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Generalization of Δ-Closed Sets in Ideal Spaces

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Abstract. In the paper, we introduce g_{D^*}-closed sets and g_D-closed sets using ideal spaces, and some of the properties and characterizations are discussed. Further, the relationships among some of the existing generalizations are investigated with the closed sets. Every $\mathcal{I}_g$-closed set is g_{D^*}-closed is proved in general and some results are investigated.

1. Introduction

Given an ideal space Y with the ideal $\mathcal{I}$ and topology τ, a local function [3] of C set of Y is defined as $C^* = \{y \in Y \mid V \cap C \notin \mathcal{I}$ for every $V \in \tau(y)\}$ where in $\tau(y) = \{V \in \tau \mid y \in V\}$. The notion of generalized closed sets was introduced by Levin [1] in 1970. A set C of space Y is said to be g-closed if $cl(C) \subseteq V$ when $C \subseteq V$ and V is open in Y. The concept of $\mathcal{I}_g$-closed sets was introduced by Dontshev. J, Ganster. M, and Noiri. T [2]in 1999. This was further studied by Navaneetha Krishnan and Paulraj Joseph [4] in 2008. A set C of an ideal space Y is said to be $\mathcal{I}_g$-closed [10] if $C^* \subseteq V$ when $C \subseteq V$ and $V \in \tau$. Δ-open sets were introduced by M. Veera Kumar [5]. All Δ-open set collections satisfying the topology criterion are given by τ^D for Y. Local function was defined by using Δ-open sets denoted by $C_{D^*}(\mathcal{I}, \tau)$ in [6]. Assume $A \subseteq Y$, then $C_{D^*}(\mathcal{I}, \tau) = \{y \in Y \mid V \cap C \notin \mathcal{I}$, $\forall\ V \in \tau^D(y)\}$ where $\tau^D(y) = \{V \in \tau^D \mid y \in V\}$ is known as D^*-local function [6] in C related to $\mathcal{I}$, τ. If $C \subseteq C_{D^*}$, then $cl(C) = cl_{D^*}(C)$. A kuratowski closure operator $cl_{D^*}(C)$ for a topology τ_{D^*} finer than τ is given by $cl_{D^*}(C) = C \cup C_{D^*}$ [6]. A set C of Y is ⋆-closed [7](resp. ⋆-dense in itself [8]) if $C^* \subseteq C$ (resp. $C \subseteq C^*$). We assume the topological space used to be always without separation properties. For $C \subseteq Y$, $int(C)$ will denote interior and $cl(C)$ closure of C in (Y, τ). Similarly $int^*(C)$ will denote the interior of C in (Y, τ_{D^*}).

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A set C of Y is a θ-closed set if $cl_\theta(C) = C$ and C is a δ-closed set if $cl_\delta(C) = C$ [9]. In [10] θg-closed sets are defined. $\alpha\mathcal{I}_g$-closed sets were defined by S. Maragathavalli and D. Vinodhini in [11]. A set C of Y is $\alpha\mathcal{I}_g$-closed [11] if $C^* \subseteq V$ when $C \subseteq V$ and V is α-open also every $\alpha\mathcal{I}_g$-closed set is a $\mathcal{I}_g$-closed set. In 2011 Antony Rex Rodrig et al. defined $\mathcal{I}\hat{g}$-closed sets in the following way: if $C^* \subseteq V$ whenever $C \subseteq V$ and V is semi-open then C is called $\mathcal{I}\hat{g}$-closed [12]. A set C of Y is said to be semi-closed if $int(cl(C)) \subseteq C$. Its complement is said to be semi-open [13]. Also all $\mathcal{I}\hat{g}$-closed sets is $\alpha\mathcal{I}g$-closed. In [14] $\mathcal{I}$-R-closed sets are defined by A. Acikgoz and S. Yuksel. They defined a set C of Y to be $\mathcal{I}$-R-closed if $C = cl^*(int(C))$. It is also proved in [15] that every $\mathcal{I}$-R-closed set is an $\mathcal{I}_g$-closed set. A set C is said to be $g\Delta^\star$-closed [16] if $C_{\delta^*} \subseteq V$ provided $C \subseteq V$ and $V \in \tau$. Also, A set C is said to be $g_s\Delta^\star$-closed [16] if $C_{\delta^*} \subseteq V$ provided $C \subseteq V$ and V is a semi-open set. Consider (Y, τ) to be a topological space and $x \in Y$ then $Ker\{x\} = \cap\{G \mid G \in \tau(x)\}$ where $\tau(x)$ is the collection of all open sets of Y [17]. Nitakshi Goyal in "On θ_I kernel of a set" in 2017 proved that for each $C \subseteq Y$, $C \subseteq Ker(C)$. The below given result is used to prove several results in this paper.The extension of the result [18] will be used .

Lemma 1.1. *Suppose E and F are subsets of Y, an ideal space. If so the given conditions can be proved [6]:*

(1) *If* $E \subseteq F, \Rightarrow E_{D^*} \subseteq F_{D^*}$.
(2) $(E_{D^*})_{D^*} \subseteq E_{D^*}$
(3) $E_{D^*} \subseteq cl_D(E)$
(4) $cl_{D^*}(E) \subseteq cl^*(E)$
(5) $E_{D^*} = cl_D(E_{D^*}) \subseteq cl_\theta(E)$
(6) *If* $E \in \mathcal{I}$, *then* $E_{D^*} = \phi$
(7) $E_{D^*} \cup F_{D^*} = (E \cup F)_{D^*}$
(8) $(E \cap F)_{D^*} \subseteq E_{D^*} \cap F_{D^*}$.

Lemma 1.2. $A_{D^*} \subseteq A^*$ *always holds [6].*

2. g_{D^*}-Closed Sets

Definition 2.1. *Suppose C is a set of Y in* $(Y, \tau, \mathcal{I})$ *then C is* g_{D^*}*-closed if* $C_{D^*} \subseteq V$ *when* $C \subseteq V$ *and* $V \in \tau$.

Definition 2.2. *A set C is said to be* g_D*-closed in* (Y, τ) *if* $cl(C) \subseteq V$ *when* $C \subseteq V$ *and* $V \in \tau^D$.

Definition 2.3. *Consider C to be a set of* $(Y, \tau, \mathcal{I})$ *then it is D★-closed if* $C_{D^*} \subseteq C$ *and D★-dense in itself if* $C \subseteq C_{D^*}$.

Theorem 2.1. *Consider* $(Y, \tau, \mathcal{I})$ *is an ideal space and C is a set of Y, then the below given can be compared.*

(1) *C is* g_{D^*}*-closed,*
(2) $cl_{D^*}(C) \subseteq V$ *when* $C \subseteq V$ *and V is open in Y,*
(3) $\forall y \in cl_{D^*}(C)$, $cl_D(x) \cap C \neq \phi$,
(4) *All closed sets of* $cl_{D^*}(C) - C$ *are empty,*
(5) *All closed sets of* $C_{D^*} - C$ *are empty.*

Proof. 1 ⇒ 2: Suppose C is g_{D^*}-closed ⇒ $C_{D^*} \subseteq V$ when C is a set of V and $V \in \tau$. Therefore $cl_{D^*}(C) \subseteq V$ when C is a set of V and V is open.
2 ⇒ 3: Assume $y \in cl_{D^*}(C)$. Suppose $cl_D(y) \cap C = \phi \Rightarrow C \subseteq Y - cl_D(y)$ and $Y - cl_D(y)$ is open in Y. Therefore $cl_{D^*}(C) \subseteq Y - cl_D(y)$ by (2). This implies $C_{D^*} \subseteq Y - cl_D(y)$ which is a contradiction since $C_{D^*} \subseteq cl_D(C)$ by Lemma [1.1,3] ⇒ $cl_D(y) \cap C \neq \phi$.
3 ⇒ 4: Assume G is a closed set, $y \in G$ and $G \subseteq cl_{D^*}(C) - C$. Hence $G \subseteq Y - C \Rightarrow C \subseteq Y - G$ also $cl_D(y) \cap C = \phi$. This is a contradiction since $\forall y \in cl_{D^*}(C)$, $cl_D(y) \cap C \neq \phi$ by (2). Thus all closed sets of $cl_{D^*}(C) - C$ are empty.
4 ⇒ 5: Suppose all closed sets of $cl_{D^*}(C) - C$ are empty. $cl_{D^*}(C) - C = (C \cup C_{D^*}) - C = (C \cup C_{D^*}) \cap (Y - C) = \phi \cup (C_{D^*} \cap (Y - C)) = C_{D^*} - C \Rightarrow$ all closed subsets of $C_{D^*} - C$ are empty.
5 ⇒ 1: Suppose C is a set of V and $V \in \tau \Rightarrow Y - V \subseteq Y - C \Rightarrow C_{D^*} \cap (Y - V) \subseteq C_{D^*} \cap (Y - C) = C_{D^*} - C$. By (5) all closed sets contained in $C_{D^*} - C$ are empty. $C_{D^*} \cap (Y - V)$ is a closed set $\Rightarrow C_{D^*} \cap (Y - V) = \phi \Rightarrow C_{D^*} \subseteq V$ when $C \subseteq V$ and V is open. □

Theorem 2.2. *Suppose Y is an ideal space, each $\mathcal{I}_g$ closed set is g_{D^*}-closed.*

Proof. Suppose C is $\mathcal{I}_g$ closed ⇒ $C^* \subseteq V$ when C is a set of V and $V \in \tau$. By Lemma [1.2] $C_{D^*} \subseteq C^*$ ⇒ $C_{D^*} \subseteq V$ when C is a set of V and $V \in \tau$ ⇒ C is g_{D^*}-closed. □

Remark 2.1. *The reverse implication is false and can be shown by the below given example.*

Example 2.1. *Suppose $Y = \{4, 5, 6\}$, $\tau = \{\phi, Y, \{6\}\}$ and $I = \{\phi\}$. I_g-closed sets are $\{\phi, Y, \{4\}, \{5\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}$. g_{D^*}-closed sets are $\{\phi, Y, \{6\}, \{4\}, \{5\}, \{4, 5\}, \{4, 6\}, \{5, 6\}\}$. $A = \{6\}$ is g_{D^*}-closed not I_g-closed.*

Theorem 2.3. *All $\alpha\mathcal{I}_g$-closed set are g_{D^*}-closed.*

Proof. We know every $\alpha\mathcal{I}_g$-closed set is $\mathcal{I}_g$-closed. Also from the previous theorem, we know every $\mathcal{I}_g$-closed set is g_{D^*}-closed set. This implies that every $\alpha\mathcal{I}_g$-closed set is g_{D^*}-closed. □

Theorem 2.4. *All ⋆-closed set are g_{D^*}-closed.*

Proof. Suppose C is ⋆-closed then C^* is a set of C. If C is a set of V and $V \in \tau \Rightarrow C^*$ is a set of V. Since $C_{D^*} \subseteq C \Rightarrow C_{D^*} \subseteq V$ when C is a set of V and $V \in \tau \Rightarrow C$ is g_{D^*}-closed. □

Remark 2.2. *The reverse implication of the above theorem is not true and is shown below in an example.*

Example 2.2. *$Y = \{1, 5, 8\}$, $\tau = \{\phi, Y, \{8\}\}$ and $\mathcal{I} = \{\phi\}$. Then $C = \{1\}$ is g_{D^*}-closed but not ⋆-closed.*

Theorem 2.5. *All g-closed sets are g_{D^*}-closed.*

Proof. Suppose C is g-closed ⇒ $cl(C) \subseteq V$ when C is a set of V and $V \in \tau$. We know by Lemma [1.1,4] $cl_{D^*}(C) \subseteq cl^*(C) \subseteq cl(C) \Rightarrow cl_{D^*}(C) \subseteq V$ when C is a set of V and $V \in \tau \Rightarrow C$ is g_{D^*}-closed by Theorem [2.1,2]. □

Remark 2.3. *The reverse of the above implication is false and is given by an example below.*

Example 2.3. *$Y = \{4, 5, 6, 7\}$, $\tau = \{\phi, Y, 4, \{4, 5\}, \{4, 6, 7\}\}$ and $\mathcal{I} = \{\phi, \{4\},$ $\{7\}, \{4, 7\}\}$. Then $C = \{4, 5\}$ is g_{D^*}-closed but not g-closed.*

Theorem 2.6. *If C is a θ-closed set then C is g_{D^*}-closed but the reverse implication is false and can be shown by an example.*

Proof. If C is θ-closed then, $C = cl_\theta(C)$. Assume C is a set of V and $V \in \tau$. we know by Lemma [1.1,5] $C_{D^*} \subseteq cl_\theta(C) \Rightarrow C_{D^*} \subseteq V$ when C is a set of V and $V \in \tau \Rightarrow C$ is g_{D^*}-closed. □

Example 2.4. *Consider $Y = \{e, i, j, f\}$, $\tau = \{\phi, Y, \{e\}, \{j\}, \{f\}, \{e, j\}, \{e, f\}\{j, f\}, \{e, j, f\}\}$ and $\mathcal{I} = \phi, \{e\}, \{i\}, \{e, i\}$. If $C = e$, then C is g_{D^*}-closed but not θ-closed.*

Theorem 2.7. *If C is a θg-closed set then C is g_{D^*}-closed but the reverse implication is false and can be shown by an example.*

Proof. If C is θg-closed set then $cl_\theta(C) \subseteq V$ provided $C \subseteq V$ and $V \in \tau$. We know $C_{D^*} \subseteq cl_\theta(C)$ by Lemma [1.1,5]. Thus $C_{D^*} \subseteq V$ when $C \subseteq V$ and $V \in \tau$. □

Example 2.5. *Consider $Y = \{l, a, j, f\}$, $\tau = \{\phi, Y, \{l\}, \{j\}, \{f\}, \{l, j\}, \{l, f\}\{j, f\}, \{l, j, f\}\}$ and $\mathcal{I} = \phi, \{l\}, \{a\}, \{l, a\}$. If $C = l$, then C is g_{D^*}-closed but not θg-closed.*

Theorem 2.8. *If C is a δ-closed set then C is g_{D^*}-closed but the reverse implication is not true and can be shown by an example.*

Proof. If C is δ-closed then, $C = cl_\delta(C)$. Assume $C \subseteq V$ and V is open. we know by Lemma [1.1,4] $cl_{D^*}(C)$ is a set of $cl^*(C)$ also $cl^*(C) \subseteq cl^*_\delta(C) \Rightarrow cl_{D^*}(C) \subseteq V$ when C is a set of V and $V \in \tau \Rightarrow C$ is g_{D^*}-closed by Theorem [2.1,2]. □

Example 2.6. *Consider $Y = \{e, i, j, f\}$, $\tau = \{\phi, Y, \{e\}, \{j\}, \{f\}, \{e, j\}, \{e, f\}, \{j, f\}, \{e, j, f\}\}$ and $\mathcal{I} = \{\phi, \{e\}, \{i\}, \{e, i\}\}$. If $C = \{j\}$, then C is g_{D^*}-closed not δ-closed.*

Theorem 2.9. *If C is a $g\Delta^\star$-closed set then C is g_{D^*}-closed but the reverse implication is not true and can be shown by an example.*

Proof. Suppose C is $g\Delta^\star$-closed $\Rightarrow C_{\delta^*} \subseteq V$ when $C \subseteq V$ and $V \in \tau$. Thus $C_{D^*} \subseteq V$ when $C \subseteq V$ and $V \in \tau$ since $C_{D^*} \subseteq C_{\delta^*}$. Therefore C is g_{D^*}-closed. □

Example 2.7. *Consider $Y = \{e, i, j, f\}$, $\tau = \{\phi, Y, \{e\}, \{j\}, \{f\}, \{e, j\}, \{e, f\}, \{j, f\}, \{e, j, f\}\}$ and $\mathcal{I} = \{\phi, \{e\}, \{i\}, \{e, i\}\}$. If $C = \{f\}$, then C is g_{D^*}-closed not g^*_Δ-closed.*

Theorem 2.10. *Every $g_s\Delta^\star$-closed set is g_{D^*}-closed.*

Proof. Suppose C is $g_s\Delta^\star$-closed implies $C_{\delta^*} \subseteq V$ when $C \subseteq V$ and V is a semi-open set. Thus $C_{D^*} \subseteq V$ when $C \subseteq V$ and $V \in \tau$ since $C_{D^*} \subseteq C_{\delta^*}$ and every open set is semi-open. Therefore C is g_{D^*}-closed. □

Remark 2.4. *Counterexample to the converse of the above theorem is given below.*

Example 2.8. *Consider* $Y = \{e, i, j, f\}$, $\tau = \{\phi, Y, \{e\}, \{j\}, \{f\}, \{e, j\}, \{e, f\}, \{j, f\}, \{e, j, f\}\}$ *and* $\mathcal{I} = \{\phi, \{e\}, \{i\}, \{e, i\}\}$. *If* $C = \{e, j\}$, *then* C *is* g_{D^*}*-closed not* $g_s\Delta^\star$*-closed.*

Theorem 2.11. *Every* g_D*-closed set is* g_{D^*}*-closed.*

Proof. Suppose C is g_D-closed $\Rightarrow cl(C) \subseteq V$ when C is a set of V and $V \in \tau^D$. We know by Lemma [1.1,4] $cl_{D^*}(C) \subseteq cl^*(C) \subseteq cl(C) \subseteq V \ \forall\ V \in \tau^D \Rightarrow cl_{D^*}(C) \subseteq V$ when $C \subseteq V$ and V is open $\Rightarrow C$ is g_{D^*}-closed by Theorem [2.1,2]. □

Remark 2.5. *Counterexample to the converse of the above theorem is given below.*

Example 2.9. $Y = \{4, 5, 6, 7\}$, $\tau = \{\phi, Y, 4, \{4, 5\}, \{4, 6, 7\}\}$ *and* $\mathcal{I} = \{\phi, \{4\}, \{7\}, \{4, 7\}\}$. *Then* $C = \{4, 6, 7\}$ *is* g_{D^*}*-closed not* g_D*-closed.*

Theorem 2.12. *Every* $\mathcal{I}$*-R-closed set is* g_{D^*}*-closed.*

Proof. We know $\mathcal{I}$-R-closed set is a subset of $\mathcal{I}_g$-closed set. Also, every $\mathcal{I}_g$-closed set is g_{D^*}-closed. Thus Every $\mathcal{I}$-R-closed set is g_{D^*}-closed. □

Theorem 2.13. *All* $\mathcal{I}\hat{g}$*-closed sets are* g_{D^*}*-closed.*

Proof. We know all $\mathcal{I}\hat{g}$-closed sets are $\alpha\mathcal{I}g$-closed sets also by Theorem [2.3] we know every $\alpha\mathcal{I}g$-closed sets is g_{D^*}-closed. Therefore all $\mathcal{I}\hat{g}$-closed set is g_{D^*}-closed. □

Theorem 2.14. *Every* g_D*-closed set is a* g*-closed set.*

Proof. Suppose G be a g_D-closed set $\Rightarrow cl(G) \subseteq V$ when $G \subseteq V$ and $V \in \tau^D$. Implies $cl(G) \subseteq V$ when $G \subseteq V$ and $V \in \tau$ since $\tau \subseteq \tau^D$. Hence G is a g-closed set. □

Theorem 2.15. *All* g_D*-closed sets are* $g\delta$*-closed. Converse of the theorem is not true and is proved using an example.*

Proof. Suppose G be a g_D-closed set that is $cl(G) \subseteq V$ provided $G \subseteq V$ and V is Δ-open in Y. Since all δ-open sets are Δ-open we get $cl(G) \subseteq V$ provided $G \subseteq V$ and V is δ-open. Therefore all g_D-closed set is $g\delta$-closed. □

Example 2.10. *Consider* $Y = \{4, 8, 3, 7\}$ *and* $\tau = \{\phi, Y, \{4\}, \{8\}, \{4, 3, 7\}\}$. *If* $C = \{4\}$, *then* C *is* $g\delta$*-closed not* g_D*-closed.*

Theorem 2.16. *Suppose* Y *is an ideal space, for all* $C \in \mathcal{I}$, C *is* g_{D^*}*-closed.*

Proof. Assume $C \subseteq V$ and $V \in \tau$. If $C \in \mathcal{I}$ then C_{D^*} is empty by Lemma [1.1,6] $\Rightarrow C_{D^*} \subseteq V$ when $C \subseteq V$ and $V \in \tau \Rightarrow C$ is g_{D^*}-closed. □

Theorem 2.17. *Suppose* Y *is an ideal space then for all* $C \subseteq Y$, C_{D^*} *is* g_{D^*}*-closed.*

Proof. Assume $C_{D^*} \subseteq V$ and $V \in \tau$. Since $(C_{D^*})_{D^*} \subseteq C_{D^*}$ by Lemma [1.1,2] $\Rightarrow (C_{D^*})_{D^*} \subseteq V$ when $C_{D^*} \subseteq V$ and $V \in \tau \Rightarrow C_{D^*}$ is g_{D^*}-closed. □

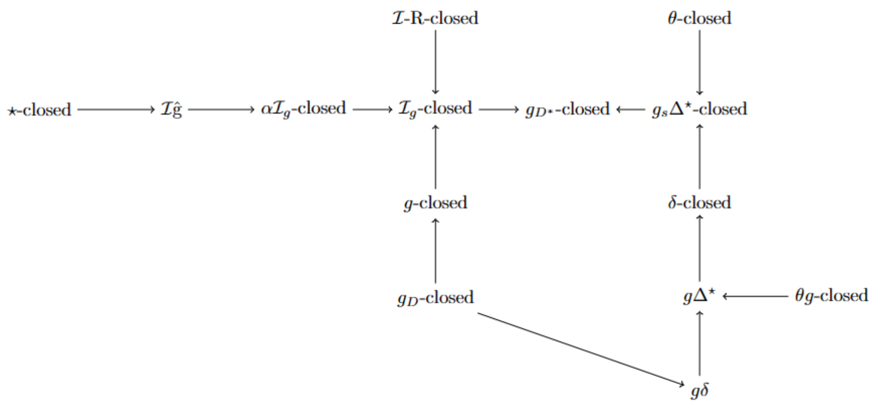

Figure 1. The diagram illustrates the relationships between various generalized closed sets that are discussed in the above stated theorems.

Theorem 2.18. *Consider C is g_{D^*}-closed set that is also open then C is D⋆-closed.*

Proof. Given that $C \in \tau$ and $C \subseteq C$ then since C is g_{D^*}-closed, $C_{D^*} \subseteq C$ implies that C is $D\star$-closed. □

Theorem 2.19. *Suppose Y is an ideal space and C is g_{D^*}-closed, Then the below given are identical.*

(1) *C is D⋆-closed,*
(2) *$cl_{D^*}(C) - C$ is a closed set,*
(3) *$C_{D^*} - C$ is a closed set.*

Proof. 1 ⇒ 2: If C is $D\star$-closed $\Rightarrow C_{D^*} \subseteq C$. $cl_{D^*}(C) - C = (C \cup C_{D^*}) - C = C_{D^*} - C = \phi$. Therefore $cl_{D^*}(C) - C$ is a closed set.
2 ⇒ 3: Suppose $cl_{D^*}(C) - C$ is a closed set. $cl_{D^*}(C) - C = C_{D^*} - C \Rightarrow C_{D^*} - C$ is a closed set.
3 ⇒ 1: Assume $C_{D^*} - C$ is a closed set and C is g_{D^*}-closed. $\Rightarrow C_{D^*} - C = \phi$ by Theorem [2.1,5]. Hence $C_{D^*} \subseteq C$. □

Theorem 2.20. *Suppose Y is an ideal space, a set C of Y is D⋆-dense in itself and C is g_{D^*}-closed implies C is g-closed.*

Proof. Assume C is $D\star$-dense in itself then $cl(C) = cl_{D^*}(C)$. Assume $C \subseteq V$ and $V \in \tau$, since C is g_{D^*}-closed $\Rightarrow cl_{D^*}(C) \subseteq V$ by Theorem [2.1,2]. Thus $cl(C) \subseteq V$ when $C \subseteq V$ and $V \in \tau \Rightarrow C$ is g-closed. □

Corollary 2.1. *Suppose Y is an ideal space and $\mathcal{I} = \phi$ then, C is g_{D^*}-closed if and only if C is g-closed.*

Proof. When $\mathcal{I} = \phi$ implies $C_{D^*} = cl(C)$ also $C \subseteq cl(C) \Rightarrow C \subseteq C_{D^*} \Rightarrow C$ is $D\star$-dense in itself. Assuming C is g_{D^*}-closed then by the above theorem C is g-closed. Conversely, assuming C is g-closed then by Theorem [2.11] C is g_{D^*}-closed. □

Theorem 2.21. *Suppose Y is an ideal space. If $H \subseteq M \subseteq H_{D^*}$ then $H_{D^*} = M_{D^*}$ and M is D⋆-dense in itself.*

Proof. If $H \subseteq M \Rightarrow H_{D^*} \subseteq M_{D^*}$. But $M \subseteq H_{D^*} \Rightarrow M_{D^*} \subseteq H_{D^*} \Rightarrow H_{D^*} = M_{D^*}$. Also $M \subseteq H_{D^*} = M_{D^*}$. Therefore $D\star$-dense in itself. □

Theorem 2.22. *Suppose Y is an ideal space with H and M as sets of Y, H is g_{D^*}-closed and $H \subseteq M \subseteq cl_{D^*}(H)$ implies M is g_{D^*}-closed.*

Proof. Assume H is g_{D^*}-closed $\Rightarrow$ all closed sets in $cl_{D^*}(H) - H$ are empty. We know $cl_{D^*}(M) - M \subseteq cl_{D^*}(H) - H$. Therefore all closed sets in $cl_{D^*}(M) - M$ is empty $\Rightarrow$ M is g_{D^*}-closed by Theorem [2.1,4]. □

Corollary 2.2. *Suppose F and R are sets in Y which is an ideal space such that F is g_{D^*}-closed and $F \subseteq R \subseteq F_{D^*} \Rightarrow F$ and R are g-closed .*

Proof. Suppose $F \subseteq R \subseteq F_{D^*}$ implies $F \subseteq R \subseteq F_{D^*} \subseteq cl_{D^*}(F)$ and assume F is g_{D^*}-closed then by the above theorem R is g_{D^*}-closed. Since $F \subseteq R \subseteq F_{D^*}$ gives $F_{D^*} = R_{D^*}$ and F, R are $D\star$-dense in itself by Theorem [2.21]. Then by Theorem [2.20] F and R are g-closed □

Theorem 2.23. *Suppose C is a set in Y which is an ideal space, C is g_{D^*}-open and $int_{D*}(C) \subseteq R \subseteq C$ implies R is g_{D^*}-open.*

Proof. Since $R \subseteq C \Rightarrow cl_{D^*}(R) \subseteq cl_{D^*}(C) \Rightarrow cl_{D^*}(Y - C) \subseteq cl_{D^*}(Y - R)$. Also since $int_{D*}(C) \subseteq R$ implies $int_{D*}(C) \subseteq int_{D*}(R) \Rightarrow cl_{D^*}(Y - R) \subseteq cl_{D^*}(Y - C)$. Thus $cl_{D^*}(Y - R) = cl_{D^*}(Y - C) \Rightarrow cl_{D^*}(Y - R) - (Y - R) \subseteq cl_{D^*}(Y - C) - (Y - C)$. Suppose C is g_{D^*}-open then $Y - C$ is g_{D^*}-closed. By Theorem [2.1,4] all closed sets of $cl_{D^*}(Y - C) - (Y - C)$ are empty $\Rightarrow$ all closed sets of $cl_{D^*}(Y - R) - (Y - R)$ are empty $\Rightarrow Y - R$ is g_{D^*}-closed $\Rightarrow R$ is g_{D^*}-open. □

Theorem 2.24. *Suppose Y is an ideal space and C is a set of Y, then the given below are comparable:*

(1) *C is g_{D^*}-closed,*
(2) *$C \cup (Y - C_{D^*})$ is g_{D^*}-closed,*
(3) *$C_{D^*} - C$ is g_{D^*}-open.*

Proof. 1 $\Rightarrow$ 2: Assume C is g_{D^*}-closed. If $V \in \tau$ such that $C \cup (Y - C_{D^*}) \subseteq V \Rightarrow Y - V \subseteq Y - (C \cup (Y - C_{D^*}) = Y \cap (C \cup (C_{D^*})^c)^c = Y \cap (C^c \cap C_{D^*}) = C_{D^*} - C$. That is $Y - V \subseteq C_{D^*} - C$ and since V is an open set $Y - V$ is a closed set. C is g_{D^*}-closed hence all closed sets in $C_{D^*} - C$ is empty by Theorem [2.1,5] implies $Y - V = \phi \Rightarrow Y = V$. Hence $C \cup (Y - C_{D^*}) \subseteq V \Rightarrow C \cup (Y - C_{D^*}) \subseteq Y \Rightarrow (C \cup (Y - C_{D^*}))_{D^*} \subseteq Y = V$ when $C \cup (Y - C_{D^*}) \subseteq V$ and $V \in \tau \Rightarrow C \cup (Y - C_{D^*})$ is g_{D^*}-closed.
2 $\Rightarrow$ 1: Suppose $C \cup (Y - C_{D^*})$ is g_{D^*}-closed. Consider a closed set G such that it is a set in $C_{D^*} - C$ which implies $G \subseteq C_{D^*}$ and G not in $C \Rightarrow G \subseteq Y - C$. Thus $Y - C_{D^*} \subseteq Y - G$ and $C \subseteq Y - G$ implies $C \cup (Y - C_{D^*}) \subseteq C \cup (Y - G) = Y - G$ also $C \cup (Y - C_{D^*})$ is g_{D^*}-closed $\Rightarrow (C \cup (Y - C_{D^*}))_{D^*} \subseteq Y - G$ $\Rightarrow C_{D^*} \subseteq Y - G \Rightarrow G \subseteq Y - C_{D^*}$ which is a contradiction. Thus $G = \phi \Rightarrow$ any closed set G in $C_{D^*} - C$ is empty $\Rightarrow C$ is g_{D^*}-closed.
2 $\Leftrightarrow$ 3: $Y - (C_{D^*} - C) = Y \cap (C_{D^*} \cap C^c)^c = Y \cap ((C_{D^*})^c \cup C) = (Y \cap (C_{D^*})^c) \cup (Y \cap C) = (Y - C_{D^*}) \cup C$. Suppose $C_{D^*} - C$ is g_{D^*}-open $\Leftrightarrow Y - (C_{D^*} - C)$ is g_{D^*}-closed $\Leftrightarrow C \cup (Y - C_{D^*})$ is g_{D^*}-closed. □

Theorem 2.25. *If Y is an ideal space then, all sets in Y is g_{D^*}-closed if and only if all open sets are $D\star$-closed.*

Proof. Suppose all sets in Y is g_{D^*}-closed. Assume $V \in \tau$ in Y then V is g_{D^*}-closed $\Rightarrow V_{D^*} \subseteq V \Rightarrow V$ is $D\star$-closed. Suppose all open sets are $D\star$-closed. If V is open and $C \subseteq V \subseteq Y$ then, $C_{D^*} \subseteq V_{D^*} \subseteq V$ $\Rightarrow C$ is g_{D^*}-closed. □

Theorem 2.26. *C is a g_{D^*}-closed set if and only if $C = F - N$ where F is $D\star$-closed and all closed sets in N are empty.*

Proof. Assume C is a g_{D^*}-closed set. Consider $N = C_{D^*} - C$, then by Theorem [2.1,5], all closed sets of N are empty. If $F = cl_{D^*}(C)$ then F is $D\star$-closed. $F - N = (C \cup C_{D^*}) - (C_{D^*} - C) = (C \cup C_{D^*}) \cap (C_{D^*} \cap C^c)^c = C \cup (C_{D^*} \cap (C_{D^*})^c) = C$. Conversely, let $C = F - N \Rightarrow C \subseteq F$. Suppose $C \subseteq V$ and $V \in \tau$. $F - N \subseteq V \Rightarrow F - V \subseteq N$. Thus $F \cap (Y - V) \subseteq N$. Since $F_{D^*} \subseteq F$, implies $C_{D^*} \subseteq F$. Therefore $C_{D^*} \cap (Y - V) \subseteq F \cap (Y - V) \subseteq N \Rightarrow C_{D^*} \cap (Y - V) = \phi \Rightarrow C_{D^*} \subseteq V$. □

Theorem 2.27. *Suppose H and G are g_{D^*}-closed sets in $(Y, \tau, \mathcal{I})$ if and only if union of H and G are g_{D^*}-closed*

Proof. Suppose $H \cup G \subseteq V$ and $V \in \tau \Rightarrow H \subseteq V$ and $G \subseteq V$ where $V \in \tau$. Since H and G are g_{D^*}-closed $\Leftrightarrow H_{D^*} \subseteq V$ and $G_{D^*} \subseteq V \Leftrightarrow H_{D^*} \cup G_{D^*} \subseteq V$. $\Leftrightarrow (H \cup G)_{D^*} \subseteq V$ by Lemma [1.1,7]. □

Theorem 2.28. *The intersection of two g_{D^*}-closed sets is g_{D^*}-closed.*

Proof. Assume G and H are g_{D^*}-closed. Consider $G \cap H \subseteq V$ and $V \in \tau$. $\Rightarrow G \subseteq V$ and $H \subseteq V$ $\Rightarrow G_{D^*} \subseteq V$ and $H_{D^*} \subseteq V$ where $V \in \tau$.Then $G_{D^*} \cap H_{D^*} \subseteq V$. Therefore $(G \cap H)^*_D \subseteq V$ since $(G \cap H)^*_D \subseteq G_{D^*} \cap H_{D^*}$ by Lemma [1.1,8]. □

Theorem 2.29. *Suppose H and G are g_{D^*}-open sets in $(Y, \tau, \mathcal{I})$ then the intersection of H and G are g_{D^*}-open.*

Proof. $Y - H$ and $Y - G$ are g_{D^*}-closed since H and G are g_{D^*}-open $\Rightarrow (Y - H) \cup (Y - G)$ is g_{D^*}-closed by using the previous theorem. Hence $Y - (H \cap G)$ is g_{D^*}-closed. Therefore $(H \cap G)$ is g_{D^*}-open. □

Definition 2.4. *Suppose C is a non-empty g_{D^*}-closed set of Y. Then C is said to be maximal g_{D^*}-closed set if any g_{D^*}-closed set containing C is either C or Y.*

Theorem 2.30. *The following conditions hold for an ideal space Y:*

(1) *Suppose E is g_{D^*}-closed and H is maximal g_{D^*}-closed then either $E \cup H = Y$ or $E \subseteq H$.*
(2) *When E and H are maximal g_{D^*}-closed sets then either $E \cup H = Y$ or $E = H$.*

Proof. (1) Since H is maximal g_{D^*}-closed it is obvious from the definition that $E \cup H = Y$. Suppose $E \cup H \neq Y$ then since H is maximal g_{D^*}-closed $E \cup H = H \Rightarrow E \subseteq H$.
(2) Suppose $E \neq H$ then since E and H are maximal g_{D^*}-closed it implies that $E \cup H = Y$. Suppose $E \cup H \neq Y$ then by (1) $E \subseteq H$ and $H \subseteq E$ implies that $E = H$. □

Theorem 2.31. *A set C of Y is g_{D^*}-closed if Ker(C) is g_{D^*}-closed.*

Proof. We know that for $C \subseteq Y$, $C \subseteq Ker(C) \Rightarrow C_{D^*} \subseteq (Ker(C))_{D^*}$. Since $Ker(C)$ is g_{D^*}-closed implies $(Ker(C))_{D^*} \subseteq V$ when $Ker(C) \subseteq V$ and $V \in \tau$. Thus $C_{D^*} \subseteq V$ when $C \subseteq V$ and $V \in \tau$. □

Conclusion

We defined g_{D^*}-closed sets and g_D-closed sets and made a comparative study between these newly defined sets and some already existing closed sets such as θ-closed sets, δ-closed sets, g-closed sets, $\star$-closed sets and $\mathcal{I}_g$-closed sets. We also discuss some characterizations and properties of g_{D^*}-closed sets using definitions of kuratowski closure operator cl_{D^*}, $D\star$-closed sets and some other sets.

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