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A New Compound Family of Weibull Order Statistics and Left k-Truncated Power Series Distributions

Mohieddine Rahmouni*

Applied College, King Faisal University, Al-Ahsa, Saudi Arabia

**Corresponding author:* mrahmouni@kfu.edu.sa

Abstract. This paper introduces a new family of distributions by compounding the left k-truncated power series distribution with the k-th order statistic of the Weibull distribution. The new family provides a flexible framework for modeling complex data, particularly in reliability engineering and survival analysis. We derive key functions, including the probability density function (PDF), cumulative distribution function (CDF), and hazard rate function (HRF). Several important special cases—such as the geometric, Poisson, binomial, and logarithmic power series distributions—are discussed. Fundamental properties of the family, including moments and quantiles, are explored. Parameter estimation is addressed using the maximum likelihood and the expectation-maximization methods. The paper concludes with potential applications and future research directions.

1. Introduction

The Weibull distribution is a cornerstone of reliability analysis for its versatility in modeling diverse hazard rates—monotonically increasing, decreasing, or constant—governed by its shape parameter. Its flexibility makes it one of the most widely used distributions for analyzing the lifespan of systems and their components. However, its utility wanes when modeling complex systems where failure times are interdependent or the number of components varies, requiring more flexible frameworks to reflect real-world complexities [1,2].

To overcome these limitations, several extensions of the Weibull distribution have been developed [3,4]. The exponentiated Weibull distribution introduces an additional shape parameter, enabling it to model non-monotonic hazard rates [5]. Similarly, the generalized Weibull distribution adds a parameter to enhance its flexibility in representing diverse failure patterns [6]. For

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systems with random or variable component counts, compound distributions offer further innovation. The Weibull-Poisson distribution integrates a Poisson process, while the Weibull-geometric distribution leverages a geometric framework, both providing robust options for intricate scenarios [7,8].

These advanced models significantly expand the applicability of Weibull-based approaches, proving effective in reliability studies and survival analysis. By addressing the shortcomings of the standard Weibull distribution, they enable more accurate modeling of multi-component systems and complex failure mechanisms, reinforcing their practical value across diverse fields. Order statistics are also essential for modeling systems where failure times depend on the sequence of events. In reliability systems with redundant components, the k-th order statistic plays a crucial role, as it represents the time of the k-th failure, which often determines the system's overall lifetime.

This paper introduces a novel family of distributions by compounding the k-th order Weibull distribution with the left k-truncated power series distribution. The left k-truncated power series distribution provides a versatile framework for modeling discrete random variables with a lower truncation point–common in applications where data below a certain threshold is unobserved. By incorporating order statistics and left truncation, this approach extends the work of Morais & Barreto-Souza [9] and Rahmouni & Orabi [10–12], significantly enhancing modeling capabilities for reliability and survival analysis. This combination increases modeling flexibility and unifies several well-known distributions, including the geometric, Poisson, binomial, and logarithmic distributions. Moreover, it accommodates diverse hazard rate shapes, such as bathtub-shaped and unimodal patterns, making it particularly useful for systems with varying numbers of components.

The rest of the paper is organized as follows: Section 2 introduces the new family of distributions, including the probability density function (PDF), cumulative distribution function (CDF), and special members of the family. Section 3 discusses the mathematical properties of the new family, including moments, quantiles, and hazard rate functions. Section 4 presents the estimation of the model parameters using the maximum likelihood method and the expectation-maximization (EM) algorithm. Finally, Section 5 concludes the paper with a discussion of potential applications and future research directions.

2. The New Family of Distributions

Let $X_{(k)}$ denote the k-th order statistic from a sample of size n drawn from a Weibull distribution with parameters α and β. The CDF of $X_{(k)}$, denoted by $F_{X_{(k)}}(y;n,\alpha,\beta)$, is given by:

$$F_{X_{(k)}}(y;n,\alpha,\beta) = \sum_{j=k}^{n} \binom{n}{j} \left(1 - e^{-(\beta y)^{\alpha}}\right)^{j} e^{-(n-j)(\beta y)^{\alpha}},$$

where $\alpha > 0$ is the shape parameter, and $\beta > 0$ is the scale parameter of the Weibull distribution. This CDF represents the probability that the k-th smallest value, $X_{(k)}$, in a sample of size n is less than or equal to x, incorporating the cumulative behavior of the Weibull distribution for the order statistic.

Let N be a discrete random variable following a left k-truncated power series distribution with the probability mass function (PMF):

$$P(N=n)=\frac{a_n\theta^n}{C_k^L(\theta)},\quad n\geq k,$$

where $a_n \geq 0$ are coefficients depending on n, and $\theta > 0$ is the power series parameter. The normalization constant $C_k^L(\theta)$ ensures the PMF sums to unity and is defined as:

$$C_k^L(\theta)=\sum_{n=k}^{\infty}a_n\theta^n.$$

Let's denote the random variable of this new family as $Y = X_{(k)}$. The CDF of the new family of distributions is obtained by summing over all values of N, weighted by the PMF of N:

$$F_Y(y)=P(Y\leq y)=\sum_{n=k}^{\infty}P(X_{(k)}\leq y|N=n)P(N=n),$$

where $F_{X_{(k)}}(y) = P(X_{(k)} \leq y|N = n)$ is the CDF of the k-th order statistic for a sample of size n. Thus,

$$F_Y(y;\alpha,\beta,\theta,k)=\sum_{n=k}^{\infty}\frac{a_n\theta^n}{C_k^L(\theta)}F_{X_{(k)}}(y;n,\alpha,\beta).$$

$$F_Y(y;\alpha,\beta,\theta,k)=\sum_{n=k}^{\infty}\frac{a_n\theta^n}{C_k^L(\theta)}\left[1-\sum_{j=0}^{k-1}\binom{n}{j}\left(1-e^{-(\beta y)^\alpha}\right)^j e^{-(n-j)(\beta y)^\alpha}\right].$$

This expression models the cumulative probability of the k-th order statistic under the left k-truncated power series distribution and the Weibull order statistics. The CDF of the Weibull order statistic can also be expressed using the regularized incomplete beta function $I(p;a,b)$:

$$F_{X_{(k)}}(y;n,\alpha,\beta)=I\left(1-e^{-(\beta y)^\alpha};k,n-k+1\right),$$

where

$$I(p;a,b)=\frac{B(p;a,b)}{B(a,b)},$$

with $B(p;a,b)$ being the incomplete beta function and $B(a,b)$ the complete beta function. Thus, the CDF of the new family becomes:

$$F_Y(y;\alpha,\beta,\theta,k)=\sum_{n=k}^{\infty}\frac{a_n\theta^n}{C_k^L(\theta)}I\left(1-e^{-(\beta y)^\alpha};k,n-k+1\right).$$

Proof. For a sample $X_1, X_2, \ldots, X_n$, the k-th order statistic $X_{(k)}$ is the k-th smallest value, so:

$$X_{(1)}\leq X_{(2)}\leq\cdots\leq X_{(n)}.$$

The CDF of $X_{(k)}$ is $P(X_{(k)} \leq y)$, the probability that the k-th smallest value is less than or equal to y. Let S be the number of observations X_i that are less than or equal to y. For each X_i, the event $X_i \leq y$

is a Bernoulli trial with success probability $p = F_{X_{(k)}}(y)$. With n independent trials, S follows a binomial distribution with parameters n and p. The probability of exactly j successes (i.e., exactly j observations $\leq y$) is:

$$P(S = j) = \binom{n}{j}[F_{X_{(k)}}(y)]^j[1 - F_{X_{(k)}}(y)]^{n-j}.$$

The event $X_{(k)} \leq y$ corresponds to $S \geq k$. Thus, the CDF is the cumulative probability of S from $j = k$ to $j = n$:

$$P(X_{(k)} \leq y) = P(S \geq k) = \sum_{j=k}^{n} P(S = j).$$

Substituting the binomial probability:

$$P(X_{(k)} \leq y) = \sum_{j=k}^{n} \binom{n}{j}[F_{X_{(k)}}(y)]^j[1 - F_{X_{(k)}}(y)]^{n-j}.$$

Note that the regularized incomplete beta function is defined as:

$$I(p; a, b) = \frac{1}{B(a, b)} \int_0^p t^{a-1}(1 - t)^{b-1}\, dt,$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function, and Γ is the gamma function. For positive integers a and b, the following identity holds:

$$I(p; a, b) = \sum_{j=a}^{n} \binom{n}{j} p^j (1 - p)^{n-j}.$$

Thus, the binomial sum becomes:

$$I(p; a, b) = \sum_{j=k}^{n} \binom{n}{j}[F_{X_{(k)}}(y)]^j[1 - F_{X_{(k)}}(y)]^{n-j} = I(F_{X_{(k)}}(y); k, n - k + 1).$$

Therefore:

$$P(X_{(k)} \leq y) = I(F_{X_{(k)}}(y); k, n - k + 1).$$

The PDF of the new family of distributions is:

$$f_Y(y; \alpha, \beta, \theta, k) = \sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \binom{n}{k-1} \left(1 - e^{-(\beta y)^\alpha}\right)^{k-1} e^{-(n-k+1)(\beta y)^\alpha} \alpha \beta^\alpha y^{\alpha-1}.$$

$$f_Y(y; \alpha, \beta, \theta, k) = \frac{\alpha \beta^\alpha y^{\alpha-1} \left(1 - e^{-(\beta y)^\alpha}\right)^{k-1} e^{(k-1)(\beta y)^\alpha}}{C_k^L(\theta)} \sum_{n=k}^{\infty} a_n \binom{n}{k-1} \left(\theta e^{-(\beta y)^\alpha}\right)^n.$$

We denote a random variable Y following the generalized Weibull-left k-truncated power series (GWPS) distribution with parameters α, β, and θ by $Y \sim \text{GWPS}(\alpha, \beta, \theta)$. This new class of distributions generalizes several existing models. The Weibull-geometric (WG) distribution is obtained by setting $C_k^L(\theta) = \theta(1 - \theta)^{-1}$ with $\theta \in (0, 1)$. Similarly, the Weibull-Poisson (WP) and Weibull-logarithmic (WL) distributions arise by taking $C_k^L(\theta) = e^\theta - 1, \theta > 0$ and $C_k^L(\theta) =$

$-\log(1-\theta), \theta \in (0,1)$, respectively. The GWPS model encompasses these as special cases while allowing for greater flexibility in modeling survival and reliability data. Additionally, extending the parameter space of θ beyond its conventional domain may lead to broader generalizations, as seen in other power series-based distributions.

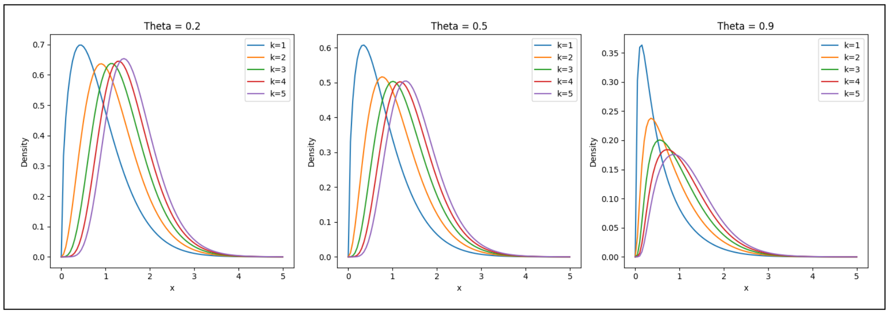
The flexibility of the new model allows various choices of power series distributions for N, leading to different sub-models. The new family includes several special cases depending on the choice of the power series distribution. Table 1 summarizes the useful quantities for some power series distributions.

Table 1. Useful quantities of some power series distributions.

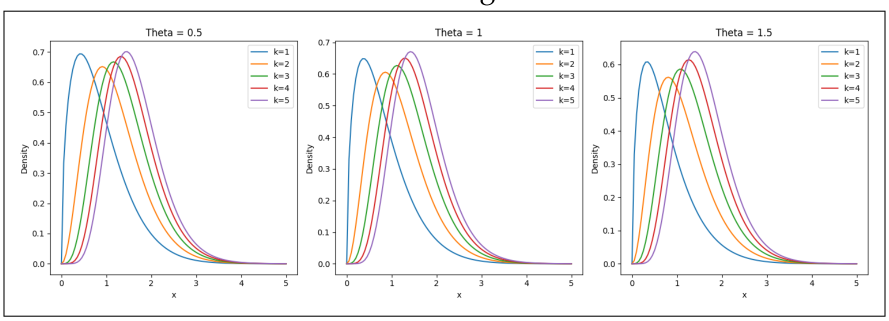
Distribution	a_n	$C_k^L(\theta)$	$C_k'^L(\theta)$	$C_k''^L(\theta)$	θ
Geometric	1	$\dfrac{\theta^k}{1-\theta}$	$\dfrac{k\theta^{k-1}}{(1-\theta)^2}$	$\dfrac{k(k-1)\theta^{k-2}}{(1-\theta)^3}+\dfrac{2k\theta^{k-1}}{(1-\theta)^3}$	$\theta \in (0,1)$
Poisson	$\dfrac{1}{n!}$	$e^\theta-\sum_{j=0}^{k-1}\dfrac{\theta^j}{j!}$	$e^\theta-\sum_{j=0}^{k-1}\dfrac{j\theta^{j-1}}{j!}$	$e^\theta-\sum_{j=0}^{k-1}\dfrac{j(j-1)\theta^{j-2}}{j!}$	$\theta \in (0,\infty)$
Binomial	$\binom{m}{n}$	$(\theta+1)^m -\sum_{j=0}^{k-1}\binom{m}{j}\theta^j$	$m(\theta+1)^{m-1} -\sum_{j=0}^{k-1}j\binom{m}{j}\theta^{j-1}$	$m(m-1)(\theta+1)^{m-2} -\sum_{j=0}^{k-1}j(j-1)\binom{m}{j}\theta^{j-2}$	$\theta \in (0,1)$
Logarithmic	$\dfrac{1}{n}$	$-\log(1-\theta) -\varphi(k)\sum_{j=1}^{k-1}\dfrac{\theta^j}{j}$	$\dfrac{1}{1-\theta}-\varphi(k)\sum_{j=1}^{k-1}\theta^{j-1}$	$\dfrac{1}{(1-\theta)^2} -\varphi(k)\sum_{j=1}^{k-1}(j-1)\theta^{j-2}$	$\theta \in (0,1)$

$$\varphi(k)=\begin{cases}0 & \text{if } k=1\\ 1 & \text{if } k=2,3,...,n\end{cases}$$

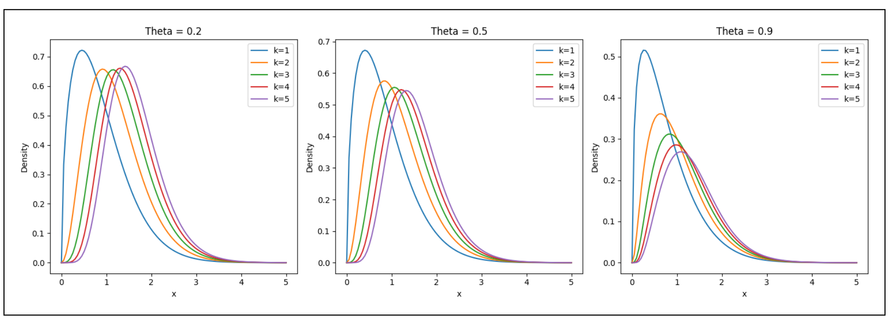
Figure 1 illustrates the PDFs of a generalized Weibull distribution mixed with some truncated discrete distributions, namely the geometric, Poisson, logarithmic, and binomial distributions under a left k-truncated power series framework. Each figure contains subplots corresponding to different values of the power series parameter θ (e.g., 0.2, 0.5, 0.9), with each subplot displaying five curves representing the order statistic parameter k (1, 2, 3, 4, 5). Lower values of θ result in PDFs concentrated around smaller y values with sharper peaks, indicating lower variability, while higher values of θ lead to a broader spread and a rightward shift, reflecting increased variability. As k increases, the PDF shifts further to the right, consistent with the behavior of higher-order statistics. The tail behavior differs across distributions: geometric and logarithmic distributions exhibit heavier tails, indicating a higher probability of larger y values, whereas Poisson and binomial distributions display a smoother decay, reflecting a more gradual transition. These visualizations provide a comprehensive understanding of how θ, k, and the underlying power series distribution collectively shape the Weibull density function's characteristics, including its spread and tail behavior.



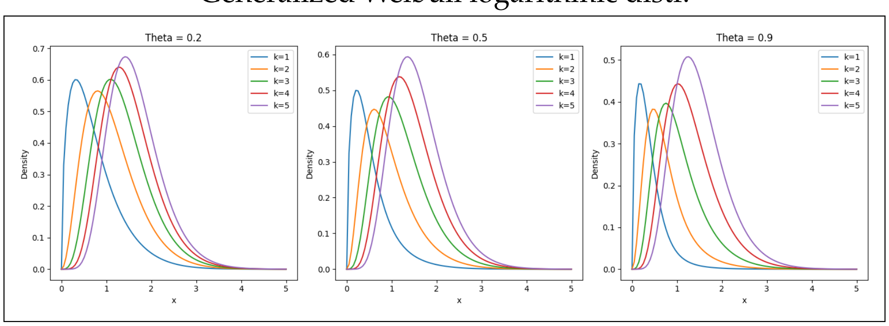
Generalized Weibull geometric distr.



Generalized Weibull Poisson distr.



Generalized Weibull logarithmic distr.



Generalized Weibull binomial distr.

Figure 1. PDF of the GWPS distribution with $\beta = 1$ and $\alpha = 1.5$.

3. Properties

3.1. **Moments.** The moments of the new family are essential for understanding its statistical properties. The first moments are derived using integral representations, and the skewness and kurtosis are analyzed to assess the distribution's tail behavior and shape characteristics.

The general expression for the r-th moment of the new family is given by:

$$\mathbb{E}[X_{(k)}^r] = \sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \mathbb{E}[X_{(k)}^r | N = n], \tag{3.1}$$

where $\mathbb{E}[X_{(k)}^r | N = n]$ represents the r-th moment of the Weibull order statistic for a fixed $N = n$. The explicit form of these moments can be obtained using recurrence relations or numerical integration.

The r-th raw moment of the k-th order Weibull statistic for a sample of size n is given by:

$$\mu_r(n) = \beta^{-r} \binom{n}{k-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (n-j)^{-r/\alpha} \Gamma\left(\frac{r}{\alpha} + 1\right),$$

where $\Gamma(\cdot)$ is the gamma function. The r-th raw moment of the new family is expressed as:

$$\mu_r = \sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \mu_r(n).$$

Table 2. Moments for different distributions

Distribution	μ_r	$\mathbb{E}[Y]$ **($r=1$)**	$\mathbb{E}[Y^2]$ **($r=2$)**
Geometric	$\sum_{n=k}^{\infty} (1-\theta)\theta^{n-k} \mu_r(n)$	$\beta^{-1}\Gamma\left(\frac{1}{\alpha}+1\right) \frac{1-\theta}{\theta} \mathrm{Li}_{1/\alpha}(\theta)$	$\beta^{-2}\Gamma\left(\frac{2}{\alpha}+1\right) \frac{1-\theta}{\theta} \mathrm{Li}_{2/\alpha}(\theta)$
Poisson	$\sum_{n=k}^{\infty} \frac{\frac{\theta^n}{n!}}{e^{\theta} - \sum_{m=0}^{k-1} \frac{\theta^m}{m!}} \mu_r(n)$	$\frac{\beta^{-1}\Gamma\left(\frac{1}{\alpha}+1\right)}{e^{\theta}-1} \sum_{n=1}^{\infty} \frac{\theta^n}{n!} n^{-1/\alpha}$	$\frac{\beta^{-2}\Gamma\left(\frac{2}{\alpha}+1\right)}{e^{\theta}-1} \sum_{n=1}^{\infty} \frac{\theta^n}{n!} n^{-2/\alpha}$
Binomial	$\sum_{n=k}^{m} \frac{\binom{m}{n}\theta^n}{(\theta+1)^m - \sum_{n=0}^{k-1} \binom{m}{n}\theta^n} \mu_r(n)$	$\frac{\beta^{-1}\Gamma\left(\frac{1}{\alpha}+1\right)}{(\theta+1)^m-1} \sum_{n=1}^{m} \binom{m}{n}\theta^n n^{-1/\alpha}$	$\frac{\beta^{-2}\Gamma\left(\frac{2}{\alpha}+1\right)}{(\theta+1)^m-1} \sum_{n=1}^{m} \binom{m}{n}\theta^n n^{-2/\alpha}$
Logarithmic	$\sum_{n=k}^{\infty} \frac{\frac{\theta^n}{n}}{-\log(1-\theta) - \varphi(k) \sum_{n=1}^{k-1} \frac{\theta^n}{n}} \mu_r(n)$	$\frac{\beta^{-1}\Gamma\left(\frac{1}{\alpha}+1\right)}{-\log(1-\theta)} \mathrm{Li}_{1+1/\alpha}(\theta)$	$\frac{\beta^{-2}\Gamma\left(\frac{2}{\alpha}+1\right)}{-\log(1-\theta)} \mathrm{Li}_{1+2/\alpha}(\theta)$

Note: $\mu_r(n) = \beta^{-r} n^{-r/\alpha} \Gamma\left(\frac{r}{\alpha}+1\right)$ for $k=1$, and $\mathrm{Li}_s(\theta) = \sum_{n=1}^{\infty} n^{-s}\theta^n$ is the polylogarithm function.

Proof. The r-th moment μ_r of $Y = X_{(k)}$ is defined as:

$$\mu_r = \mathbb{E}[X_{(k)}^r] = \int_0^{\infty} y^r f_{X_{(k)}}(y; \alpha, \beta, \theta, k)\, dx.$$

Substituting the PDF:

$$\mu_r = \int_0^{\infty} x^r \left(\frac{\alpha\beta^{\alpha} y^{\alpha-1} \left(1 - e^{-(\beta y)^{\alpha}}\right)^{k-1} e^{(k-1)(\beta y)^{\alpha}}}{C_k^L(\theta)} \sum_{n=k}^{\infty} a_n \binom{n}{k-1} \left(\theta e^{-(\beta y)^{\alpha}}\right)^n \right) dy.$$

$$\mu_r = \frac{\alpha\beta^\alpha}{C_k^L(\theta)} \sum_{n=k}^{\infty} a_n \binom{n}{k-1} \theta^n \int_0^\infty y^{\alpha+r-1} \left(1 - e^{-(\beta y)^\alpha}\right)^{k-1} e^{-(n-k+1)(\beta y)^\alpha} \, dy.$$

Let

$$t = (\beta y)^\alpha \Rightarrow dt = \alpha\beta^\alpha y^{\alpha-1} dy.$$

Since $y = (\beta^{-1} t^{1/\alpha})$, we substitute:

$$dy = \frac{dt}{\alpha\beta^\alpha y^{\alpha-1}} = \frac{dt}{\alpha\beta^\alpha} t^{(1-\alpha)/\alpha}.$$

Thus,

$$\mu_r = \frac{1}{C_k^L(\theta)} \sum_{n=k}^{\infty} a_n \binom{n}{k-1} \theta^n \beta^{-r} \int_0^\infty t^{r/\alpha} (1 - e^{-t})^{k-1} e^{-(n-k+1)t} \, dt.$$

The moment expression is:

$$\mu_r(n) = \beta^{-r} \binom{n}{k-1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} (n-j)^{-r/\alpha} \Gamma\left(\frac{r}{\alpha} + 1\right).$$

Thus,

$$\mu_r = \sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \mu_r(n).$$

This expression shows that the r-th moment of the new family is a weighted sum of the r-th raw moments of the k-th order Weibull statistic, with the weights given by $\frac{a_n \theta^n}{C_k^L(\theta)}$.

3.2. **Quantiles.** The quantile function $Q_k(p)$ of the new family is the value of y that satisfies:

$$F_X(Q_k(p); \alpha, \beta, \theta, k) = p,$$

where $F_X(y; \alpha, \beta, \theta, k)$ is the CDF, and p is a given probability in the range $(0, 1)$. To determine $Q_k(p)$, we should solve the equation:

$$p = \sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \left[1 - \sum_{j=0}^{k-1} \binom{n}{j} \left(1 - e^{-(\beta Q_k(p))^\alpha}\right)^j e^{-(n-j)(\beta Q_k(p))^\alpha}\right].$$

3.3. **Hazard rate function.** The hazard rate function (HRF), also known as the *failure rate* or *force of mortality*, is a key concept in survival analysis and reliability theory. It represents the instantaneous failure rate at a given time y, conditioned on survival up to that point. The hazard rate $h_Y(y)$ is defined as:

$$h_Y(y) = \frac{f_Y(y)}{S_Y(y)} = \frac{f_Y(y)}{1 - F_Y(y)},$$

where $S_Y(y) = 1 - F_Y(y)$ is the survival function. Substituting the expressions for the PDF $f_{X_{(k)}}(y)$ and CDF $F_{X_{(k)}}(y)$, we obtain:

$$h_Y(y; \alpha, \beta, \theta, k) = \frac{\sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \binom{n}{k-1} \left(1 - e^{-(\beta y)^\alpha}\right)^{k-1} e^{-(n-k+1)(\beta y)^\alpha} \alpha\beta^\alpha y^{\alpha-1}}{1 - \sum_{n=k}^{\infty} \frac{a_n \theta^n}{C_k^L(\theta)} \left[1 - \sum_{j=0}^{k-1} \binom{n}{j} \left(1 - e^{-(\beta y)^\alpha}\right)^j e^{-(n-j)(\beta y)^\alpha}\right]}.$$

The flexibility of the GWPS model allows various choices of the power series distribution for N, leading to different sub-models. For $C_K(\theta) = \frac{\theta}{1-\theta}, \theta \in (0,1)$, the hazard rate corresponds to the Weibull-geometric distribution. The hazard rate reduces to the Weibull-Poisson distribution when $C_K(\theta) = e^{\theta} - 1, \theta > 0$. When $C_K(\theta) = -\log(1-\theta), \theta \in (0,1)$, the hazard rate corresponds to the Weibull-Logarithmic distribution. These special cases demonstrate the flexibility of the GWPS distribution, making it a versatile tool for modeling survival and reliability data across different fields.

4. Estimation

4.1. **Maximum likelihood estimation (MLE).** The parameters of the proposed distribution can be estimated using the maximum likelihood estimation method. This approach involves deriving the likelihood function and maximizing the corresponding log-likelihood function to obtain the parameter estimates.

Let $y_1, y_2, \ldots, y_m$ be a random sample from the GWPS distribution. The likelihood function $L(\alpha, \beta, \theta, k)$ is the product of the individual probability densities evaluated at each sample point:

$$L(\alpha,\beta,\theta,k) = \prod_{i=1}^{m} f_Y(y_i;\alpha,\beta,\theta,k).$$

The log-likelihood function is the natural logarithm of the likelihood function:

$$\ell(\alpha,\beta,\theta,k) = \sum_{i=1}^{m} \log\left(f_Y(y_i;\alpha,\beta,\theta,k)\right).$$

Substituting the PDF of the new distribution, the log-likelihood function then becomes:

$$\ell(\alpha,\beta,\theta,k) = \sum_{i=1}^{m} \log\left(\frac{\alpha\beta^{\alpha} y_i^{\alpha-1}\left(1-e^{-(\beta y_i)^{\alpha}}\right)^{k-1} e^{(k-1)(\beta y_i)^{\alpha}}}{C_k^L(\theta)} \sum_{n=k}^{\infty} a_n \binom{n}{k-1}\left(\theta e^{-(\beta y_i)^{\alpha}}\right)^n\right).$$

The first derivative of the log-likelihood function with respect to α is:

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{m}\left(\frac{1}{\alpha} + \log(y_i) + (k-1)\log\left(1-e^{-(\beta y_i)^{\alpha}}\right) - (k-1)(\beta y_i)^{\alpha}\right).$$

Setting $\frac{\partial \ell}{\partial \alpha} = 0$ results in the equation for α:

$$\sum_{i=1}^{m}\left(\frac{1}{\alpha} + \log(y_i) + (k-1)\log\left(1-e^{-(\beta y_i)^{\alpha}}\right) - (k-1)(\beta y_i)^{\alpha}\right) = 0.$$

This equation is nonlinear in α and generally requires numerical methods for solution.

The first derivative of the log-likelihood function with respect to β is:

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{m}\left(\frac{\alpha}{\beta} + (k-1)\frac{e^{-(\beta y_i)^{\alpha}}}{1-e^{-(\beta y_i)^{\alpha}}} - (k-1)(\beta y_i)^{\alpha}\log(y_i)\right).$$

Setting $\frac{\partial \ell}{\partial \beta} = 0$ gives the equation for β:

$$\sum_{i=1}^{m}\left(\frac{\alpha}{\beta}+(k-1)\frac{e^{-(\beta y_i)^{\alpha}}}{1-e^{-(\beta y_i)^{\alpha}}}-(k-1)(\beta y_i)^{\alpha}\log(y_i)\right)=0.$$

The partial derivative of the log-likelihood function with respect to θ is:

$$\frac{\partial \ell}{\partial \theta}=\sum_{i=1}^{m}\left(\sum_{n=k}^{\infty}\frac{a_n}{C_k^L(\theta)}\left(\theta e^{-(\beta y_i)^{\alpha}}\right)^n\right).$$

Setting $\frac{\partial \ell}{\partial \theta}=0$ gives the equation for θ:

$$\sum_{i=1}^{m}\left(\sum_{n=k}^{\infty}\frac{a_n}{C_k^L(\theta)}\left(\theta e^{-(\beta y_i)^{\alpha}}\right)^n\right)=0.$$

The MLEs for the parameters α, β, and θ of the proposed distribution are obtained by solving the system of nonlinear equations derived from the log-likelihood function. These equations can be solved numerically using optimization methods such as Newton-Raphson or the expectation-maximization (EM) algorithm. The Fisher information matrix (FIM) is a key quantity in statistical inference, representing the expected value of the observed information. It is used to assess the precision of parameter estimates and plays a fundamental role in deriving asymptotic properties such as the Cramér-Rao lower bound. FIM is computed from the second-order derivatives of the log-likelihood function. For $f(y;\alpha,\beta,\theta)$ with parameter vector $\boldsymbol{\varphi}=(\alpha,\beta,\theta)$, the FIM, $\mathcal{I}(\boldsymbol{\varphi})$, is a symmetric matrix whose elements are defined as:

$$\mathcal{I}_{ij}(\boldsymbol{\varphi})=\mathbb{E}\left[\left(\frac{\partial}{\partial \varphi_i}\log f(X_{(k)};\boldsymbol{\varphi})\right)\left(\frac{\partial}{\partial \varphi_j}\log f(X_{(k)};\boldsymbol{\varphi})\right)\right],$$

where φ_i and φ_j are parameters in the vector $\boldsymbol{\varphi}=(\alpha,\beta,\theta)$. Alternatively, this can be written as:

$$\mathcal{I}_{ij}(\boldsymbol{\varphi})=-\mathbb{E}\left[\frac{\partial^2}{\partial \varphi_i \partial \varphi_j}\log f(X_{(k)};\boldsymbol{\varphi})\right].$$

The second derivatives of the log-likelihood function are essential for calculating the FIM components. The log-likelihood function for a single observation y is given by:

$$\begin{aligned}\log f(y;\alpha,\beta,\theta)&=\log\alpha+\alpha\log\beta+(\alpha-1)\log y+(k-1)\log\left(1-e^{-(\beta y)^{\alpha}}\right)\\&+(k-1)(\beta y)^{\alpha}-\log C_k^L(\theta)+\log\left(\sum_{n=k}^{\infty}a_n\binom{k-1}{n}\left(\theta e^{-(\beta y)^{\alpha}}\right)^n\right),\end{aligned}$$

where $C_k^L(\theta)=\sum_{n=k}^{\infty}a_n\theta^n$ is the normalization constant, and $\mathcal{S}(y)=\sum_{n=k}^{\infty}a_n\binom{k-1}{n}\left(\theta e^{-(\beta y)^{\alpha}}\right)^n$. The FIM for the parameter vector $\boldsymbol{\varphi}=(\alpha,\beta,\theta)$ is a 3×3 matrix:

$$\mathcal{I}(\boldsymbol{\varphi})=\begin{bmatrix}\mathcal{I}_{\alpha\alpha} & \mathcal{I}_{\alpha\beta} & \mathcal{I}_{\alpha\theta}\\ \mathcal{I}_{\beta\alpha} & \mathcal{I}_{\beta\beta} & \mathcal{I}_{\beta\theta}\\ \mathcal{I}_{\theta\alpha} & \mathcal{I}_{\theta\beta} & \mathcal{I}_{\theta\theta}\end{bmatrix},$$

where the diagonal elements $\mathcal{I}_{\alpha\alpha}$, $\mathcal{I}_{\beta\beta}$, and $\mathcal{I}_{\theta\theta}$ measure the amount of information about each parameter individually, and the off-diagonal elements $\mathcal{I}_{\alpha\beta}$, $\mathcal{I}_{\alpha\theta}$, and $\mathcal{I}_{\beta\theta}$ capture the interactions between pairs of parameters.

To compute the FIM, we need the second-order partial derivatives of the log-likelihood function with respect to the parameters α, β, and θ.

The first derivative with respect to α is:

$$\frac{\partial}{\partial\alpha}\log f = \frac{1}{\alpha} + \log\beta + \log y + (k-1)\left[\frac{(\beta y)^{\alpha}\log(\beta y)}{1-e^{-(\beta y)^{\alpha}}} + (\beta y)^{\alpha}\log(\beta y)\right] - \frac{\partial}{\partial\alpha}\log C_k^L(\theta) + \frac{\partial}{\partial\alpha}\log\mathcal{S}(y).$$

The second derivative with respect to α is:

$$\frac{\partial^2}{\partial\alpha^2}\log f = -\frac{1}{\alpha^2} + (k-1)\left[\frac{(\beta y)^{\alpha}\log^2(\beta y)}{(1-e^{-(\beta y)^{\alpha}})^2} + \frac{(\beta y)^{\alpha}\log(\beta y)}{1-e^{-(\beta y)^{\alpha}}}\right].$$

The first derivative with respect to β is:

$$\frac{\partial}{\partial\beta}\log f = \frac{\alpha}{\beta} - (k-1)\alpha(\beta y)^{\alpha-1}x\left[\frac{1}{1-e^{-(\beta y)^{\alpha}}} + 1\right] + \frac{\partial}{\partial\beta}\log\mathcal{S}(y).$$

The second derivative with respect to β is:

$$\frac{\partial^2}{\partial\beta^2}\log f = -\frac{\alpha}{\beta^2} - (k-1)\alpha(\alpha-1)(\beta y)^{\alpha-2}y^2\left[\frac{1}{1-e^{-(\beta y)^{\alpha}}} + 1\right]$$
$$-(k-1)\alpha(\beta y)^{\alpha-1}y\left[\frac{\alpha(\beta y)^{\alpha-1}}{(1-e^{-(\beta y)^{\alpha}})^2}\right].$$

The first derivative with respect to θ is:

$$\frac{\partial}{\partial\theta}\log f = -\frac{\partial}{\partial\theta}\log C_k^L(\theta) + \frac{\sum_{n=k}^{\infty} a_n n\theta^{n-1}\binom{n}{k-1}e^{-n(\beta y)^{\alpha}}}{\mathcal{S}(y)}.$$

The second derivative with respect to θ is:

$$\frac{\partial^2}{\partial\theta^2}\log f = -\frac{\partial^2}{\partial\theta^2}\log C_k^L(\theta) + \frac{\sum_{n=k}^{\infty} a_n n(n-1)\theta^{n-2}\binom{n}{k-1}e^{-n(\beta y)^{\alpha}}}{\mathcal{S}(y)}.$$

We can compute the elements of the FIM. These elements are the expected values of the second-order partial derivatives.

$$\mathcal{I}_{\alpha\alpha} = \mathbb{E}\left[\frac{\partial^2}{\partial\alpha^2}\log f\right].$$

$$\mathcal{I}_{\beta\beta} = \mathbb{E}\left[\frac{\partial^2}{\partial\beta^2}\log f\right].$$

$$\mathcal{I}_{\theta\theta} = \mathbb{E}\left[\frac{\partial^2}{\partial\theta^2}\log f\right].$$

$$\mathcal{I}_{\alpha\beta} = \mathbb{E}\left[\frac{\partial}{\partial\alpha}\log f \cdot \frac{\partial}{\partial\beta}\log f\right].$$

$$\mathcal{I}_{\alpha\theta} = \mathbb{E}\left[\frac{\partial}{\partial\alpha}\log f \cdot \frac{\partial}{\partial\theta}\log f\right].$$

$$\mathcal{I}_{\beta\theta} = \mathbb{E}\left[\frac{\partial}{\partial\beta}\log f \cdot \frac{\partial}{\partial\theta}\log f\right].$$

4.2. **Expectation-maximization (EM) algorithm.** Let $\{x_i\}_{i=1}^n$ denote the k-th order Weibull statistics with scale parameter $\beta > 0$ and shape parameter $\alpha > 0$. These are obtained from samples of size N_i, where $N_i \geq k$ follows a left k-truncated power series distribution. The parameter vector is:

$$\boldsymbol{\varphi} = (\alpha, \beta, \theta).$$

If we observed N_i alongside x_i, the joint density would be:

$$f_{X,N}(x_i, N_i; \alpha, \beta, \theta) = \frac{a_{N_i}\theta^{N_i}}{C_k^L(\theta)} f_{X_{(k)}}(x_i; N_i, \alpha, \beta).$$

The complete-data likelihood is:

$$L_c(\boldsymbol{\varphi}) = \prod_{i=1}^{n}\left\{\frac{a_{N_i}\theta^{N_i}}{C_k^L(\theta)} f_{X_{(k)}}(x_i; N_i, \alpha, \beta)\right\},$$

and the log-likelihood is:

$$\ell_c(\boldsymbol{\varphi}) = \sum_{i=1}^{n}\left[\log a_{N_i} + N_i\log\theta - \log C_k^L(\theta) + \log f_{X_{(k)}}(x_i; N_i, \alpha, \beta)\right].$$

E-step. At the r-th iteration with current parameters $\boldsymbol{\varphi}^{(r)} = (\alpha^{(r)}, \beta^{(r)}, \theta^{(r)})$, compute:

$$P(N_i = n \mid x_i; \boldsymbol{\varphi}^{(r)}) = \frac{a_n\theta^{(r)n} f_{X_{(k)}}(x_i; n, \alpha^{(r)}, \beta^{(r)})}{\sum_{m=k}^{\infty} a_m\theta^{(r)m} f_{X_{(k)}}(x_i; m, \alpha^{(r)}, \beta^{(r)})}, \quad n \geq k$$

Define the conditional expectations:

$$O_i^{(r)} = E(N_i \mid x_i; \boldsymbol{\varphi}^{(r)}) = \sum_{n=k}^{\infty} nP(N_i = n \mid x_i; \boldsymbol{\varphi}^{(r)}),$$

and

$$T_i^{(r)} = \sum_{n=k}^{\infty}(n-k+1)P(N_i = n \mid x_i; \boldsymbol{\varphi}^{(r)}).$$

The expected complete-data log-likelihood is:

$$Q(\boldsymbol{\varphi} \mid \boldsymbol{\varphi}^{(r)}) = \sum_{i=1}^{n}\sum_{n=k}^{\infty} P(N_i = n \mid x_i; \boldsymbol{\varphi}^{(r)})\left[\log a_n + n\log\theta + \log f_{X_{(k)}}(x_i; n, \alpha, \beta)\right] - n\log C_k^L(\theta).$$

M-step. Maximize $Q(\boldsymbol{\varphi} \mid \boldsymbol{\varphi}^{(r)})$ with respect to $\boldsymbol{\varphi}$.

(a) Update for θ:

$$Q_\theta(\theta) = S^{(r)}\log\theta - n\log C_k^L(\theta) + \text{constant}, \quad S^{(r)} = \sum_{i=1}^{n} O_i^{(r)}.$$

Solve:

$$\frac{\partial Q_\theta}{\partial\theta} = \frac{S^{(r)}}{\theta} - n\frac{C_k^L(\theta)'}{C_k^L(\theta)} = 0.$$

(b) Updates for α and β:

$$\log f_{X_{(k)}}(x_i; n, \alpha, \beta) = \log \frac{n!}{(k-1)!(n-k)!} + (k-1)\log(1 - e^{-(\beta x_i)^\alpha})$$
$$- (n-k+1)(\beta x_i)^\alpha + \log\alpha + \alpha\log\beta + (\alpha-1)\log x_i.$$

$$Q_{\alpha,\beta}(\alpha,\beta) = \sum_{i=1}^{n}\sum_{n=k}^{\infty} P(N_i = n \mid x_i; \boldsymbol{\varphi}^{(r)})\Big[\log\frac{n!}{(k-1)!(n-k)!}$$
$$+ (k-1)\log(1 - e^{-(\beta x_i)^\alpha}) - (n-k+1)(\beta x_i)^\alpha + \log\alpha + \alpha\log\beta + (\alpha-1)\log x_i\Big].$$

$$Q_{\alpha,\beta}(\alpha,\beta) = \sum_{i=1}^{n}\Big[(k-1)\log(1 - e^{-(\beta x_i)^\alpha}) - T_i^{(r)}(\beta x_i)^\alpha + \log\alpha + \alpha\log\beta + (\alpha-1)\log x_i\Big] + \text{constant},$$

where the constant includes $\log\frac{n!}{(k-1)!(n-k)!}$ terms. Maximize by taking partial derivatives:

$$\frac{\partial Q_{\alpha,\beta}}{\partial\beta} = \sum_{i=1}^{n}\left[-\frac{(k-1)\alpha\beta^{\alpha-1}x_i^\alpha e^{-(\beta x_i)^\alpha}}{1 - e^{-(\beta x_i)^\alpha}} - T_i^{(r)}\alpha\beta^{\alpha-1}x_i^\alpha + \frac{\alpha}{\beta}\right] = 0.$$

$$\beta^{(r+1)} = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha^{(r+1)}}\left(T_i^{(r)} - \frac{(k-1)e^{-(\beta^{(r+1)}x_i)^{\alpha^{(r+1)}}}}{1 - e^{-(\beta^{(r+1)}x_i)^{\alpha^{(r+1)}}}}\right)\right]^{-1/\alpha^{(r+1)}}.$$

$$\beta^{(r+1)} = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{\alpha^{(r+1)}}\left\{T_i^{(r)} - \frac{(k-1)a_i^{(r+1)}}{1 - a_i^{(r+1)}}\right\}\right]^{-1/\alpha^{(r+1)}}.$$

For α, the expression is complex and typically solved numerically or iteratively with β, often requiring the previous iteration's values. Using $T_i^{(r)}$, updates are:

$$\alpha^{(r+1)} = \left[\sum_{i=1}^{n}\ln(b_i^{(r+1)})\left\{(T_i^{(r)}+1)(b_i^{(r+1)})^{\alpha^{(r+1)}} - \frac{(k-1)(b_i^{(r+1)})^{\alpha^{(r+1)}}a_i^{(r+1)}}{1 - a_i^{(r+1)}} - 1\right\}\right]^{-1},$$

where $a_i^{(r+1)} = e^{-(\beta^{(r+1)}x_i)^{\alpha^{(r+1)}}}$, $b_i^{(r+1)} = \beta^{(r+1)}x_i$.

5. Conclusion

This paper presents a novel generalization of the Weibull power series family by compounding k-th order statistics and a left truncated power series distribution, significantly enhancing flexibility for modeling survival, reliability, and lifetime data. The proposed new family of distributions captures ordered failure mechanisms in multi-component systems while accounting for left-truncated sample sizes common in real-world applications. By unifying these concepts, our framework extends several classical lifetime distributions and provides a robust tool for complex data analysis in engineering, finance, and medical research. Future work should explore Bayesian estimation methods, model selection criteria, and applications to competing risks and extreme value analysis, further broadening the model's practical utility.

This research bridges theoretical advancements with practical needs, offering a versatile approach for analyzing lifetime data where traditional distributions fall short. The integration of order statistics and truncated count distributions opens new possibilities for reliability modeling, risk assessment, and beyond.

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