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# ANALYTIC SOLUTIONS OF SPECIAL FUNCTIONAL EQUATIONS

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ABSTRACT. We recall some of our earlier results on the construction of a mapping defined implicitly, without using the implicit function theorem. All these considerations work in the real case, for functions and operators. Then we consider the complex case, proving the analyticity of the function defined implicitly, under certain hypothesis. Some consequences are given. An approximating formula for the analytic form of the solution is also given. Finally, one illustrates the preceding results by an application to a concrete functional and operatorial equation. Some related examples are given.

## 1 Introduction

The equation

$$g = g \circ f,$$

where $g$ is given, while $f$ is the unknown function, always has the trivial solution

$$f(x) = x, \quad \forall x \in D,$$

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where $D$ is the domain of definition for $f$.

When $g$ is firstly decreasing and then increasing (or $-g$ firstly increases and then decreases), there exists exactly one decreasing nontrivial solution $f$, with the qualities stated in theorem 2.1 from below. Such functions $-g$ appear as probability densities associated to some random variables. These equations appeared firstly in [5]. For concrete functions $g$, one obtains special qualities of the corresponding solutions $f$.

The present approach allows the construction of the solutions of such functional and operatorial equations, without using the implicit function theorem. In the operatorial case the solution $F$ is a function of $U \in D \subset X$, where $X$ is the commutative algebra of selfadjoint operators (2). We essentially use the fact that $X$ is also an order-complete vector lattice, with respect to the natural order relation on the real vector space $A(H)$ of all selfadjoint operators acting on $H$.

This work continues theorems published in [4]-[7] and contains new results too. The background is partially contained in [1], [2], [3], [8]. In Section 2 we recall some known results on the subject, especially related to the real case. Section 3 contains the proof of the analyticity of the solution and some related consequences. An approximating explicit formula is also given. In Section 4, we apply the theoretical results to a concrete functional equation and to the corresponding operatorial equation.

## 2. General known results

**Theorem 2.1.** (see also [4]-[7]). *Let* $u, v \in \overline{\mathbb{R}}$, $u < v$, $\alpha \in ]u, v[$ *and let* $g : ]u, v[ \to \mathbb{R}$ *be a continuous function. Assume that*

(a) $\lim_{x \downarrow u} g(x) = \lim_{x \uparrow v} g(x) = w \in \overline{\mathbb{R}}$,

(b) *$g$ is strictly decreasing on* $]u, \alpha[$ *and strictly increasing on* $[\alpha, v[$.

*Then there exists* $f : ]u, v[ \to ]u, v[$ *such that*

$$g(x) = g(f(x)), \quad \forall\, x \in ]u, v[$$

*and f has the following qualities:*

(i) *f is strictly decreasing on* $]u, v[$ *and we have*

$$\lim_{x \downarrow u} f(x) = v, \quad \lim_{x \uparrow v} f(x) = u;$$

(ii) $\alpha$ *is the unique fixed point of f;*

(iii) *we have* $f^{-1} = f$ *on* $]u, v[$;

(iv) *f is continuous on* $]u, v[$;

(v) *if* $g \in C^n(]u, v[ \setminus \{\alpha\})$, $n \in \mathrm{N} \cup \{\infty\}$, $n \geq 1$, *then* $f \in C^n(]u, v[ \setminus \{\alpha\})$;

(vi) *if g is derivable on* $]u, v[ \setminus \{\alpha\}$, *so is f;*

(vii) *if* $g \in C^2(]u, v[)$, $g''(\alpha) \neq 0$ *and there exists* $\rho_1 := \lim_{x \to \alpha} f'(x) \in \overline{\mathbf{R}}$ *then* $f \in C^1(]u, v[) \cap C^2(]u, v[ \setminus \{\alpha\})$ *and* $f'(\alpha) = -1$;

(viii) *if* $g \in C^3(]u, v[)$, $g''(\alpha) \neq 0$ *and there exist* $\rho_1 := \lim_{x \to \alpha} f'(x) \in \overline{\mathbf{R}}$ *and* $\rho_2 := \lim_{x \to \alpha} f''(x) \in R$, *then* $f \in C^2(]u, v[) \cap C^3(]u, v[ \setminus \{\alpha\})$ *and*

$$f''(\alpha) = \rho_2 = -\frac{2}{3} \cdot \frac{g'''(\alpha)}{g''(\alpha)};$$

(ix) *if* $g$ *is analytic at* $\alpha$, *then* $f$ *is derivable at* $\alpha$ *and* $f'(\alpha) = -1$;

(x) *let* $g_l := g|_{]u, \alpha[}$, $g_r := g|_{]\alpha, v[}$; *then we have*

$$f(x_0) = (g_r^{-1} \circ g_l)(x_0) = \sup\{\, x \in [\alpha, v[\,;\, g_r(x) \leq g_l(x_0)\} \;\; \forall\, x_0 \in ]u, \alpha]$$

*and*

$$f(x_0) = (g_l^{-1} \circ g_r)(x_0) = \inf\{\, x \in ]u, \alpha]\,;\, g_l(x) \leq g_r(x_0)\} \;\; \forall\, x_0 \in [\alpha, v[$$

The proof of this theorem is similar to that of Theorem 1.1. [5] (see Figure 1 [5], p. 62).

We recall the geometric meaning of the construction of $f$. If $x \in ]u, v[ \setminus \{\alpha\}$, consider the horizontal passing through the point $(x, g(x))$. Thanks to the qualities of $g$, this straight line intersect once again the graph of $g$ at exactly one point

$$(x_1, g(x_1)) = (x_1, g(x)),\; x_1 \neq x.$$

We define

$$f(x) = x_1.$$

Then we have

$$g(x) = g(x_1) = g(f(x)),\ x \in ]u, v[ \setminus \{\alpha\},\ f(\alpha) := \alpha.$$

When $x$ runs over the interval $]u, \alpha],$ $f(x)$ runs over the interval $[\alpha, v[,$ in the decreasing sense, from $v$ to $\alpha$. When $x$ runs over the interval $[\alpha, v[$ (in the increasing sense), $f(x)$ runs over the interval $]u, \alpha],$ in the decreasing sense, from $\alpha$ to $u$.

Let $\alpha \in ]0, \infty[$ and denote by $G$ the set of all continuous functions $g : ]0, \infty[ \to ]0, \infty[,$ $g(\alpha) = 0,$ which are decreasing on $]0, \alpha]$ and increasing on $[\alpha, \infty[,$ such that

$$\lim_{x \downarrow 0} g(x) = \lim_{x \uparrow \infty} g(x) = \infty.$$

For $g, h \in G,$ an interesting problem is the following one: find necessary and sufficient conditions on $g, h$ for the equality:

$$f_g = f_h,$$

where $f_g, f_h$ are the corresponding functions attached to $g,$ respectively to $h$ by Theorem 2.1. The following statement is giving the answer.

**Theorem 2.2.** (Theorem 1.4 [5]). *Let* $g, h \in G,\ \lambda \in ]0, \infty[.$ *Then* $g + h,\ \lambda g,\ g \cdot h$ *are also elements of* $G$ *and the following statements are equivalent*

(a) $f_g = f_h;$

(b) $h_r \circ g_r^{-1} = h_l \circ g_l^{-1},\ g_l := g|_{]0,\alpha]},\ g_r := g|_{[\alpha,\infty[},\quad g \in G;$

(c) *there exists* $\varphi : [0, \infty[ \to [0, \infty[$ *such that* $\varphi(0) = 0,$ $\varphi$ *is continuous and increasing, verifying the relation*

$$h = \varphi \circ g.$$

Next we consider the abstract operatorial version of Theorem 2.1. In the sequel, $X$ will be an order-complete vector lattice, and $Izom_+(X)$ will be the set of all vector space isomorphisms $T : X \to X$ which apply $X_+$ onto itself.

**Theorem 2.3.** *Let $X$ be an order-complete vector lattice, $X_+$ its positive cone, $\alpha \in X$, $D_l$ a convex subset such that*

$$\alpha \in D_l \subset \{x \in X; \quad x \leq \alpha\};$$

*$D_r$ a convex subset such that*

$$\alpha \in D_r \subset \{x \in X; \quad x \geq \alpha\};$$

*Let $g_l : D_l \to X$ be a convex operator such that*

$$\partial g_l(x) \cap (-Izom_+(X)) \neq \Phi, \quad \forall x \in D_l \setminus \{\alpha\}.$$

*Let $g_r : D_r \to X$ be a convex operator such that*

$$\partial g_r(x) \cap (Izom_+(X)) \neq \Phi, \quad \forall x \in D_r \setminus \{\alpha\}.$$

We also assume that

$$g_l(\alpha) = g_r(\alpha) \text{ and } R(g_l) = R(g_r),$$

*where $R(g)$ is the range of $g$, while $\partial g(x)$ is the set of all subgradients of $g$ at $x$. Let*

$$g : D = D_l \cup D_r \to X,$$
$$g(x) = g_l(x)\ \forall x \in D_l, \quad g(x) = g_r(x)\ \forall x \in D_r.$$

*Then there exists $F : D \to D$ such that*

$$g(x) = g(F(x)), \quad \forall x \in D$$

*$F$ is strictly decreasing in $D$ and it has the following properties:*

(i) $\alpha$ *is the only fixed point of F;*
(ii)*there exists $F^{-1}$ and $F^{-1} = F$ on $D$;*
(iii) we have

$$F(x_0) = g_r^{-1}(g_l(x_0)) = \sup\{ x \in D_r ; g_r(x) \leq g_l(x_0)\} \ \forall x_0 \in D_l,$$

$$F(x_0) = g_l^{-1}(g_r(x_0)) = \inf\{ x \in D_l ; g_l(x) \leq g_r(x_0)\} \ \forall x_0 \in D_r.$$

The proof of this theorem is similar to that from [5].

## 3. On the analyticity of the solution. The complex case

Application of the complex form of the implicit function theorem for a holomorphic function $\tilde{g}$, that is the extension of the real function $g$ of Theorem 2.1, might be difficult around $\alpha$. Namely, considering the equation

$$H(z,w) = \tilde{g}(z) - \tilde{g}(w) = 0,$$

we have

$$\frac{\partial H}{\partial w}(\alpha,\alpha) = 0, \ \lim_{z \to \alpha} \frac{\tilde{g}'(z)}{\tilde{g}'(w(z))} = \frac{0}{0} = \lim_{z \to \alpha} w'(z).$$

Note that around points from $]u,v[\setminus\{\alpha\}$, application of the implicit function theorem leads to the analyticity of $f$ at such points. Therefore, we only have to study the analyticity at $\alpha$.

**Theorem 3.1.** *Let* $\tilde{g}$ *be the extension of the function* $g$ *from Theorem 2.1, such that* $\tilde{g}$ *is holomorphic in a complex neighborhood of* $]u,v[$. *Then there is a unique holomorphic solution* $\tilde{f}$ *of the equation* $\tilde{g} = \tilde{g} \circ \tilde{f}$, *such that* $\tilde{f}$ *is the extension of* $f$ *from Theorem* 2.1 *to a complex neighborhood of* $]u,v[$.

**Proof.** From the preceding remarks, we have to prove the analyticity of $\tilde{f}$ at $\alpha$. To this end, let us write the following expansions:

$$\tilde{g}(z)-\tilde{g}(\alpha)=(z-\alpha)^{2k}\left[\frac{\tilde{g}^{2k}(\alpha)}{(2k)!}+\frac{\tilde{g}^{2k+1}(\alpha)}{(2k+1)!}(z-\alpha)+0((z-\alpha))\right],$$

$$\tilde{g}(f(z))-\tilde{g}(\alpha)=(f(z)-\alpha)^{2k}\left[\frac{\tilde{g}^{2k}(\alpha)}{(2k)!}+\frac{\tilde{g}^{2k+1}(\alpha)}{(2k+1)!}(f(z)-\alpha)+0((f(z)-\alpha))\right].$$

Here $m=2k$ is the smallest natural number, which for the derivative of order $m$ of $\tilde{g}$ at $\alpha$ is not vanishing. By Taylor formula, it must be an even number, since $\alpha$ is a minimum point for $g$. On the other hand, since $\tilde{g}=\tilde{g}\circ\tilde{f}$, elementary computations and the preceding expansions lead to

$$\lim_{z\to\alpha} f(z)=\alpha\,(=f(\alpha)).$$

Further computations yield

$$\frac{\tilde{f}(z)-\tilde{f}(\alpha)}{z-\alpha}=\left(\frac{\tilde{g}^{(2k)}(\alpha)/(2k)!+\varphi(z)(z-\alpha)}{\tilde{g}^{(2k)}(\alpha)/(2k)!+\varphi\left(\tilde{f}(z)\right)\left(\tilde{f}(z)-\alpha\right)}\right)^{1/2k},$$

where $\varphi$ is holomorphic around $\alpha$. It follows that $\tilde{f}'(\alpha)$ is a $2k$-order root of the unity. Using the fact that $\tilde{f}$ applies intervals of the real line into the real line and that it is decreasing, we deduce that $\tilde{f}'(\alpha)=f'(\alpha)=-1$. This concludes the proof. □

In the sequel, for a complex neighborhood $V$ of $\alpha$ we denote:

$$V_l^+ = \{z \in V ; \operatorname{Re} z < \alpha , \operatorname{Im} z > 0\}, V_l^- = \{z \in V ; \operatorname{Re} z < \alpha , \operatorname{Im} z < 0\},$$

$$V_r^+ = \{z \in V ; \operatorname{Re} z > \alpha , \operatorname{Im} z > 0\}, V_r^- = \{z \in V ; \operatorname{Re} z > \alpha , \operatorname{Im} z < 0\}.$$

**Corollary 3.1.** *There is a neighborhood* $V$ *of* $\alpha$ *such that*

$$\tilde{f} : V \to \tilde{f}(V)$$

*is a one to one mapping and we have*

$$\tilde{f} \circ \tilde{f} = id , \ \tilde{f}\left(V_l^+\right) = (f(V))_r^- , \ \tilde{f}\left(V_l^-\right) = (f(V))_r^+ , \ \tilde{f}\left(V_r^+\right) = (f(V))_l^- , \ \tilde{f}\left(V_r^-\right) = (f(V))_l^+ .$$

**Proof.** The first two assertions follow by the local inversion theorem and respectively from the analytic continuation principle ($\tilde{f}$ is a holomorphic extension of $f$). The last four relations are consequences of the fact that $\tilde{f}$ is conformal at $x_0 = \alpha$, also using the qualities of $f$ (see the comments following Theorem 2.1). The conclusions follow. □

**Corollary 3.2.** *There is a complex neighborhood* $W$ *of* $]u, v[$ *such that* $\tilde{f}$ *is holomorphic on* $W$.

**Corollary 3.3.** *The function* $\frac{1}{z-\alpha} + (f(z) - \alpha)$ *is univalent in the open disc* $|z-\alpha| < 1$ *if and only if* $f(z) = -z + 2\alpha$

**Proof.** For the only if part, assume that

$$\frac{1}{z-\alpha} + (f(z) - \alpha) = \frac{1}{z-\alpha} - (z-\alpha) + \frac{\tilde{f}''(\alpha)}{2}(z-\alpha)^2 + \cdots$$

is univalent in the open disc $|z-\alpha| < 1$. Then by the area theorem 14.13 [8], we should have

$$\sum_{n=1}^{\infty} n \cdot |a_n|^2 \leq 1,$$

where $a_n, n \geq 1$ are the coefficients of the holomorphic part of the preceding expansion. Since $a_1 = -1,$ it follows that all the other $a_n, n \geq 2$ are vanishing, so that:

$$f(z) - \alpha = -z + \alpha, \; f(z) = -z + 2\alpha.$$

Conversely, if this last relation is verified, then a straightforward computation shows that the function

$$\frac{1}{z-\alpha} + (f(z) - \alpha)$$

is univalent in the unit open disc centered at $\alpha$. This concludes the proof.
□

*Remark 3.1.* The function $f(z) = -z + \alpha$ is an extreme point of the convex set of all holomorphic functions with real coefficients $\psi(z) = \sum_{n=1}^{\infty} a_n (z-\alpha)^n, \sum_{n=1}^{\infty} n \cdot a_n^2 \leq 1.$ It is also an extreme point of the convex subset of all functions

$$h(z) = \sum_{n=1}^{\infty} a_n (z-\alpha)^n, \; a_1^2 + \sum_{n=2}^{\infty} p_n |a_n|^{m_n} \leq 1, \; p_n \geq 0, \; m_n > 1, \; \forall n \geq 2,$$

where $p_n, m_n$ are given numbers with the properties from above.

**Theorem 3.2.** *In a small neighborhood of* $\alpha$, *we have*:

$$\tilde{f}(z) \approx z - 2\frac{\tilde{g}'(z)}{\tilde{g}''(z)}.$$

**Proof.** The following relations hold true:

$$0 = \tilde{g}(f(z)) - \tilde{g}(z) = \tilde{g}'(z)(f(z) - z) + \frac{\tilde{g}''(z)}{2}(f(z) - z)^2 + 0\left((f(z) - z)^2\right).$$

Dividing by $f(z) - z \neq 0$ and neglecting the nonlinear remained terms in $f(z) - z$, the conclusion follows.
□

**Corollary 3.4.** *For* $\tilde{g}$ *as above and any linear bounded operator* $U$ *acting on a Hilbert space* $H$, *with spectrum* $\sigma(U)$ *in a small neighborhood of* $\alpha$, *there is a holomorphic function* $\tilde{f}$ *such that*

$$\tilde{g}\left(\tilde{f}(U)\right) = \tilde{g}(U),\ \tilde{f}\left(\tilde{f}(U)\right) = U,$$
$$\sigma(U) \subset \{\operatorname{Im} z \geq 0\} \Leftrightarrow \sigma\left(\tilde{f}(U)\right) \subset \{\operatorname{Im} z \leq 0\},$$
$$\tilde{f}(U) \approx U - 2\cdot\tilde{g}'(U)\cdot[\tilde{g}''(U)]^{-1},$$

*for all such operators* $U$.

**Proof.** Part of the relations follows by analytic functional calculus. For the third relation, one applies Corollary 3.1. For the last relation one uses Theorem 3.2. □

## 4. Examples and applications

We consider the functional equation

$$x^{a}\exp(-\beta\cdot x)=(f(x))^{a}\exp(-\beta\cdot f(x)),\ x>0,\ a>0,\ \beta>0.$$

This equation is equivalent to the following one:

$$x\exp(-bx)=f(x)\exp(-bf(x)),\ x>0,b:=\beta/a>0. \tag{1}$$

**Theorem 4.1.** *There exists a unique decreasing solution* $f:]0,\infty[\to]0,\infty[$ *of the equation* (1) *and this solution has the following properties:*

(i) $f(0+)=\infty-,\ f(\infty-)=0+;$

(ii) $\alpha=1/b$ *is the unique fixed point of* $f;$

(iii) $f^{-1}=f$ *on* $]0,\infty[;$

(iv) *the following constructive formulae for* $f(x)$ *hold*

$$f(x_0)=\sup\{x\in[1/b,\infty[;\ x\exp(-bx)\ge x_0\cdot\exp(-bx_0)\},\ \forall x_0\in]0,1/b],$$

$$f(x_0)=\inf\{x\in]0,1/b];\ x\exp(-bx)\ge x_0\cdot\exp(-bx_0)\},\ \forall x_0\in[1/b,\infty[;$$

(v) $f$ *is the restriction of a holomorphic function* $\tilde{f}$ *on a complex neighborhood of* $]0,\infty[$, *such that* $\tilde{f}'(1/b)=-1,\ \tilde{f}\circ\tilde{f}=id$, *and* $\tilde{f}$ *has the qualities mentioned in Corollary* 3.1*;*

(vi) *in a small neighborhood of* $1/b$, *we have*: $\tilde{f}(z)\approx z+\frac{2}{b}\cdot\frac{1-bz}{2-bz}.$

**Proof.** The function $g(x)=-x\cdot\exp(-bx)$ decreases from $0$ to $g(1/b)=-e^{-1}b^{-1}$ in the interval $\left[0,b^{-1}\right]$, and increases from $g\left(b^{-1}\right)$ to $0-$ in the interval $[b^{-1},\infty[$. Hence, the conclusions (i)-(iv) follow from theorem 2.1. The function $g$ is the restriction of a holomorphic function $\tilde{g}$, with

$$\tilde{g}'\left(b^{-1}\right)=0,\ \tilde{g}^{(n)}(z)=(-1)^{n}b^{n-1}\exp(-bz)(n-bz),\ n\in\mathrm{N},\ n\ge1,\ z\in C.$$

In particular, $\tilde{g}''\left(b^{-1}\right)\ne0$, so that we can apply Corollary 3.1, that leads to the conclusion (v) of the present statement. The assertion (vi) follows from Theorem 3.2. This concludes the proof.

□

Let $g$ be the function from the proof of Theorem 4.1. Then there exists appropriate intervals $]u,v[$, containing $b^{-1}$, such that $g$ is convex on $]u,v[$, $g(u)=g(v)$. The convexity is required in theorem 2.3 in order to deduce the existence of a subgradient, which allow the construction of the solution. Next we apply theorem 2.3 to the operatorial equation corresponding to (1). The case of arbitrary linear bounded operators follows from corollary 3.4, for $\tilde{g}(z)=-z\cdot\exp(-bz)$, $\operatorname{Re} z>0$, $\alpha=b^{-1}$, $\tilde{f}$ being the holomorphic extension of $f$ from theorem 4.1 to a complex neighborhood of $]0,\infty[$ (see also theorem 3.1). Now we consider the case of the associated operator equation in a commutative algebra of selfadjoint operators $X=X(A)$, $A\in \mathrm{A}(H)$ being a fixed selfadjoint operator acting of the Hilbert space $H$. We define;

$$X_1=\{U\in \mathrm{A}(H);UA=AU\},\ X=X(A)=\{V\in X_1;VU=UV,\ \forall U\in X_1\},$$
$$X_+=\{V\in X;\langle Vh,h\rangle\geq 0\ \forall h\in H\}. \tag{2}$$

It is known [2] that $X$ is an order complete Banach lattice and a commutative algebra. We denote:

$$D_l=\{V\in X;\sigma(V)\subset[u,1/b[\}\cup\{b^{-1}I\},$$
$$D_r=\{V\in X;\sigma(V)\subset[1/b,v]\}\cup\{b^{-1}I\},\ D=D_l\cup D_r,$$

where $\sigma(V)$ is the spectrum of $V$.

**Theorem 4.2.** There exists a decreasing mapping $F:D\to D$, such that

$$U\exp(-bU)=F(U)\exp(-bF(U)),\ U\in D.$$

This mapping has the following properties:

(i) $b^{-1}I$ is the unique fixed point of $F$;

(ii) $F$ is invertible and $F^{-1}=F$ on $D$;

(iii) $F$ applies $D_l$ onto $D_r$ and $D_r$ onto $D_l$;

(iv) the following constructive formulae for $F$ hold:

$$F(U_0) = \sup\{U \in D_r; U\exp(-bU) \geq U_0\exp(-bU_0)\}, \forall U_0 \in D_l,$$

$$F(U_0) = \inf\{U \in D_l; U\exp(-bU) \geq U_0\exp(-bU_0)\}, \forall U_0 \in D_r;$$

(v) for $U \in A$, with the spectrum in a small neighborhood of $b^{-1}$, we have:

$$F(U) \approx U + (2b^{-1})(I - bU)(2I - bU)^{-1}.$$

**Proof.** We have to verify the conditions from the statement of theorem 2.3. To prove the convexity of $g(U) = -U\exp(-bU)$ on the convex subsets $D_l, D_r$, we use the convexity of the scalar function $g$ on $]u, v[$ and the positivity of the spectral measures attached to the elements $U_1, U_2 \in D_l$, respectively $U_j \in D_r$, $j = 1,2$. The fact that $U_1, U_2$ are commuting operators is essential. One can use Fubini's theorem. Namely, for $t \in [0,1]$ we have

$$g((1-t)U_1 + tU_2) = \iint_{\sigma(U_1)\times\sigma(U_2)} g((1-t)x_1 + tx_2)dE_1 dE_2 \leq$$

$$(1-t)\iint_{\sigma(U_1)\times\sigma(U_2)} g(x_1)dE_1 dE_2 + t\iint_{\sigma(U_1)\times\sigma(U_2)} g(x_2)dE_1 dE_2 =$$

$$(1-t)g(U_1) + tg(U_2).$$

Now the convexity on the subsets $D_l, D_r$ is proved. On the other hand, for $U \in D_l \setminus \{b^{-1}I\}$, we have:

$$g_l'(U) = \exp(-bU)(bU - I) < 0$$

as a product of two commuting selfadjoint operators, the first one being positive and the second one being negative. In the same way, one shows that

$$g_r'(U) > 0, \forall U \in D_r \setminus \{b^{-1}I\}.$$

It remains to prove that the ranges of the ranges of the two operators $g_l, g_r$ coincide, and also the assertions (iii) and (v) of the present statement. Let

$g_l(U_1) \in r(g_l), U_2 := F(U_1),$ where $F$ is associated to $f$ by means of functional calculus. Using the qualities of $f$, we obtain:

$$\sigma(U_2) = \sigma(F(U_1)) = f(\sigma(U_1)) \subset [1/b, v] \Rightarrow U_2 \in D_r.$$

From the equality $g_l(t) = g_r(f(t)), t \in [u, 1/b] \supset \sigma(U_1),$ by integration with respect to the spectral measure attached to $U_1$, one obtains

$$g_l(U_1) = \int_{\sigma(U_1)} g_l(t) dE_{U_1} = \int_{\sigma(U_1)} g_r(f(t)) dE_{U_1} = g_r(f(U_1)) = g_r(U_2) \in R(g_r).$$

Similarly, one proves that $r(g_r) \subset r(g_l)$. Now all the conclusion (except (v)), follow by application of theorem 2.3. The assertion (v) is a consequence of (vi), theorem 4.1. This concludes the proof. □

We next give some examples for which the exact analytic expression of the solution $f$ can be determined explicitly.

**Theorem 4.3.** *The nontrivial solution of the equation*

$$e^{-x}\left(1 - e^{-x}\right) = e^{-f(x)}\left(1 - e^{-f(x)}\right), \ x > 0,$$

*is given by*

$$f(x) = -\ln\left(1 - e^{-x}\right), \ x > 0.$$

*Its holomorphic extension is* $\tilde{f}(z) = -\ln\left(1 - e^{-z}\right), \ \mathrm{Re}\ z > 0.$

**Proof.** One can prove easily that the function

$$g(x) = -e^{-x}\left(1 - e^{-x}\right), \ x \in ]0, \infty[$$

verifies the conditions of theorem 2.1, where $\alpha = \ln 2$. To find the analytic expression of $f$, we rewrite our functional equation as

$$e^{-f(x)} - e^{-x} = e^{-2f(x)} - e^{-2x} = \left(e^{-f(x)} - e^{-x}\right)\left(e^{-f(x)} + e^{-x}\right).$$

Dividing by

$$e^{-f(x)} - e^{-x} \neq 0,\ x \neq \ln 2,$$

and doing some straightforward computations, one obtains the result. This concludes the proof.
□

*Example 4.1.* The unique nontrivial solution $\tilde{f}$ of the equation

$$\tilde{g}(z) = \tilde{g}\left(\tilde{f}(z)\right),\ |z| < 1,\ \tilde{g}(z) = \left(\frac{z}{2-z}\right)^2$$

is given by $\tilde{f}(z) = \dfrac{z}{z-1}$.

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