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Properties and Characterizations of Controlled K-g-Fusion Frames within Hilbert C^*-Modules

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Abstract. This paper investigates several aspects of controlled K-g-fusion frames within the setting of Hilbert C^*-modules. We provide detailed characterizations of these frames, highlighting their structural properties and demonstrating how they adapt under transformations by various operators. A significant focus is placed on the relationship between the quotient operator and the controlled K-g-fusion frames, exploring their algebraic properties extensively. The results enrich the theoretical understanding of fusion frames.

1. Introduction

The theory of frames for Hilbert spaces, initially introduced in 1952 by Duffin and Schaeffer [3], was originally developed to address challenges in nonharmonic Fourier series. This theory was revitalized in 1986 through the work of Daubechies, Grossman, and Meyer [2], which led to its widespread application in signal processing, data compression, and more recently, quantum communication. Since then, frame theory has gained significant popularity and has been extended to a wide variety of mathematical and applied fields, including engineering, medicine, and functional analysis.

In the decades that followed, various generalizations of frame theory emerged, proposed by numerous authors. Among the most notable are the g-frame [16], fusion frame [6], g-fusion frame [10], K-g-frame [12], and K-g-fusion frame [17]. These generalizations broadened the scope of frame theory, allowing for the handling of more complex and structured settings, particularly in the context of Hilbert C^*-modules [1], which serve as a natural setting for operator theory. For more detailed information on biframes theory, readers are recommended to consult: [4,5,7–9,14,19–28].

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Recently, the study of controlled frames has been gaining traction. The notion of controlled frames has seen the introduction of several new types, such as controlled K-frames [13], controlled g-frames [16], controlled fusion frames [6], controlled g-fusion frames [10], and controlled K-g-fusion frames [17]. These controlled versions of frames introduce additional stability conditions, which are essential for applications where robustness to perturbations is crucial.

This paper focuses on the progression from K-g-fusion frames to controlled K-g-fusion frames. Specifically, we introduce a novel form of controlled g-fusion frames derived from existing controlled K-g-fusion frames by applying an invertible bounded linear operator. We also establish a necessary and sufficient condition for a controlled g-fusion Bessel sequence to qualify as a controlled K-g-fusion frame. Furthermore, we explore the stability of controlled g-fusion frames and their duals, presenting new results that deepen our understanding of the behavior of these frames in the presence of various operators.

This paper is organized as follows: In the first section, we present the preliminaries, introducing the necessary definitions and results for the remainder of the paper. The following section provides two examples that help illustrate the concepts introduced earlier, aiding in the understanding of the results discussed in subsequent sections. Finally, the last section, which forms the core of the paper, presents the fundamental results and focuses on the main theorems. Specifically, we investigate the core results related to controlled (P,Q) g-frames and their transformation into controlled (P,Q) K-fusion frames. We discuss the algebraic, analytical, and order conditions on K that are required for a Bessel controlled (P,Q) g-frame to qualify as a controlled (P,Q) K-fusion frame. These conditions establish a framework for understanding how certain sequences can be classified within the context of fusion frames. Additionally, we examine the commutation conditions for the synthesis operators P, Q, and other relevant operators, as these are essential for deriving further results and insights. Finally, the paper addresses stability results, exploring how these frames behave under various transformations and perturbations.

Throughout this paper, let $\mathcal{A}$ be a unital C^*-algebra and J a countable index set. We consider $\mathcal{H}$ and $\mathcal{K}$ as countably generated Hilbert $\mathcal{A}$-modules, with $\left\{\mathcal{H}_j\right\}_{j\in J}$ representing a sequence of submodules within $\mathcal{K}$. For each j in J, $End^*_{\mathcal{A}}\left(\mathcal{H},\mathcal{H}_j\right)$ denotes the set of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ to $\mathcal{H}_j$, and the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{H}$ to itself is denoted by $End^*_{\mathcal{A}}(\mathcal{H})$. Additionally, $GL^+(\mathcal{H})$ denotes the set of all positive bounded linear invertible operators on $\mathcal{H}$ with a bounded inverse. For more detailed information and fundamental results on C^*-algebras and Hilbert C^*-modules, the reader is referred to the work of Manuilov and Troitsky [11], as well as Arambasic's studies [1].

2. Preliminaries

In this section, we revisit some essential definitions and theorems. We begin with Hilbert C^*-modules and their operators. Then, we recall the definition of (P,Q)-controlled K-g-fusion frames and some special cases.

Theorem 2.1. *[11, Proposition 2.13] Consider an operator $U : \mathcal{H} \to \mathcal{H}$. The following conditions are equivalent:*

1. *U is a positive operator within $End^*_{\mathcal{A}}(\mathcal{H})$.*
2. *For every $\xi \in \mathcal{H}$, the inequality $\langle U\xi, \xi\rangle \geq 0$ holds in $\mathcal{A}$.*

Note that from the previous theorem one can conclude that every positive operator $T \in End^*_{\mathcal{A}}(\mathcal{H})$, for an Hilbert C^*-module $\mathcal{H}$, has a square root, this means that there is a unique positive operator $S \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $S^2 = T$. We will note $S = T^{\frac{1}{2}}$.

The following result is a Douglas's theorem version relative to C*-Hilbert modules.

Theorem 2.2. *[30]*

Consider $\mathcal{H}$ an Hilbert $\mathcal{A}$-module over a C^*-algebra $\mathcal{A}$. Suppose $T, S \in End^*_{\mathcal{A}}(\mathcal{H})$ and that the range of S, denoted $\mathcal{R}ang(S)$, is closed. Then, the following conditions are equivalent:

1. *$\mathcal{R}ang(T) \subseteq \mathcal{R}ang(S)$.*
2. *There exists a non-negative $\lambda \geq 0$ such that $TT^* \leq \lambda^2 SS^*$.*
3. *There exists an operator $Q \in End^*_{\mathcal{A}}(\mathcal{H})$ for which $T = SQ$.*

Lemma 2.1. *[12, Lemma 2.1] Let $\left(W_j\right)_{j\in J}$ denote a sequence of orthogonally complemented closed submodules of $\mathcal{H}$. Suppose $U \in End^*_{\mathcal{A}}(\mathcal{H})$ is invertible and satisfies $U^*UW_j \subseteq W_j$ for each $j \in J$. Under these conditions, the sequence $\left(UW_j\right)_{j\in J}$ also forms a sequence of orthogonally complemented closed submodules, and the relation $\pi_{W_j}U^* = \pi_{W_j}U^*\pi_{UW_j}$ holds for each $j \in J$.*

The next theorem characterize the operators on Hilbert C*-modules that possesses a Moore-Penrose inverse

Theorem 2.3. *[29, Theorem 2.2] Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, and let $T \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:*

1. *The range of T is closed.*
2. *The Moore-Penrose inverse $T^\dagger$ of T exists; this means that $T^\dagger$ is an element of $End^*_{\mathcal{A}}(\mathcal{K}, \mathcal{H})$ which satisfies:*

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger \text{ and } (T^\dagger T)^* = T^\dagger T.$$

Now, we present some fundamental definitions and key results concerning frames in Hilbert C*-modules, which will be essential for the development of the paper.

Definition 2.1. *[18, Definition 4.2] Assume $P, Q \in GL^+(\mathcal{H})$ and $K \in End^*_{\mathcal{A}}(\mathcal{H})$. Let $\{W_j\}_{j\in J}$ represent a collection of orthogonally complemented closed submodules of $\mathcal{H}$, and $\{\mathcal{H}_j\}_{j\in J}$ denote a sequence of submodules of a countably generated Hilbert $\mathcal{A}$-module. Additionally, let $\{v_j\}_{j\in J}$ be a sequence of weights in $\mathcal{A}$; this means that for each v_j, $j \in J$, is a positive invertible element from the center of the algebra $\mathcal{A}$. We also assume that $\Gamma_j \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_j)$ for every $j \in J$.*

The collection $\Gamma_{PQ} = \{W_j, \Gamma_j, v_j\}_{j\in J}$ is said to form a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ if it satisfies the following condition: there exist real constants $0 < C \le D < \infty$ such that for all $\xi \in \mathcal{H}$,

$$C\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} P\xi, \Gamma_j \pi_{W_j} Q\xi\rangle \le D\langle \xi, \xi\rangle. \tag{2.1}$$

Here, the constants C and D are called the lower and upper bounds of the (P,Q)-controlled K-g-fusion frame, respectively. If the left inequality holds as an equality for all ξ, then Γ_{PQ} is referred to as a tight (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$.

Some special cases of the previous definition (Definition 2.1) correspond to well-known classical ones; see [12,18]. Specifically:

- If $P = Q = I_{\mathcal{H}}$, then every (P,Q)-controlled K-g-fusion frame is simply called a K-g-fusion frame.
- If $K = I_{\mathcal{H}}$, then every (P,Q)-controlled K-g-fusion frame is referred to as a (P,Q)-controlled g-fusion frame.
- If $\Gamma_j = P = Q = K = I_{\mathcal{H}}$ for all $j \in J$ in the previous definition, then Γ acts as a fusion frame for $\mathcal{H}$.
- If only the upper bound inequality holds in (2.1), then Γ_{PQ} is described as a (P,Q)-controlled g-fusion Bessel sequence for $\mathcal{H}$.

Suppose Γ_{PQ} is a (P,Q)-controlled g-fusion Bessel sequence for $\mathcal{H}$. The adjointable bounded linear operator $T_{(P,Q)} : \ell^2(\{\mathcal{H}_j\}_{j\in J}) \to \mathcal{H}$ defined by

$$T_{(P,Q)}\left(\{\xi_j\}_{j\in J}\right) = \sum_{j\in J} v_j (PQ)^{\frac{1}{2}} \pi_{W_j} \Gamma_j^* \xi_j, \quad \forall \{\xi_j\}_{j\in J} \in l^2(\{\mathcal{H}_j\}_{j\in J})$$

is known as the synthesis operator for Γ_{PQ}. The adjoint of $T_{(P,Q)}$ is the operator $T^*_{(P,Q)} : \mathcal{H} \to \ell^2(\{\mathcal{H}_j\}_{j\in J})$ is given by

$$T^*_{(P,Q)}(g) = \{v_j \Gamma_j \pi_{W_j} (QP)^{\frac{1}{2}} g\}_{j\in J}$$

and is referred to as the analysis operator for Γ_{PQ}. When P and Q commute with each other, and with the operator $\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}$ for each $j \in J$, the (P,Q)-controlled g-fusion frame operator $S_{(P,Q)} : \mathcal{H} \to \mathcal{H}$ is defined as

$$S_{(P,Q)}(\xi) = T_{(P,Q)} T^*_{(P,Q)}(\xi) = \sum_{j\in J} v_j^2 Q \pi_{W_j} \Gamma_j^* \Gamma_j \pi_{W_j} P\xi, \quad \forall \xi \in \mathcal{H}$$

and we have

$$\langle S_{(P,Q)}(\xi), \xi\rangle = \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} P\xi, \Gamma_j \pi_{W_j} Q\xi\rangle, \quad \forall \xi \in \mathcal{H}.$$

Henceforth, it is assumed that P and Q commute with each other and with the operator $\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}$ for each $j \in J$.

Lemma 2.2. *[18, Lemma 3.4] Suppose Γ_{PQ} constitutes a (P,Q)-controlled g-fusion frame for $\mathcal{H}$. It follows that the (P,Q)-controlled g-fusion frame operator $S_{(P,Q)}$ is positive, self-adjoint, and invertible.*

Theorem 2.4. *[12, Theorem 2.5] Assume $T \in End^*_{\mathcal{A}}(\mathcal{H})$ is an invertible operator on $\mathcal{H}$, and let $\Gamma_{P,Q} = \left\{W_j, \xi_j, v_j\right\}_{j\in J}$ denote a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$, with K being an operator in $End^*_{\mathcal{A}}(\mathcal{H})$. If $T^*TW_j \subseteq W_j$ and both P and Q commute with T, then the transformed set $\Lambda_{P,Q} = \left\{TW_j, \xi_j\pi_{W_j}T^*, v_j\right\}_{j\in J}$ forms a (P, Q)-controlled TKT^*-g-fusion frame for $\mathcal{H}$.*

Theorem 2.5. *[12, Theorem 2.6] Consider T as an invertible operator from $End^*_{\mathcal{A}}(\mathcal{H})$ on $\mathcal{H}$, and let $\Lambda_{P,Q} = \left\{TW_j, \xi_j\pi_{W_j}T^*, v_j\right\}_{j\in J}$ represent a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$, where K is also an operator from $End^*_{\mathcal{A}}(\mathcal{H})$. If $T^*TW_j \subseteq W_j$, for all $j \in J$, and both control operators P and Q are commutative with T, then the set $\Gamma_{P,Q} = \left\{W_j, \xi_j, v_j\right\}_{j\in J}$ qualifies as a (P, Q)-controlled $T^{-1}KT$-g-fusion frame for $\mathcal{H}$.*

Theorem 2.6. *[12, Theorem 2.7] Assume $K \in End_{\mathcal{A}}{}^*(\mathcal{H})$ is an invertible operator on $\mathcal{H}$, and let $\Gamma = \left\{W_j, \Gamma_j, v_j\right\}_{j\in J}$ define a g-fusion frame for $\mathcal{H}$ with bounds C and D. Let S_Γ denote the associated g-fusion frame operator. If $U^*UW_j \subseteq W_j$ for all $j \in J$, where $U = KS_\Gamma^{-1}$, then the transformed set $\left\{KS_\Gamma^{-1}W_j, \Lambda_j\pi_{W_j}S_\Gamma^{-1}K^*, v_j\right\}_{jinJ}$ forms a K-g-fusion frame for $\mathcal{H}$. The corresponding g-fusion frame operator for this frame is $KS_\Gamma^{-1}K^*$.*

3. Examples

In this section, we will present two types of examples of frames on Hilbert C*-modules. The first one deals with frames on a Hilbert C*-module over a C*-algebra of finite dimension ($\mathbb{A} = \mathbb{C}^2$). The second example presents frames on Hilbert C*-modules over a commutative infinite dimensional C*-algebra. Other examples of frames on Hilbert C*-modules over non-commutative C*-algebras will be provided as counterexamples in the following section.

Example 3.1. *Let $\mathcal{A}$ be the C*-algebra $\mathbb{C}^2$, and let $\mathcal{H} = \mathbb{C}^6$ be the Hilbert C*-module over $\mathcal{A}$, equipped with the module operations defined as follows: for $(x, y) \in \mathcal{A}^2$ and $(a_k)_{1\le k\le n} \in \mathcal{H}$, the action is given by*

$$(x, y)(a_k)_{1\le k\le n} = (b_k)_{1\le k\le n},$$

where $b_{2k-1} = xa_{2k-1}$ and $b_{2k} = ya_{2k}$ for $1 \le k \le n$. The C-inner product on $\mathcal{H}$ is defined by*

$$\langle a, b\rangle = \left(\sum_{k=1}^{n} a_{2k}\overline{b_{2k}}, \sum_{k=1}^{n} a_{2k-1}\overline{b_{2k-1}}\right) = \sum_{k=1}^{n} \left(a_{2k}\overline{b_{2k}}, a_{2k-1}\overline{b_{2k-1}}\right) \in \mathcal{A}.$$

Denote

$$W_j = \left\{(a_k)_{1\le k\le n} \ :\ a_{2j-1} = a_{2j} = 0\right\} \quad \text{and} \quad \mathcal{H}_j = \mathcal{A}$$

for $i = 1, 2, 3$. Those spaces are seeing as Hilbert C-modules over $\mathcal{A}$. The operators $\Lambda_j : W_j \to \mathcal{H}_j$, $j = 1, 2, 3$ are defined by*

$$\Lambda_1(a_k)_{1\le k\le n} = 11(a_5, a_6), \quad \Lambda_2(a_k)_{1\le k\le n} = 3(a_1, a_2) \text{ and } \Lambda_3(a_k)_{1\le k\le n} = 7(a_3, a_4)$$

for all $(a_k)_{1\le k\le n} \in \mathcal{H}$. One can check that

$$\begin{aligned} \langle \Lambda_1(a_k)_{1\le k\le n}, (x, y)\rangle &= \langle 11(a_5, a_6), (x, y)\rangle \\ &= 11(a_5\overline{x}, a_6\overline{y}) \\ &= \langle (a_k)_{1\le k\le n}, 11(0, 0, 0, 0, x, y)_{1\le k\le n}\rangle. \end{aligned}$$

Thus $\Lambda_1^* : \mathcal{H}_1 \to \mathcal{H}$ *is defined by* $\Lambda_1^*(x,y) = 11(0,0,0,0,x,y)$ *for all* $(x,y) \in \mathcal{H}_1$. *We conclude that*

$$\Lambda_1^*\Lambda_1(a_k)_{1\le k\le n} = 11^2(0,0,0,0,a_5,a_6).$$

Similarly we obtain

$$\Lambda_2^*(x,y) = 3(x,y,0,0,0,0) \text{ and } \Lambda_3^*(x,y) = 7(0,0,x,y,0,0)$$

for all $(x,y) \in \mathcal{H}_2 = \mathcal{H}_3$. *Therefore*

$$\Lambda_2^*\Lambda_2(a_k)_{1\le k\le n} = 3^2(a_1,a_2,0,0,0,0) \text{ and } \Lambda_3^*\Lambda_3(a_k)_{1\le k\le n} = 7^2(0,0,a_3,a_4,0,0).$$

Observe that $\Lambda_j\pi_{W_j} = \Lambda_j$ *for* $j = 1,2,3$.

Let P, Q *the operators defined on* $\mathcal{H}$ *by*

$$P(a_k)_{1\le k\le n} = (ka_k)_{1\le k\le n} \text{ and } Q(a_k)_{1\le k\le n} = (\frac{1}{k}a_k)_{1\le k\le n}.$$

In fact $P,Q \in GB^+(\mathcal{H})$ *and* Γ_{PQ} *serves as a* (P,Q)*-controlled g-fusion frame for* $\mathcal{H}$ *if and only if* $\mathcal{R}ang(K) \subseteq \mathcal{R}ang(T_C)$.

$$\begin{aligned}\langle T_C a, a\rangle &= \sum_{j=1}^{3}\langle \Lambda_j^*\Lambda_j Pa, \Lambda_j Qa\rangle \\ &= \sum_{j=1}^{3}\langle \Lambda_j Pa, \Lambda_j Qa\rangle \\ &= 11^2(a_5\overline{a_5}, a_6\overline{a_6}) + 3^2(a_1\overline{a_1}, a_2\overline{a_2}) + 7^2(a_3\overline{a_3}, a_4\overline{a_4})\end{aligned}$$

Thus $3^2\langle a,a\rangle \le \langle T_C a,a\rangle \le 11^2\langle a,a\rangle$. *Now let* K *be the operator defined on* $\mathcal{H}$ *by*

$$K(a_1,a_2,a_3,a_4,a_5,a_6) = \frac{3}{11}(a_3,a_6,a_1,a_4,a_3,a_2).$$

We can see that Γ_{PQ} *serves as a* (P,Q)*-controlled tight K-g-fusion frame for* $\mathcal{H}$.

Example 3.2. *Let* $\mathcal{A} = C([-1,1],\mathbb{C})$ *be the* C^**-algebra of all scalar continuous functions on the compact space* $[-1,1]$. *Denote by* ℓ_n, *for an integer* $n \ge 5$, *the classical Hilbert space* $\mathbb{C}^n$ *with its canonical basis* $(e_1,e_2,e_3,e_4,e_5,...,e_n)$. *Let* $\mathcal{H} = C([-1,1],\ell_n)$ *be the space of continuous functions from* $[-1,1]$ *into* ℓ_n. *It is known that* $(\mathcal{H},\langle,\rangle)$ *is a Hilbert* $\mathcal{A}$*-module with* $\langle f,g\rangle(x) = \langle f(x),g(x)\rangle$, *where the last notation refers to the usual inner product of* ℓ_n.

We need now to define the following sub-Hilbert $\mathcal{A}$*-modules of* $\mathcal{H}$:

$$\begin{aligned} W_1 &= C([-1,1], span(e_1,e_2,e_3)), & W_2 &= C([-1,1], span(e_2,e_3,e_4)),\\ W_3 &= C([-1,1], span(e_3,e_4,e_5)), & W_4 &= C([-1,1], span(e_1,e_4,e_5)),\\ H_1 &= C([-1,1], span(e_4,e_5)), & H_2 &= C([-1,1], span(e_1,e_5)),\\ H_3 &= C([-1,1], span(e_1,e_2)), & H_4 &= C([-1,1], span(e_2,e_3)).\end{aligned}$$

Next, we define the maps $\Lambda_j : \mathcal{H} \to \mathcal{H}_j$ *as follows:*

$$\Lambda_j(f) = \langle f, E_{\sigma^j(1)}\rangle E_{\sigma^{j+2}(1)} + \langle f, E_{\sigma^{j+1}(1)}\rangle E_{\sigma^{j+3}(1)},$$

where $E_k : [-1,1] \to H$ is the constant function $x \mapsto e_k$, $k = 1, ..., n$, with σ being the cycle $(1,2,3,4,5)$. Therefore the adjoint of the operator Λ_j is given by:

$$\Lambda_j^* : H_j \to H;\ \Lambda_j^*(g) = \langle g, E_{j+3}\rangle E_{j+1} + \langle g, E_{j+4}\rangle E_{j+2}.$$

Thus, the positive $\Lambda_j^*\Lambda_j$ is defined on $\mathcal{H}$ by:

$$\Lambda_j^*\Lambda_j(f) = \langle f, E_{j+1}\rangle E_{j+1} + \langle f, E_{j+2}\rangle E_{j+2}.$$

This can be written as:

$$\Lambda_j^*\Lambda_j = \pi_{span(E_{j+1},E_{j+2})}.$$

Now, for a given positive invertible commuting operators $P, Q \in End_{\mathcal{A}}^*(\mathcal{H})$, we denote $S_{\Lambda_{P,Q}}$ the associated operator frame to $\Lambda_{P,Q} = \{W_j, \Lambda_j, v_j\}_{j\in J}$, where $J = \{1,2,3,4\}$ and v_1, v_2, v_3, v_4 are positive real numbers. Now let us compute the following sum:

$$\begin{aligned}
\langle S_{\Lambda_{P,Q}} f, f\rangle &= \sum_{j=1}^{4} v_j^2 \langle \Lambda_j \pi_{W_j} P f, \Lambda_j \pi_{W_j} Q f\rangle \\
&= \sum_{j=1}^{4} v_j^2 \langle Q \pi_{W_j} \Lambda_j^* \Lambda_j \pi_{W_j} P f, f\rangle \\
&= \sum_{j=1}^{4} v_j^2 \langle C' \pi_{span(E_{j+1},E_{j+2})} C f, f\rangle. \\
&= \sum_{j=1}^{4} v_j^2 \langle \pi_{span(E_{j+1},E_{j+2})} C f, \pi_{span(E_{j+1},E_{j+2})} C' f\rangle. \\
&= \sum_{j=1}^{4} v_j^2 \left(\langle C f, E_{j+1}\rangle\langle E_{j+1}, C' f\rangle + \langle P f, E_{j+2}\rangle\langle E_{j+2}, Q f\rangle \right).
\end{aligned}$$

Now, let P and Q be defined on $\mathcal{H}$ as:

$$Pf(x) = e^x f(x) \quad and \quad Qf(x) = (x+2)^2 f(x),$$

for all $f \in \mathcal{H}$ and $x \in [-1,1]$. Substitute these expressions into the previous sum we obtain:

$$\langle \Theta f, f\rangle(x) = \sum_{j=1}^{4} v_j^2 \left(\langle e^x f(x), E_{j+1}\rangle\langle E_{j+1}, (x+2)^2 f(x)\rangle + \langle e^x f(x), E_{j+2}\rangle\langle E_{j+2}, (x+2)^2 f(x)\rangle \right).$$

Using the notations $f_j = \langle f, E_j\rangle$ for every $f \in \mathcal{H}$ and $1 \le j \le n$. This simplifies the above expression to:

$$\langle S_{\Lambda_{P,Q}} f, f\rangle(x) = e^x (2+x)^2 \sum_{j=1}^{4} v_j^2 \left(|f_{\sigma^j(1)}(x)|^2 + |f_{\sigma^{j+1}(1)}(x)|^2 \right).$$

for all $f \in \mathcal{H}$ and $x \in [-1,1]$. Finally, the expression can be written as:

$$\begin{aligned}
\langle S_{\Lambda_{P,Q}} f, f\rangle(x) &= e^x(2+x)^2(v_4^2+v_5^2)|f_1(x)|^2 + e^x(2+x)^2(v_5^2+v_1^2)|f_2(x)|^2 \\
&+ e^x(2+x)^2(v_1^2+v_2^2)|f_3(x)|^2 + e^x(2+x)^2(v_2^2+v_3^2)|f_4(x)|^2 \\
&+ e^x(2+x)^2(v_3^2+v_4^2)|f_5(x)|^2
\end{aligned}$$

for all $f \in \mathcal{H}$ and $x \in [-1,1]$. This expression tells us that $\Lambda_{P,Q}$ is a (P,Q)-controlled g-fusion Bessel sequence for $\mathcal{H}$ with:

$$\langle S_{\Lambda_{P,Q}} f, f\rangle \leq 9e\left(\sum_{j=1}^{5} v_j^2\right)\langle f, f\rangle.$$

Furthermore we have:

(1) *If $n = 5$, then $\Lambda_{P,Q} = \left\{W_j, \Lambda_j, v_j\right\}_{j\in J}$ is a (P,Q)-controlled g-fusion frame for $\mathcal{H}$ with lower and upper bounds A and B given by $0 < A = e^{-1}\min_j(v_j^2) \leq B = 9e\left(\sum_{j=1}^{4} v_j^2\right) < \infty$, more precisely:*

$$e^{-1}\min_{1\leq j\leq 4}(v_j^2)\langle f, f\rangle \leq \langle S_{\Lambda_{P,Q}} f, f\rangle \leq 9e\left(\sum_{j=1}^{4} v_j^2\right)\langle f, f\rangle.$$

(2) *If $n \geq 6$, for $K = \pi_{\text{sapn}(E_1,\ldots,E_5)}$, $K^* = K$ and we have $\Lambda_{P,Q} = \left\{W_j, \Lambda_j, v_j\right\}_{j\in J}$ is not a (P,Q)-controlled g-fusion frame. But it is a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ with lower and upper bounds A and B given by $0 < A = e^{-1}\min_j(v_j^2) \leq B = 9e\left(\sum_{j=1}^{4} v_j^2\right) < \infty$, more precisely:*

$$e^{-1}\min_{1\leq j\leq 4}(v_j^2)\langle K^* f, K^* f\rangle \leq \langle S_{\Lambda_{P,Q}} f, f\rangle \leq 4e\left(\sum_{j=1}^{5} v_j^2\right)\langle f, f\rangle.$$

(3) *If $Q = P$ and $(Pf)(x) = f(-x)$, $x \in [-1,1]$, then the following properties hold: $P^2 = P$, $P \notin End_{\mathcal{A}}^*(\mathcal{H})$ and P is an isometry on the normed Banach space $(\mathcal{H}, \|\cdot\|)$, where $\|f\| := \max_{x\in[-1,1]}\|f(x)\|_{\ell_n}$.*

In this case, $\Lambda_{P,Q} = \{\mathcal{H}, I_{\mathcal{H}}, 1\}$ is not a (P,Q)-controlled Bessel sequence for $\mathcal{H}$. To see this, assume the contrary and let $f_k(x) = e^{kx}e_1$, for $x \in [-1,1]$.

Then, there exists a real scalar $D > 0$ such that the following inequality holds:

$$\begin{aligned} e^{-2kx} = \langle (Pf_k)(x), (Qf_k)(x)\rangle &= \langle Pf_k, Qf_k\rangle(x) \\ &\leq D\langle f_k, f_k\rangle(x) \\ &\leq D\langle f_k(x), f_k(x)\rangle = De^{2kx}. \end{aligned}$$

This inequality holds for all $x \in [-1,1]$ and $k \in \mathbb{N}$. However, this leads to a contradiction. Specifically, for $x = -1$, we obtain $e^{4k} \leq D$ for all integers k. Therefore, the assumption that $\Lambda_{P,Q}$ is a (P,Q)-controlled Bessel sequence must be false.

4. Main results

Here, we present our investigation on the (P,Q)-controlled K-g-fusion frame for the Hilbert C*-module $\mathcal{H}$. We begin by discussing some algebraic, analytic, and order conditions that allow us to view a (P,Q)-controlled g-fusion Bessel sequence for $\mathcal{H}$ as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$. Additionally, we will discuss some perturbation results related to frames. We will also provide counterexamples that show that certain assumptions, which may seem restrictive in our results, are actually necessary.

Theorem 4.1. *Consider $K \in End^*_{\mathcal{A}}(\mathcal{H})$, and let $\Gamma_{PQ} = \{W_j, \Gamma_j, v_j\}_{j\in J}$ be a (P,Q)-controlled g-fusion Bessel sequence for $\mathcal{H}$, with the synthesis operator T_C associated with Γ_{PQ}. Assuming that the range $\mathcal{R}ang(T_C)$ is closed, then the following propositions hold:*

(1) *Γ_{PQ} serves as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ if and only if $\mathcal{R}ang(K) \subseteq \mathcal{R}ang(T_C)$.*
(2) *The equality $\mathcal{R}ang(K) = \mathcal{R}ang(T_C)$ holds if and only if there are constants $0 < A \leq B < \infty$ such that:*
$$A\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} P\xi, \Gamma_j \pi_{W_j} Q\xi\rangle \leq B\langle K^*\xi, K^*\xi\rangle, \quad \forall \xi \in \mathcal{H}$$
(3) *Γ_{PQ} is a (P,Q)-controlled tight K-g-fusion frame for $\mathcal{H}$, then $\mathcal{R}ang(K) = \mathcal{R}ang(T_C)$.*

Proof. (1) Assume Γ_{PQ} qualifies as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$. Consequently, for any $\xi \in \mathcal{H}$, the following inequality holds:
$$\begin{aligned} A\langle K^*\xi, K^*\xi\rangle &\leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \\ &= \langle (v_j \Gamma_j \pi_{W_j} Q\xi)_{j\in J}, (v_j \Gamma_j \pi_{W_j} P\xi)_{j\in J}\rangle \\ &= \langle T_C^*\xi, T_C^*\xi\rangle \end{aligned}$$
This implies:
$$\langle AKK^*\xi, \xi\rangle \leq \langle T_C T_C^*\xi, \xi\rangle$$
for all $\xi \in \mathcal{H}$. Thus:
$$AKK^* \leq T_C T_C^*$$
Therefore, according to Theorem 2.2 and since the rang of T_C is closed, we have the suitable inequality $\mathcal{R}ang(K) \subseteq \mathcal{R}ang(T_C)$.

(2) If it is given that $\mathcal{R}ang(K) = \mathcal{R}ang(T_C)$, per Theorem 2.2, constants $A, B > 0$ exist such that:
$$AKK^* \leq T_C T_C^* \leq BKK^*$$
This implies, for every $\xi \in \mathcal{H}$:
$$\langle AKK^*\xi, \xi\rangle \leq \langle T_C T_C^*\xi, \xi\rangle \leq \langle BKK^*\xi, \xi\rangle$$
Then
$$A\langle KK^*\xi, \xi\rangle \leq \langle T_C T_C^*\xi, \xi\rangle \leq B\langle KK^*\xi, \xi\rangle$$
Consequently:
$$A\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \leq B\langle K^*\xi, K^*\xi\rangle$$
Assuming the existence of constants $A, B > 0$ satisfying the above inequalities for every $\xi \in \mathcal{H}$, then:
$$AKK^* \leq T_C T_C^* \leq BKK^*$$
This, by Theorem 2.2, affirms that $\mathcal{R}ang(T_C) = \mathcal{R}ang(K)$.

(3) Assume that Γ_{PQ} qualifies as a (P,Q)-controlled tight K-g-fusion frame for $\mathcal{H}$. Consequently, for any $\xi \in \mathcal{H}$, the following equalities hold:

$$\begin{aligned} A\langle K^*\xi, K^*\xi\rangle &= \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \\ &= \langle (v_j \Gamma_j \pi_{W_j} Q\xi)_{j\in J}, (v_j \Gamma_j \pi_{W_j} P\xi)_{j\in J}\rangle \\ &= \langle T_C^*\xi, T_C^*\xi\rangle \end{aligned}$$

This implies:

$$\langle AKK^*\xi, \xi\rangle = \langle T_C T_C^*\xi, \xi\rangle$$

Thus:

$$AKK^* = T_C T_C^*$$

Therefore, according to Theorem 2.2, the ranges $\mathcal{R}ang(T_C)$ and $\mathcal{R}ang(K)$ are equal.

□

The following example, quoted from [15, Example 3.1], illustrates that the closeness of the range of T_C is necessary in Theorem 4.1, even if $\overline{\mathcal{R}ang(T_C)}$ is orthogonally complemented in $\mathcal{H}$.

Example 4.1. *Let $\mathcal{H}_1 = \mathcal{A}$ be the C^*-algebra of all bounded linear operators on the Hilbert space ℓ^2, and let $\mathcal{H}$ denote the ideal of compact operators on ℓ^2, which we also denote by $\mathcal{A}$. Then both $\mathcal{H}$ and $\mathcal{H}_1$ are Hilbert C^*-modules over $\mathcal{A}$, equipped with the inner product $\langle U, V\rangle := UV^*$.*

Let $\Lambda_1 : \mathcal{H} \to \mathcal{H}_1$ be the right multiplication operator, defined by $\Lambda_1(\xi) = L\xi$, where L is the compact positive operator on ℓ^2 defined by $Le_n = \frac{1}{n}e_n$ for all $n \in \mathbb{N}$.

Now, denote $W_1 = \mathcal{H}$, $P = Q = I_{\mathcal{H}}$, and $v_1 = 1$. Let $\Gamma_{PQ} = \{W_1, \Gamma_1, v_1\}$ be defined as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$, where $K : \mathcal{H} \to \mathcal{H}$ is the operator defined by $K\xi = L\xi$.

It can be verified that the synthesis operator $T_C : \mathcal{H}_1 \to \mathcal{H}$ is given by $T_C\xi = L\xi$. Therefore, by [15, Example 3.1], the adjoint operator $T_C^* : \mathcal{H} \to \mathcal{H}_1$ is given by $T_C^*\xi = L\xi$, and $\overline{\mathcal{R}ang(T_C)}$ is complemented in $\mathcal{H}$, as the two spaces are equal. Hence, we obtain the equality $KK^* = T_C T_C^*$. This shows that Γ_{PQ} is a (P,Q)-controlled tight K-g-fusion frame for $\mathcal{H}$. On the other hand, from the proof of [15, Corollary 3.2], we deduce that the inclusion $\mathcal{R}ang(K) \subseteq \mathcal{R}ang(T_C)$ does not hold. Therefore, assertions (1) and (3) of Theorem 4.1 do not remain valid if the range of T_C is not closed, even if its closure is orthogonally complemented.

Theorem 4.2. *Let $K \in End^*_{\mathcal{A}}(\mathcal{H})$. Under this assumption the following statements hold:*

(i) *Any (P,Q)-controlled g-fusion frame is also a (P,Q)-controlled K-g-fusion frame.*
(ii) *If $\mathcal{R}ang(K)$ is closed, then any (P,Q)-controlled K-g-fusion frame is also a (P,Q)-controlled g-fusion frame relative to $\mathcal{R}ang(K)$.*

Proof. (i) Consider Γ_{PQ} as a (P,Q)-controlled g-fusion frame for $\mathcal{H}$ with bounds C and D. For each $\xi \in \mathcal{H}$, the following inequality holds:

$$\frac{C}{\|K\|^2}\langle K^*\xi, K^*\xi\rangle \leq C\langle \xi, \xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \leq D\langle \xi, \xi\rangle$$

Consequently, Γ_{PQ} also functions as a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$ with adjusted bounds $\frac{C}{\|K\|^2}$ and D.

(ii) Assuming Γ_{PQ} operates as a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$ with bounds C and D, and given that $\mathcal{R}ang(K)$ is closed, according to Theorem 2.3, there exists an operator $K^\dagger \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $(K^\dagger)^* K^* = (KK^\dagger)^* = KK^\dagger$ and $KK^\dagger\xi = \xi$ for all ξ within $\mathcal{R}ang(K)$. Therefore, for each ξ in $\mathcal{R}ang(K)$, we derive:

$$\begin{aligned} \frac{C}{1+\|K^\dagger\|^2}\langle \xi, \xi\rangle &= \frac{C}{1+\|K^\dagger\|^2}\langle KK^\dagger\xi, KK^\dagger\xi\rangle \\ &= \frac{C}{1+\|K^\dagger\|^2}\langle (K^\dagger)^* K^*\xi, (K^\dagger)^* K^*\xi\rangle \\ &\leq \frac{\|(K^\dagger)^*\|^2}{1+\|K^\dagger\|^2} C\langle K^*\xi, K^*\xi\rangle \\ &\leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \quad \leq \quad D\langle \xi, \xi\rangle \end{aligned}$$

This establishes that Γ_{PQ} acts as a (P, Q)-controlled g-fusion frame for $\mathcal{R}ang(K)$ with modified bounds $\frac{C}{1+\|K^\dagger\|^2}$ and D.

□

Theorem 4.3. *Assume $K \in End^*_{\mathcal{A}}(\mathcal{H})$ with closed range, and that Γ_{PQ} represents a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$ with bounds C and D. If $V \in End^*_{\mathcal{A}}(\mathcal{H})$ such that $\mathcal{R}ang(V)$ is included within $\mathcal{R}ang(K)$, then Γ_{PQ} is also a (P, Q)-controlled V-g-fusion frame for $\mathcal{H}$.*

Proof. Given that Γ_{PQ} constitutes a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$, it holds that for each $\xi \in \mathcal{H}$,

$$C\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \leq D\langle \xi, \xi\rangle$$

Furthermore, considering that $\mathcal{R}ang(V) \subset \mathcal{R}ang(K)$, according to Theorem 2.2, there exists a coefficient $\lambda > 0$ such that $VV^* \leq \lambda KK^*$. Consequently, for any $\xi \in \mathcal{H}$, we can deduce:

$$\frac{C}{\lambda}\langle V^*\xi, V^*\xi\rangle \leq C\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \leq D\langle \xi, \xi\rangle$$

This establishes that Γ_{PQ} also serves as a (P, Q)-controlled V-g-fusion frame for $\mathcal{H}$. The subsequent theorem elucidates that any controlled K-g-fusion frame inherently functions as a K-g-fusion frame, and vice versa, under specific conditions. □

In the following theorem, we establish a necessary and sufficient condition that allows a controlled g-fusion Bessel sequence to qualify as a controlled K-g-fusion frame by utilizing the quotient operator. Recall that if $U \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_1)$ and $V \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_2)$, where $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ are Hilbert

C*-modules over a C*-algebra $\mathcal{A}$, with the kernel condition $\mathcal{N}(U) \subseteq \mathcal{N}(V)$, the quotient $[V/U]$ of the bounded operators U and V is defined from $\mathcal{R}ang(U)$ to $\mathcal{R}ang(V)$ by

$$[V/U](U\xi) = V\xi \quad \text{for all} \quad \xi \in \mathcal{H}.$$

We note that $[U/V] \circ U = V$ and that the quotient of two bounded operators is not necessarily bounded.

Theorem 4.4. *Assume $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and Γ_{PQ} represents a (P,Q)-controlled g-fusion Bessel sequence in $\mathcal{H}$ with a frame operator $S_{P,Q}$. The sequence Γ_{PU} is a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ if and only if the quotient operator $\left[K^*/S^{1/2}_{P,Q}\right]$ is well-defined and bounded.*

Proof. First, let Γ_{PQ} be assumed to act as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ with bounds C and D. For every element ξ in $\mathcal{H}$, the following inequality holds:

$$C\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle \leq D\langle \xi, \xi\rangle$$

Consequently, for every ξ in $\mathcal{H}$:

$$C\langle K^*\xi, K^*\xi\rangle \leq \langle S_{P,Q}\xi, \xi\rangle = \langle S^{1/2}_{P,Q}\xi, S^{1/2}_{P,Q}\xi\rangle$$

Therefore

$$C\|K^*\xi\|^2 \leq \|S^{1/2}_{P,Q}\xi\|^2$$

It is straightforward to confirm that the quotient operator $U : \mathcal{R}ang(S^{1/2}_{P,Q}) \to \mathcal{R}ang(K^*)$, defined by $U(S^{1/2}_{P,Q}\xi) = K^*\xi$ for all ξ in $\mathcal{H}$, is both well-defined and bounded.

Conversely, if the quotient operator $U := [K^*/S^{1/2}_{P,Q}]$ is well defined and bounded from $\mathcal{R}ang(S^{1/2}_{P,Q})$ into $\mathcal{R}ang(K^*)$. Then $US^{1/2}_{P,Q} = K^*$ and

$$\|Ug\| = \|U\pi_{\mathcal{R}ang(S^{1/2}_{P,Q})} g\| \leq \|U\pi_{\mathcal{R}ang(S^{1/2}_{P,Q})}\|\|g\|$$

for all g in $\mathcal{R}ang(S^{1/2}_{P,Q})$. This implies that

$$\langle K^*\xi, K^*\xi\rangle = \langle US^{1/2}_{P,Q}\xi, US^{1/2}_{P,Q}\xi\rangle \leq \|U\pi_{\mathcal{R}ang(S^{1/2}_{P,Q})}\|^2 \langle S^{1/2}_{P,Q}\xi, S^{1/2}_{P,Q}\xi\rangle$$

for all ξ in $\mathcal{H}$. Hence

$$\frac{1}{\|U\pi_{\mathcal{R}ang(S^{1/2}_{P,Q})}\|^2 + 1}\langle K^*\xi, K^*\xi\rangle \leq \langle S^{1/2}_{P,Q}\xi, S^{1/2}_{P,Q}\xi\rangle = \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle$$

for all ξ in $\mathcal{H}$. Therefore, Γ_{PQ} can be seen as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$. □

We now prove that the boundedness of a quotient operator correlates directly with the transformation of a controlled K-g-fusion frame into a controlled VK-g-fusion frame, for some given $V \in \text{End}^*_{\mathcal{A}}(\mathcal{H})$.

Theorem 4.5. *Suppose $K \in End^*_{\mathcal{A}}(\mathcal{H})$, and Γ_{PQ} is a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$, with its associated frame operator $S_{P,Q}$. Let V be an invertible operator in $End^*_{\mathcal{A}}(\mathcal{H})$ that commutes with both P and Q and assume that $V(W_j)$ is complemented in $\mathcal{H}$ for every $j \in J$. Under these conditions, the following statements are equivalent:*

(1) $\Gamma_{PQ} = \left\{\left(VW_j, \Gamma_j\pi_{W_j}V^*, v_j\right)\right\}_{j\in J}$ *acts as a (P,Q)-controlled VK-g-fusion frame for $\mathcal{H}$, and its associated frame operator is $VS_{P,Q}V^*$.*
(2) *The quotient operator $\left[(VK)^*/S^{1/2}_{P,Q}V^*\right]$ is bounded.*
(3) *The quotient operator $\left[(VK)^*/(VS_{P,Q}V^*)^{1/2}\right]$ is bounded.*

Proof. The equivalence (2) $\iff$ (3) follows immediately from the definition of a quotient operator and the fact that $\langle S^{1/2}_{P,Q}V^*\xi, S^{1/2}_{P,Q}V^*\xi\rangle = \langle (VS_{P,Q}V^*)^{1/2}\xi, (VS_{P,Q}V^*)^{1/2}\xi\rangle$ for all $\xi \in \mathcal{H}$.

Now, we have to prove the equivalence (1) $\iff$ (2). First, note that we have some consequences of Lemma 2.1. Precisely, for each $\xi \in \mathcal{H}$, we have:

$$\begin{aligned}
\sum_{j\in J} v_j^2\langle \Gamma_j\pi_{W_j}V^*\pi_{VW_j}Q\xi, \Gamma_j\pi_{W_j}V^*\pi_{VW_j}P\xi\rangle &= \sum_{j\in J} v_j^2\langle \Gamma_j\pi_{W_j}V^*Q\xi, \Gamma_j\pi_{W_j}V^*P\xi\rangle \\
&= \sum_{j\in J} v_j^2\langle \Gamma_j\pi_{W_j}QV^*\xi, \Gamma_j\pi_{W_j}PV^*\xi\rangle \\
&= \langle S_{P,Q}V^*\xi, V^*\xi\rangle.
\end{aligned}$$

Thus

$$\sum_{j\in J} v_j^2\langle \Gamma_j\pi_{W_j}V^*\pi_{VW_j}Q\xi, \Gamma_j\pi_{W_j}V^*\pi_{VW_j}P\xi\rangle = \langle S^{1/2}_{P,Q}V^*\xi, S^{1/2}_{P,Q}V^*\xi\rangle. \tag{4.1}$$

Proof of (1) $\implies$ (2):Assume that Γ_{PQ} acts as a (P,Q)-controlled VK-g-fusion frame for $\mathcal{H}$ with bounds C and D. For any $\xi \in \mathcal{H}$, the following inequalities hold:

$$C\langle (VK)^*\xi, (VK)^*\xi\rangle \le \sum_{j\in J} v_j^2\langle \Gamma_j\pi_{W_j}V^*\pi_{VW_j}Q\xi, \Gamma_j\pi_{W_j}V^*\pi_{VW_j}P\xi\rangle \le D\langle \xi, \xi\rangle$$

Thus from (4.1) owe obtain that:

$$C\langle (VK)^*\xi, (VK)^*\xi\rangle \le \langle S^{1/2}_{P,Q}V^*\xi, S^{1/2}_{P,Q}V^*\xi\rangle \le D\langle \xi, \xi\rangle,$$

for all $\xi \in \mathcal{H}$. This ensures the quotient operator $U : \mathcal{R}ang(S^{1/2}_{P,Q}V^*) \to \mathcal{R}ang((VK)^*)$, defined by

$$U(S^{1/2}_{P,Q}V^*\xi) = (VK)^*\xi \quad \text{for all} \quad \xi \in \mathcal{H},$$

is well-defined and bounded.

Proof of (2) $\Rightarrow$ (1): If the quotient operator $U := \left[(VK)^*/S^{1/2}_{P,Q}V^*\right]$ is well defined on $\mathcal{R}ang(S^{1/2}_{P,Q}V^*)$ and bounded, then for each $\xi \in \mathcal{H}$, we have:

$$\langle (VK)^*\xi, (VK)^*\xi\rangle = \langle US^{1/2}_{P,Q}V^*\xi, US^{1/2}_{P,Q}V^*\xi\rangle \le \|U\pi_{\mathcal{R}ang(S^{1/2}_{P,Q}V^*)}\|^2\langle S^{1/2}_{P,Q}V^*\xi, S^{1/2}_{P,Q}V^*\xi\rangle$$

Therefore, returning to (4.1) one can see that Γ_{PQ} functions as a (P,Q)-controlled VK-g-fusion frame for $\mathcal{H}$. This concludes the proof. □

In the upcoming theorem, we explore algebraic properties en K of controlled K-g fusion frames.

Theorem 4.6. *$\Gamma_{PQ} = \{W_j, \Gamma_j, v_j\}_{j\in J}$ is defined as a (P,Q)-controlled g-fusion Bessel sequence for $\mathcal{H}$ with the synthesis operator $S_{P,Q}$ associated with Γ_{PQ}. Denote $\mathcal{K}(\Gamma_{PQ})$ the set of all operators $K \in End^*_{\mathcal{A}}(\mathcal{H})$ such that Γ_{PQ} operates as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$. Then the following statements hold:*

1. *$\mathcal{K}(\Gamma_{PQ})$ is a right ideal in $End^*_{\mathcal{A}}(\mathcal{H})$.*
2. *$\mathcal{K}(\Gamma_{PQ})$ is closed whenever $\mathcal{R}ang(T_{P,Q})$ is closed.*

Proof. (1) First we claim that for $L_1, ..., L_n \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, $n \in \mathbb{N}$, we have for every $\xi \in \mathcal{H}$:

$$\langle (\sum_{k=1}^{n} L_k)\xi, (\sum_{k=1}^{n} L_k)\xi\rangle \leq 2^{n-1}\sum_{k=1}^{m}\langle L_k\xi, L_k\xi\rangle$$

$$\langle L_1L_2\xi, L_1L_2\xi\rangle \leq \|L_1\|^2\langle L_2\xi, L_2\xi\rangle.$$

Indeed, By Theorem 2.1 the first inequality is equivalent to the inequality:

$$(L_1 + ... + L_n)^*(L_1 + ... + L_n) \leq 2^{n-1}\left(L_1^*L_1 + ... + L_n^*L_n\right),$$

which follows, by induction, from the fact that for every $u, v \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$:

$$u^*u + v^*v - u^*v - v^*u = (u-v)^*(u-v)$$

is positive a positive element in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, thus

$$(u+v)^*(u+v) = u^*u + v^*v + u^*v + v^*u \leq 2(u^*u + v^*v).$$

The second inequality is obvious.

Now, given that Γ_{PQ} is a (P,Q)-controlled K_i-g-fusion frame for $\mathcal{H}$ for each $i \in \{1, ..., n\}$. Then there exist constants $C_1, ...C_n, D > 0$ such that:

$$C_k\langle K_k^*\xi, K_k^*\xi\rangle \leq \langle S_{P,Q}\xi, \xi\rangle = \sum_{j\in J} v_j^2\langle \Gamma_j\pi_{W_j}Q\xi, \Gamma_j\pi_{W_j}P\xi\rangle \leq D\langle \xi, \xi\rangle$$

for $k \in \{1, ..., n\}$. Now, for operator $K = K_1L_1 + \cdots + K_nL_n$, where $L_1, ..., L_n \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, we have

$$\langle L_k^*K_k^*\xi, L_k^*K_k^*\xi\rangle \leq \|L_k^*\|^2\langle K_k^*\xi, K_k^*\xi\rangle \leq \|L_k^*\|^2C_k^{-1}\langle S_{P,Q}\xi, \xi\rangle$$

for all $k \in \{1, ..., n\}$. Denote

$$C' = \max\left\{\|L_k^*\|^2C_k^{-1} \;:\; 1 \leq k \leq n\right\},$$

thus we have

$$\langle K^*\xi, K^*\xi\rangle \leq 2^{n-1}nC'\langle S_{P,Q}\xi, \xi\rangle,$$

$$\begin{aligned}
\langle K^*\xi, K^*\xi\rangle &= \langle (L_1^*K_1^* + ... + L_n^*K_n^*)\xi, (L_1^*K_1^* + ... + L_n^*K_n^*)\xi\rangle \\
&\leq 2^{n-1}(\langle L_1^*K_1^*\xi, L_1^*K_1^*\xi\rangle + ... + \langle L_n^*K_n^*\xi, L_n^*K_n^*\xi\rangle) \\
&\leq 2^{n-1}(C'\langle S_{P,Q}\xi, \xi\rangle + ... + C'\langle S_{P,Q}\xi, \xi\rangle) \\
&= 2^{n-1}nC'\langle S_{P,Q}\xi, \xi\rangle,
\end{aligned}$$

for all $\xi \in \mathcal{H}$. Therefore, Γ_{PQ} operates as a (P,Q)-controlled $\sum_{k=1}^{n} K_kL_k$-g-fusion frame for $\mathcal{H}$. This proves the theorem.

(2) Assume that $\mathcal{R}ang(T_{P,Q})$ is closed and let $(K_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{K}(\Gamma_{PQ})$ that converges to K in $\mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$. Then, by Theorem 4.1, we have $\mathcal{R}ang(K_n) \subseteq \mathcal{R}ang(T_{P,Q})$ for all $n \in \mathbb{N}$. This implies that for every $\xi \in \mathcal{H}$, we have

$$K\xi = \lim_{n\to\infty} K_n\xi \in \mathcal{R}ang(T_{P,Q}).$$

Thus, $\mathcal{R}ang(K) \subseteq \mathcal{R}ang(T_{P,Q})$. Again, by Theorem 4.1, Γ_{PQ} acts as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$.

□

As consequence one can derive:

Corollary 4.1. *Assume each $K_i \in End^*_{\mathcal{A}}(\mathcal{H})$ and Λ_{TU} operates as a (T,U)-controlled K_i-g-fusion frame for $\mathcal{H}$ for indices $i = 1,2,\ldots,n$. The following properties then hold:*

(1) *Given a set of scalars a_i, for $i = 1,2,\ldots,n$, Λ_{TU} also forms a (T,U)-controlled $\sum_{i=1}^{n} a_iK_i$-g-fusion frame for $\mathcal{H}$.*

(2) *Λ_{TU} constitutes a (T,U)-controlled $K_1\cdots K_n$-g-fusion frame for $\mathcal{H}$.*

The following example shows that, without the assumption of the closeness of the synthesis operator of Λ_{TU}, the right ideal $\mathcal{K}(\Lambda_{TU})$ introduced in the previous theorem is not necessarily closed, even in the classical setting of Hilbert spaces over $\mathbb{C}$.

Example 4.2. *Let $\mathcal{H} = \ell^2$ be the classical Hilbert space with its basis $(e_n)_{n\in\mathbb{N}}$. Let Λ_1 be the operator defined on ℓ^2 by*

$$\Lambda_1(e_n) = \frac{1}{n}e_n \quad \textit{for all} \quad n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$, let $K_n := e_n \otimes e_n$ be the operator on $\mathcal{H}$ defined by $K_n(\xi) = \langle \xi, e_n\rangle e_n$ for all $\xi \in \mathcal{H}$. It can be shown that the following limit exists

$$K = \lim_{n\to\infty} \sum_{k=1}^{n} k^{-2/3} e_k \otimes e_k.$$

Let S_C be the operator frame associated with

$$\Gamma = \{(W_1, \Lambda_1, 1)\}, \quad \textit{where} \quad W_1 = \mathcal{H}.$$

Now, for every $n \in \mathbb{N}$, one can easily see that Γ is a $(I_{\mathcal{H}}, I_{\mathcal{H}})$-controlled K_n-g-fusion frame for $\mathcal{H}$. On the other hand, we have

$$S_C e_n = n^{-2} e_n \quad \textit{for all} \quad n \in \mathbb{N}.$$

Thus, for every positive real number A, the inequality $AKK^* \leq S_C$ does not hold. Indeed, assuming the contrary, we get

$$An^{-4/3} = A\langle KK^* e_n, e_n\rangle \leq \langle S_C e_n, e_n\rangle = n^{-2} \quad \textit{for all} \quad n \in \mathbb{N},$$

which leads to a contradiction.

This theorem explores how certain operators interact within a controlled K-g-fusion frame setting when they commute with each other

Theorem 4.7. *Assume $K \in End^*_{\mathcal{A}}(\mathcal{H})$ and both $P, Q \in GL^+(\mathcal{H})$, with K commuting with P and Q. Additionally, suppose that the K-g-fusion frame operator S_Γ commutes with P, i.e., $S_\Gamma P = PS_\Gamma$. Under these conditions, Γ_{PQ} qualifies as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ if and only if Γ_{PQ} serves as a K-g-fusion frame for $\mathcal{H}$. The frame operator S_Γ is given by:*

$$S_\Gamma \xi = \sum_{j\in J} v_j^2 \pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\xi, \quad \text{for all} \quad \xi \in \mathcal{H}$$

Proof. Initially, we assume that Γ_{PQ} operates as a K-g-fusion frame for $\mathcal{H}$ with established bounds C and D. For any element ξ within $\mathcal{H}$, the following relation holds:

$$C\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in J} v_j^2 \langle \Gamma_j\pi_{W_j}\xi, \Gamma_j\pi_{W_j}\xi\rangle \le D\langle \xi, \xi\rangle$$

Referencing Corollary 2.2 of [1], we infer:

$$mm'CKK^* \le PS_\Gamma Q \le MM'DI_{\mathcal{H}},$$

where m, m', M and M' are positive real numbers. Therefore, for each $\xi \in \mathcal{H}$:

$$mm'C\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in J} v_j^2 \langle P\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi, \xi\rangle \le MM'D\langle \xi, \xi\rangle.$$

Thus

$$mm'C\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in J} v_j^2 \langle \Gamma_j\pi_{W_j}Q\xi, \Gamma_j\pi_{W_j}P\xi\rangle \le MM'D\langle \xi, \xi\rangle$$

Consequently, Γ_{PQ} is established as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$.

Conversely, if we assume Γ_{PQ} is a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$, then there exist constants $C, D > 0$ such that:

$$C\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in J} v_j^2 \langle \Gamma_j\pi_{W_j}Q\xi, \Gamma_j\pi_{W_j}P\xi\rangle \le D\langle \xi, \xi\rangle$$

Now, analyzing further for each $\xi \in \mathcal{H}$:

$$\begin{aligned}
C\langle K^*\xi, K^*\xi\rangle &= C\langle (PQ)^{1/2}(PQ)^{-1/2}K^*\xi, (PQ)^{1/2}(PQ)^{-1/2}K^*\xi\rangle \\
&= C\langle (PQ)^{1/2}K^*(PQ)^{-1/2}\xi, (PQ)^{1/2}K^*(PQ)^{-1/2}\xi\rangle \\
&\le \|(PQ)^{1/2}\|^2 \sum_{j\in J} v_j^2 \langle \Gamma_j\pi_{W_j}Q(PQ)^{-1/2}\xi, \Gamma_j\pi_{W_j}P(PQ)^{-1/2}\xi\rangle \\
&= \|(PQ)^{1/2}\|^2 \langle S_\Gamma\xi, \xi\rangle \\
&= \|(PQ)^{1/2}\|^2 \sum_{j\in J} v_j^2 \langle \pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\xi, \xi\rangle.
\end{aligned}$$

Then we deduce that

$$\frac{C}{\|(PQ)^{1/2}\|^2}\langle K^*\xi, K^*\xi\rangle \le \sum_{j\in J} v_j^2 \langle \Gamma_j\pi_{W_j}\xi, \Gamma_j\pi_{W_j}\xi\rangle$$

In summary, these calculations affirm that Γ_{PQ} is indeed a K-g-fusion frame for $\mathcal{H}$. The proof is thus complete.

□

The following theorem provides a detailed characterization of a controlled K-g-fusion frame.

Theorem 4.8. *Assume $K \in End_{\mathcal{A}}(\mathcal{H})$, with $P, Q \in GL^+(\mathcal{H})$, and note that K commutes with both P and Q. Furthermore, assume $S_\Gamma P = PS_\Gamma$. Under these conditions, Γ_{PQ} qualifies as a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$ if and only if it can be characterized as a $(PQ, I_{\mathcal{H}})$-controlled K-g-fusion frame for $\mathcal{H}$. The K-g-fusion frame operator S_Γ is defined as:*

$$S_\Gamma \xi = \sum_{j\in J} v_j^2 \pi_{W_j} \Gamma_j^* \Gamma_j \pi_{W_j} \xi, \quad \text{for all} \quad \xi \in \mathcal{H}$$

Proof. For any ξ in $\mathcal{H}$, it can be shown that:

$$\begin{aligned}
\sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle &= \langle \sum_{j\in J} v_j^2 P \pi_{W_j} \Gamma_j^* \Gamma_j \pi_{W_j} Q\xi, \xi\rangle \\
&= \langle PS_\Gamma Q\xi, \xi\rangle = \langle S_\Gamma PQ\xi, \xi\rangle \\
&= \langle \sum_{j\in J} v_j^2 \pi_{W_j} \Gamma_j^* \Gamma_j \pi_{W_j} PQ\xi, \xi\rangle \\
&= \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} PQ\xi, \Gamma_j \pi_{W_j} \xi\rangle
\end{aligned}$$

Therefore, the condition that Γ_{PQ} is a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$ with bounds C and D translates into:

$$C\langle K^*\xi, K^*\xi\rangle \leq \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} PQ\xi, \Gamma_j \pi_{W_j} \xi\rangle \leq D\langle \xi, \xi\rangle, \forall \xi \in \mathcal{H}$$

From this, we deduce that Γ_{PQ} indeed qualifies as a $(PQ, I_{\mathcal{H}})$-controlled K-g-fusion frame for $\mathcal{H}$ with the specified bounds C and D. This affirmation concludes the proof.

□

Corollary 4.2. *Assume $K \in End_{\mathcal{A}}(\mathcal{H})$, with $P, Q \in GL^+(\mathcal{H})$, and that K commutes with both P and Q. Additionally, suppose $S_\Gamma P = PS_\Gamma$. Under these conditions, Γ_{PQ} qualifies as a (P, Q)-controlled K-g-fusion frame for $\mathcal{H}$ if and only if it can also be characterized as a $((PQ)^{1/2}, (PQ)^{1/2})$-controlled K-g-fusion frame for $\mathcal{H}$.*

Proof. Following the proof presented in Theorem 4.8, for each function ξ in $\mathcal{H}$, the computation proceeds as follows:

$$\begin{aligned}
\sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} Q\xi, \Gamma_j \pi_{W_j} P\xi\rangle &= \langle S_\Gamma (PQ)^{1/2}\xi, (PQ)^{1/2}\xi\rangle \\
&= \langle \sum_{j\in J} v_j^2 \pi_{W_j} \Gamma_j^* \Gamma_j \pi_{W_j} (PQ)^{1/2}\xi, (PQ)^{1/2}\xi\rangle \\
&= \sum_{j\in J} v_j^2 \langle \Gamma_j \pi_{W_j} (PQ)^{1/2}\xi, \Gamma_j \pi_{W_j} (PQ)^{1/2}\xi\rangle
\end{aligned}$$

Therefore, Γ_{PQ} qualifies as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ if and only if Γ_{PQ} is a $((PQ)^{1/2},(PQ)^{1/2})$-controlled K-g-fusion frame for $\mathcal{H}$.

□

The following theorem investigates the stability of controlled K-g-fusion frames in the presence of perturbations, confirming that certain modifications do not compromise their effectiveness within a controlled setting

Theorem 4.9. *Suppose $\Gamma_{PQ}=\left\{\left(W_j,\Gamma_j,v_j\right)\right\}_{j\in J}$ functions as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ with established bounds C and D. Consider $\Lambda_{PQ}=\left\{\left(V_j,\Lambda_j,v_j\right)\right\}_{j\in J}$. Assume there are constants λ_1,λ_2,μ where $0\leq\lambda_1,\lambda_2<1$ and $0\leq\mu<C(1-\lambda_1)$. If for every $\xi\in\mathcal{H}$, the following inequality holds:*

$$\begin{aligned}
0 &\leq \sum_{j\in J} v_j^2\langle P^*\left(\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}-\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\right)Q\xi,\xi\rangle \\
&\leq \mu\langle K^*\xi,K^*\xi\rangle+\lambda_1\sum_{j\in J} v_j^2\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\rangle \\
&\quad +\lambda_2\sum_{j\in J} v_j^2\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\rangle
\end{aligned}$$

then Λ_{PQ} qualifies as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$.

Proof. Given that Γ_{PQ} operates as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ with bounds C,D, for each $\xi\in\mathcal{H}$ the following relation is observed:

$$C\langle K^*\xi,K^*\xi\rangle\leq\sum_{j\in J} v_j^2\langle\Gamma_j\pi_{W_j}Q\xi,\Gamma_j\pi_{W_j}P\xi\rangle\leq D\langle\xi,\xi\rangle$$

Examining further, for each $\xi\in\mathcal{H}$:

$$\begin{aligned}
\sum_{j\in J} v_j^2\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\rangle &= \sum_{j\in J} v_j^2\langle P^*\left(\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}-\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\right)Q\xi,\xi\rangle \\
&\quad +\sum_{j\in J} v_j^2\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\rangle
\end{aligned}$$

Hence

$$(1-\lambda_2)\sum_{j\in J} v_j^2\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\rangle\leq(1+\lambda_1)\sum_{j\in J} v_j^2\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\rangle+\mu\langle K^*\xi,K^*\xi\rangle$$

This implies that

$$\sum_{j\in J} v_j^2\langle\Lambda_j\pi_{V_j}Q\xi,\Lambda_j\pi_{V_j}P\xi\rangle\leq\left[\frac{(1+\lambda_1)D+\mu\|K\|^2}{(1-\lambda_2)}\right]\langle\xi,\xi\rangle.$$

Now we have to prove the left inequality, pick $\xi\in\mathcal{H}$, then we have:

$$\sum_{j\in J} v_j^2\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\rangle$$
$$\geq$$
$$\sum_{j\in J} v_j^2\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\rangle - \sum_{j\in J} v_j^2\langle P^*\left(\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j} - \pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\right)Q\xi,\xi\rangle$$

This implies that

$$(1+\lambda_2)\sum_{j\in J} v_j^2\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\rangle \geq (1-\lambda_1)\sum_{j\in J} v_j^2\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\rangle - \mu\langle K^*\xi,K^*\xi\rangle$$

Hence

$$\sum_{j\in J} v_j^2\langle\Lambda_j\pi_{V_j}Q\xi,\Lambda_j\pi_{V_j}P\xi\rangle \geq \left[\frac{(1-\lambda_1)C-\mu}{(1+\lambda_2)}\right]\langle K^*\xi,K^*\xi\rangle$$

Therefore, Λ_{PQ} confirms its role as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$. □

Corollary 4.3. *Assume that Γ_{PQ} is a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$ with bounds C and D. Let $\Lambda_{PQ} = \left\{\left(V_j,\Lambda_j,v_j\right)\right\}_{j\in J}$. If a constant $0 < E < C$ ensures that for every $\xi \in \mathcal{H}$,*

$$0 \leq \sum_{j\in J} v_j^2\langle P^*\left(\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j} - \pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\right)Q\xi,\xi\rangle \leq E\langle K^*\xi,K^*\xi\rangle$$

then Λ_{PQ} qualifies as a (P,Q)-controlled K-g-fusion frame for $\mathcal{H}$.

Proof. For each $\xi \in \mathcal{H}$, we have

$$\begin{aligned}
\sum_{j\in J} v_j^2\left\langle\Lambda_j\pi_{V_j}Q\xi,\Lambda_j\pi_{V_j}P\xi\right\rangle &= \sum_{j\in J} v_j^2\left\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\right\rangle \\
&= \sum_{j\in J} v_j^2\left\langle P^*\left(\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j} - \pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\right)Q\xi,\xi\right\rangle \\
&\quad + \sum_{j\in J} v_j^2\left\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\right\rangle \\
&\leq \left(D + E\|K\|^2\right)\langle\xi,\xi\rangle
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\sum_{j\in J} v_j^2\left\langle P^*\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j}Q\xi,\xi\right\rangle &\geq \sum_{j\in J} v_j^2\left\langle P^*\pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}Q\xi,\xi\right\rangle \\
&\quad - \sum_{j\in J} v_j^2\left\langle P^*\left(\pi_{V_j}\Lambda_j^*\Lambda_j\pi_{V_j} - \pi_{W_j}\Gamma_j^*\Gamma_j\pi_{W_j}\right)Q\xi,\xi\right\rangle \\
&\geq (C-E)\langle\xi,\xi\rangle,
\end{aligned}$$

for all $\xi \in \mathcal{H}$. This completes the proof. □

Theorem 4.10. *Suppose that Γ_{PQ} is a (P,Q)-controlled g-fusion frame for $\mathcal{H}$ with the frame operator $S_{P,Q}$. If $S_{P,Q}^{-1}$ commutes with P and Q, then $\Lambda_{PQ} = \left\{\left(S_{P,Q}^{-1}W_j,\Gamma_j\pi_{W_j}S_{P,Q}^{-1},v_j\right)\right\}_{j\in J}$ constitutes a (P,Q)-controlled g-fusion frame for $\mathcal{H}$, where the corresponding frame operator is $S_{P,Q}^{-1}$.*

Proof. The proof of this theorem follows directly from applying Theorem 2.6, by setting $K = I_{\mathcal{H}}$. □

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