widetilde{\Theta} \widetilde{\supseteq} (\widetilde{\Theta} * \mathfrak{h}_S) \widetilde{\cap} (\mathfrak{h}_S * \widetilde{\Theta}) \widetilde{\supseteq} \mathfrak{p}_S$$

Thereby, $\mathfrak{p}_S * \widetilde{\Theta} \widetilde{\supseteq} \mathfrak{p}_S$, $\mathfrak{p}_S$ is an S-uni ʀ-ideal. Therefore, $\mathfrak{p}_S$ is an S-uni ideal. Thus, by Theorem 3.28, $\mathfrak{p}_S$ is an S-uni ƁQ ideal.

Theorem 3.54. Let ϑ_S be a nonempty SS over U. Then, every SS containing ϑ_S which is the soft superset of $\widetilde{\Theta} * \vartheta_S$ is an S-uni ʟ- ƁQ ideal.

Proof: Let $\mathfrak{h}_S \widetilde{\supseteq} \vartheta_S$ and $\mathfrak{h}_S \widetilde{\supseteq} \widetilde{\Theta} * \vartheta_S$. Since, $\widetilde{\Theta} * \mathfrak{h}_S \widetilde{\supseteq} \widetilde{\Theta} * \vartheta_S \widetilde{\supseteq} \mathfrak{h}_S$, $\widetilde{\Theta} * \mathfrak{h}_S \widetilde{\supseteq} \mathfrak{h}_S$ is obtained. Hence, $\mathfrak{h}_S$ is an S-uni ʟ-ideal. Thus, by Theorem 3.23, $\mathfrak{h}_S$ is an S-uni ƁQ ideal.

Theorem 3.55. Let ϑ_S be a nonempty SS over U. Then, every SS containing ϑ_S, and contained by $(\widetilde{\Theta} * \vartheta_S) \widetilde{\cap} (\vartheta_S * \widetilde{\Theta} * \vartheta_S)$ is an S-uni ʟ-ƁQ ideal.

Proof: Let $\mathfrak{h}_S \widetilde{\supseteq} \vartheta_S$ and $\mathfrak{h}_S \widetilde{\supseteq} (\widetilde{\Theta} * \vartheta_S) \widetilde{\cap} (\vartheta_S * \widetilde{\Theta} * \vartheta_S)$. Then, $\widetilde{\Theta} * \mathfrak{h}_S \widetilde{\supseteq} \widetilde{\Theta} * \vartheta_S$ and $\mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S \widetilde{\supseteq} \vartheta_S * \widetilde{\Theta} * \vartheta_S$. Since,

$$(\widetilde{\Theta} * \mathfrak{h}_S) \widetilde{\cap} (\mathfrak{h}_S * \widetilde{\Theta} * \mathfrak{h}_S) \widetilde{\supseteq} (\widetilde{\Theta} * \vartheta_S) \widetilde{\cap} (\vartheta_S * \widetilde{\Theta} * \vartheta_S) \widetilde{\supseteq} \mathfrak{h}_S$$

$\mathfrak{h}_S$ is an S-uni ʟ-ideal.

Proposition 3.56. Let ρ_S, be an S-uni subsemigroup over U, σ be a subset of U, $Im(\rho_S)$ be the image of ρ_S such that $\sigma \in Im(\rho_S)$. If ρ_S is an S-uni ʟ −(ʀ-) ƁQ ideal of S, then $\mathcal{L}(\rho_S;\sigma)$ is a ʟ-(ʀ-) ƁQ ideal.

Proof: The proof is presented only for S-uni ʟ- ƁQ ideal, as the proof for S-uni ʀ-ƁQ ideal can be shown similarly. Since, $\rho_S(\mathrm{x}) = \sigma$ for some $\mathrm{x} \in S$, $\emptyset \neq \mathcal{L}(\rho_S;\sigma) \subseteq S$. Let $\kappa \in (S.\mathcal{L}(\rho_S;\sigma)) \cup (\mathcal{L}(\rho_S;\sigma).S.\mathcal{L}(\rho_S;\sigma))$. Then, there exist $\mathrm{x}, y, z \in \mathcal{L}(\rho_S;\sigma)$ and $r, s \in S$ such that $\kappa = s\mathrm{x} = yrz$. Thus, $\rho_S(x) \subseteq \sigma$, $\rho_S(y) \subseteq \sigma$ and $\rho_S(z) \subseteq \sigma$. Since ρ_S is an S-uni ʟ-ƁQ ideal,

$$\begin{aligned}(\widetilde{\Theta} * \rho_S)(\kappa) &= \bigcap_{\kappa=\mathfrak{m}n} \{\widetilde{\Theta}(\mathfrak{m}) \cup \rho_S(n)\} \\ &\subseteq \widetilde{\Theta}(s) \cup \rho_S(\mathrm{x}) \\ &= \emptyset \cup \rho_S(\mathrm{x}) = \rho_S(\mathrm{x}) \\ &\subseteq \sigma\end{aligned}$$

and

$$\begin{aligned}(\rho_S * \widetilde{\Theta} * \rho_S)(\kappa) &= \bigcap_{\kappa=\mathfrak{m}n}\{\rho_S(\mathfrak{m}) \cup (\widetilde{\Theta} * \rho_S)(n)\} \\ &\subseteq \rho_S(\mathrm{x}) \cup (\widetilde{\Theta} * \rho_S)(yz) \\ &= \rho_S(\mathrm{x}) \cup \left[\bigcap_{yz=pq}\{\widetilde{\Theta}(p) \cup \rho_S(q)\}\right] \\ &\subseteq \rho_S(\mathrm{x}) \cup \tilde{\theta}(y) \cup \rho_S(z) \\ &\subseteq \sigma \cup \emptyset \cup \sigma = \sigma.\end{aligned}$$

Thus, $(\widetilde{\Theta} * \rho_S)(\kappa) \cup (\rho_S * \widetilde{\Theta} * \rho_S)(\kappa)) \subseteq \sigma$. Since ρ_S is an S-uni L-BQ ideal, $\rho_S(\kappa) \subseteq (\widetilde{\Theta} * \rho_S)(\kappa) \cup (\rho_S * \widetilde{\Theta} * \rho_S)(\kappa) \subseteq \sigma$. Thus, $\kappa \in \mathcal{L}(\rho_S;\sigma)$. Therefore, $[S.\mathcal{L}(\rho_S;\sigma)] \cup [\mathcal{L}(\rho_S;\sigma).S.\mathcal{L}(\rho_S;\sigma)]$. Hence, $\mathcal{L}(\rho_S;\sigma)$ is a BQ ideal.

Theorem 3.57. Let ρ_S, be an S-uni subsemigroup over U, σ be a subset of U, $Im(\rho_S)$ be the image of ρ_S such that $\sigma \in Im(\rho_S)$. If ρ_S is an S-uni BQ ideal of S, then $\mathcal{L}(\rho_S;\sigma)$ is a BQ ideal.

We illustrate Theorem 3.57 with Example 3.58.

Example 3.58. Consider the SS η_S in Example 3.2. By considering the image set of η_S, that is,

$$Im(\eta_S) = \{\{e,x\},\{e,x,x^2\},\{e,x,x^2,y\}\}$$

we obtain the following:

$$\mathcal{L}(\eta_S;\sigma) = \begin{cases}\{\mathfrak{r}\}, & \sigma = \{e,x\} \\ \{\mathfrak{f},\mathfrak{r}\}, & \sigma = \{e,x,x^2\} \\ \{\mathfrak{f},h,\mathfrak{r}\}, & \sigma = \{e,x,x^2,y\}\end{cases}$$

Here, $\{f,h,\mathfrak{r}\},\{f,\mathfrak{r}\}$ and $\{\mathfrak{r}\}$ are all BQ ideals of S. In fact, since

$$\{\mathfrak{r}\}.\{\mathfrak{r}\} \subseteq \{\mathfrak{r}\}, \{\mathfrak{f},\mathfrak{r}\}.\{\mathfrak{f},\mathfrak{r}\} \subseteq \{\mathfrak{f},\mathfrak{r}\}, \{\mathfrak{f},h,\mathfrak{r}\}.\{\mathfrak{f},h,\mathfrak{r}\} \subseteq \{\mathfrak{f},h,\mathfrak{r}\}$$

each $\mathcal{L}(\eta_S;\sigma)$ is a subsemigroup of S. Similarly, since

$$(S.\{\mathfrak{r}\}) \cap (\{\mathfrak{r}\}.S.\{\mathfrak{r}\}) \subseteq \{\mathfrak{r}\} \cap \{\mathfrak{r}\} \subseteq \{\mathfrak{r}\}$$
$$(S.\{\mathfrak{f},\mathfrak{r}\}) \cap (\{\mathfrak{f},\mathfrak{r}\}.S.\{\mathfrak{f},\mathfrak{r}\}) \subseteq \{\mathfrak{f},\mathfrak{r}\} \cap \{\mathfrak{f},\mathfrak{r}\} \subseteq \{\mathfrak{f},\mathfrak{r}\}$$
$$(S.\{\mathfrak{f},h,\mathfrak{r}\}) \cap (\{\mathfrak{f},h,\mathfrak{r}\}.S.\{\mathfrak{f},h,\mathfrak{r}\}) \subseteq \{\mathfrak{f},h,\mathfrak{r}\} \cap \{\mathfrak{f},h,\mathfrak{r}\} \subseteq \{\mathfrak{f},h,\mathfrak{r}\}$$

each $\mathcal{L}(\eta_S;\sigma)$ is a L-BQ ideal of S. Similarly, since

$$(\{\mathfrak{r}\}.S) \cap (\{\mathfrak{r}\}.S.\{\mathfrak{r}\}) \subseteq \{\mathfrak{r}\} \cap \{\mathfrak{r}\} \subseteq \{\mathfrak{r}\}$$
$$(\{\mathfrak{f},\mathfrak{r}\}.S) \cap (\{\mathfrak{f},\mathfrak{r}\}.S.\{\mathfrak{f},\mathfrak{r}\}) \subseteq \{\mathfrak{f},\mathfrak{r}\} \cap \{\mathfrak{f},\mathfrak{r}\} \subseteq \{\mathfrak{f},\mathfrak{r}\}$$
$$(\{\mathfrak{f},h,\mathfrak{r}\}.S) \cap (\{\mathfrak{f},h,\mathfrak{r}\}.S.\{\mathfrak{f},h,\mathfrak{r}\}) \subseteq \{\mathfrak{f},h,\mathfrak{r}\} \cap \{\mathfrak{f},h,\mathfrak{r}\} \subseteq \{\mathfrak{f},h,\mathfrak{r}\}$$

each $\mathcal{L}(\eta_S;\sigma)$ is a R-BQ ideal of S, and thus each of $\mathcal{L}(\eta_S;\sigma)$ is a BQ ideal of S.

Now, consider the SS $\mathfrak{d}_S$ in Example 3.2. By taking into account

$$Im(\mathfrak{d}_S) = \{\{e,x\},\{e,x,y\},\{e,x^2,y,yx^2\}\}$$

we obtain the following:

$$\mathcal{L}(\mathfrak{d}_S;\sigma) = \begin{cases}\{\mathfrak{r}\}, & \sigma = \{e,x\} \\ \{\mathfrak{f},\mathfrak{r}\}, & \sigma = \{e,x,y\} \\ \{h\}, & \sigma = \{\mathrm{e},x^2,y,yx^2\}\end{cases}$$

Here, $\{h\}$ is not a BQ ideal of S. In fact, since

$$(S.\{h\}) \cap (\{h\}.S.\{h\}) \subseteq \{h,\mathfrak{r}\} \cap \{h,\mathfrak{r}\} \not\subseteq \{h\}$$

one of the $\mathcal{L}(\mathcal{S}_S;\sigma)$ is not a L-ƁQ ideal of S, hence it is not a ƁQ ideal of S, It is seen that each of $\mathcal{L}(\mathcal{S}_S;\sigma)$ is not a ƁQ ideal of S. On the other hand, in Example 3.2 it was shown that $\mathcal{S}_S$ is not an S-uni ƁQ ideal of S.

Proposition 3.59. Let S be a regular semigroup. Then, $\eta_S = (\widetilde{\Theta} * \eta_S) \tilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S)$ for every S-uni L-ƁQ ideal η_S.

Proof: Let S be a regular semigroup, η_S be an S-uni L-ƁQ ideal and $x \in S$. Then, $(\widetilde{\Theta} * \eta_S) \tilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S) \tilde{\supseteq} \eta_S$ and there exist an element $y \in S$ such that $x = xyx$. Since

$$\begin{aligned}(\widetilde{\Theta} * \eta_S)(x) &= \bigcap_{x=kn}\{\widetilde{\Theta}(k) \cup \eta_S(n)\}\\ &\subseteq \widetilde{\Theta}(xy) \cup \eta_S(x)\\ &= \emptyset \cup \eta_S(x)\\ &= \eta_S(x)\end{aligned}$$

and

$$\begin{aligned}(\eta_S * \widetilde{\Theta} * \eta_S)(x) &= \cap_{x=kn}\{\eta_S(k) \cup (\widetilde{\Theta} * \eta_S)(n)\}\\ &\subseteq \eta_S(x) \cup (\widetilde{\Theta} * \eta_S)(yx)\\ &= \eta_S(x) \cup \bigcap_{yx=rs}\{\widetilde{\Theta}(r) \cup \eta_S(s)\}\\ &\subseteq \eta_S(x) \cup \widetilde{\Theta}(y) \cup \eta_S(x)\\ &= \eta_S(x) \cup \emptyset \cup \eta_S(x)\\ &= \eta_S(x)\end{aligned}$$

Thus, $(\widetilde{\Theta} * \eta_S)(x) \cup (\eta_S * \widetilde{\Theta} * \eta_S)(x) \subseteq \eta_S(x) \cup \eta_S(x) \subseteq \eta_S(x)$ implying that $\eta_S \tilde{\supseteq} (\widetilde{\Theta} * \eta_S) \tilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S)$. Therefore, $\eta_S = (\widetilde{\Theta} * \eta_S) \tilde{\cup} (\eta_S * \widetilde{\Theta} * \eta_S)$.

Proposition 3.60. Let S be a regular semigroup. Then, $\flat_S = (\flat_S * \widetilde{\Theta}) \tilde{\cup} (\flat_S * \widetilde{\Theta} * \flat_S)$ for every S-uni R-ƁQ ideal $\flat_S$.

Proof: Let S be a regular semigroup, $\flat_S$ be an S-uni R-ƁQ ideal and $x \in S$. Then, $(\flat_S * \widetilde{\Theta}) \tilde{\cup} (\flat_S * \widetilde{\Theta} * \flat_S) \tilde{\supseteq} \flat_S$ and there exist an element $t \in S$ such that $x = xtx$. Since,

$$\begin{aligned}(\flat_S * \widetilde{\Theta})(x) &= \bigcap_{x=kn}\{\flat_S(k) \cup \widetilde{\Theta}(n)\}\\ &\subseteq \flat_S(x) \cup \widetilde{\Theta}(tx)\\ &= \flat_S(x) \cup \emptyset = \flat_S(x)\end{aligned}$$

and

$$\begin{aligned}(\flat_S * \widetilde{\Theta} * \flat_S)(x) &= \bigcap_{x=kn}\{\flat_S(k) \cup (\widetilde{\Theta} * \flat_S)(n)\}\\ &\subseteq \flat_S(x) \cup (\widetilde{\Theta} * \flat_S)(tx)\\ &= \flat_S(x) \cup \bigcap_{tx=qs}\{\widetilde{\Theta}(q) \cup \flat_S(s)\}\end{aligned}$$

$$\subseteq \mathfrak{b}_S(x) \cup \widetilde{\theta}(t) \cup \mathfrak{b}_S(x)$$
$$= \mathfrak{b}_S(x) \cup \emptyset \cup \mathfrak{b}_S(x)$$
$$= \mathfrak{b}_S(x).$$

Thus, $(\mathfrak{b}_S * \widetilde{\theta})(x) \cup (\mathfrak{b}_S * \widetilde{\theta} * \mathfrak{b}_S)(x) \subseteq \mathfrak{b}_S(x) \cup \mathfrak{b}_S(x) \subseteq \mathfrak{b}_S(x)$ implying that $\mathfrak{b}_S \,\widetilde{\supseteq}\, (\mathfrak{b}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\mathfrak{b}_S * \widetilde{\theta} * \mathfrak{b}_S)$. Therefore, $\mathfrak{b}_S = (\mathfrak{b}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\mathfrak{b}_S * \widetilde{\theta} * \mathfrak{b}_S)$.

Theorem 3.61. Let S be a regular semigroup. Then, $\eta_S = (\widetilde{\theta} * \eta_S) \,\widetilde{\cap}\, (\eta_S * \widetilde{\theta} * \eta_S) = (\eta_S * \widetilde{\theta}) \,\widetilde{\cap}\, (\eta_S * \widetilde{\theta} * \eta_S)$ for every S-uni BQ ideal.

Proof: It is followed by Proposition 3.59 and Proposition 3.60.

Proposition 3.62. Let S be a regular semigroup. Then every S-uni L-BQ ideal of a semigroup S is an S-uni quasi ideal of a semigroup.

Proof: Let f_S be an S-uni L-BQ ideal of S. Then, $(\widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\cup}\, (\mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, \mathfrak{q}_S$. We know that $\mathfrak{q}_S * \widetilde{\theta}$ and $\widetilde{\theta} * \mathfrak{q}_S$ are S-uni R-and S-uni L-ideals of the semigroup S respectively. By Corollary 2.16, we have

$$(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) = \mathfrak{q}_S * \widetilde{\theta} * \widetilde{\theta} * \mathfrak{q}_S$$

Thus, $(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, \widetilde{\theta} * \mathfrak{q}_S$ and $(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) = \mathfrak{q}_S * \widetilde{\theta} * \widetilde{\theta} * \mathfrak{q}_S \,\widetilde{\supseteq}\, \mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S$. Hence,

$$(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, (\widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\cup}\, (\mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, \mathfrak{q}_S$$

Therefore, $\mathfrak{q}_S$ is an S-uni quasi ideal.

Proposition 3.63. Let S be a regular semigroup. Then every S-uni R-BQ ideal of a semigroup S is an S-uni quasi ideal of a semigroup.

Proof: Let $\mathfrak{q}_S$ be an S-uni R-BQ ideal of S. Then, $(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, \mathfrak{q}_S$. We know that $\mathfrak{q}_S * \widetilde{\theta}$ and $\widetilde{\theta} * \mathfrak{q}_S$ are S-uni R-and S-uni L-ideals of the semigroup S respectively. By Corollary 2.16, we have

$$(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) = \mathfrak{q}_S * \widetilde{\theta} * \widetilde{\theta} * \mathfrak{q}_S$$

Thus, $(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, \mathfrak{q}_S * \widetilde{\theta}$ and $(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) = \mathfrak{q}_S * \widetilde{\theta} * \widetilde{\theta} * \mathfrak{q}_S \,\widetilde{\supseteq}\, \mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S$. Hence,

$$(\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, (\mathfrak{q}_S * \widetilde{\theta}) \,\widetilde{\cup}\, (\mathfrak{q}_S * \widetilde{\theta} * \mathfrak{q}_S) \,\widetilde{\supseteq}\, \mathfrak{q}_S$$

Therefore, $\mathfrak{q}_S$ is an S-uni quasi ideal.

Theorem 3.64. Let S be a regular semigroup. Then every S-uni BQ ideal of a semigroup S is an S-uni quasi ideal of semigroup.

Proof: It follows by Proposition 3.62 and Proposition 3.6.

The relation between several S-uni ideals and their generalized ideals is depicted in the following figure, where $\mathcal{A} \to \mathcal{B}$ denotes that $\mathcal{A}$ is $\mathcal{B}$ but $\mathcal{B}$ may not always be $\mathcal{A}$.

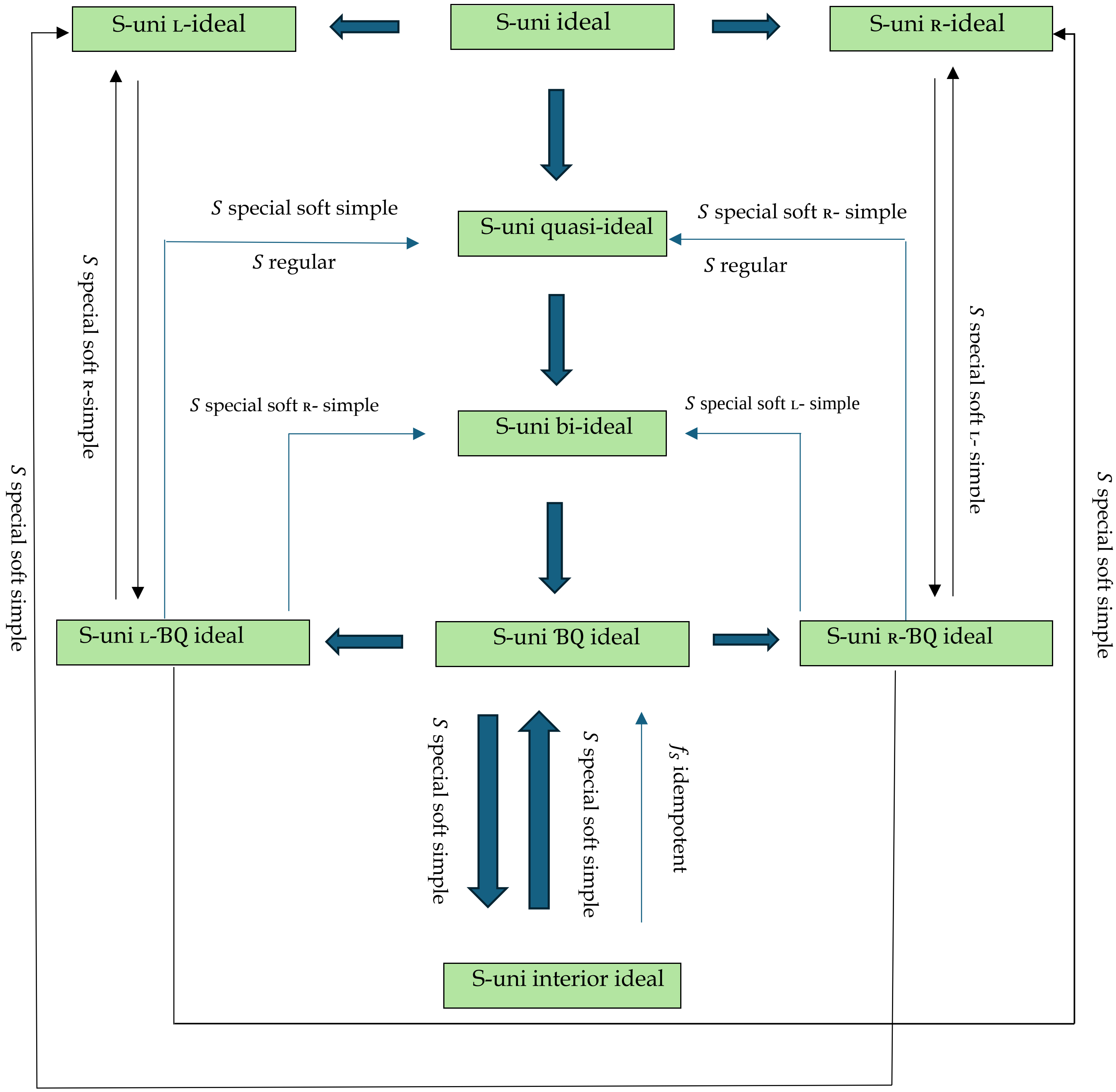

Figure 1. Diagram illustrating the relationships between some S-uni ideals.

4. Conclusion

Rao [8] expanded the notions of quasi-ideal, bi-ideal, L-(R-) ideal, and ideal in semigroups by defining ƁQ ideals and examining their characteristics. In this study, we applied the concept of "S-uni ƁQ ideals of semigroups" to both SS theory and semigroup theory. It has been shown that every S-uni bi-ideal, S-uni ideal, S-uni quasi-ideal, and S-uni interior ideal of an idempotent SS is an S-uni ƁQ ideal. Counterexamples show that the reverse is not always true, and for the reverse to hold, the semigroup must be special soft simple, or regular. It has also been demonstrated that in a special soft simple semigroup, the S-uni ƁQ ideal coincides with the S-uni bi-ideal, S-uni L- (R-) ideal, S-uni quasi-ideal, and S-uni interior ideal. The finite soft union of S-uni ƁQ ideals is shown to be S-uni ƁQ ideals, as are the soft union of S-uni ideals. Additionally, the relationship between regular semigroups and S-uni ƁQ ideals is explored. In later studies, various semigroup types can be used to characterize S-uni ƁQ ideals.

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