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FRACTIONAL DIFFERENTIAL EQUATIONS AND INCLUSIONS WITH NONLOCAL GENERALIZED RIEMANN-LIOUVILLE INTEGRAL BOUNDARY CONDITIONS

BASHIR AHMAD[1,*], SOTIRIS K. NTOUYAS[1,2] AND AHMED ALSAEDI[1]

Abstract. In this paper, we study a new kind of nonlocal boundary value problems of nonlinear fractional differential equations and inclusions supplemented with nonlocal and generalized Riemann-Liouville fractional integral boundary conditions. In case of single valued maps (equations), we make use of contraction mapping principle, fixed point theorem due to Sadovski, Krasnoselskii-Schaefer fixed point theorem due to Burton and Kirk, and fixed point theorem due to O'Regan to obtain the desired existence results. On the other hand, the existence results for inclusion case are based on Krasnoselskii's fixed point theorem for multivalued maps and nonlinear alternative for contractive maps. Examples illustrating the main results are also constructed.

1. Introduction

Fractional order differential and integral operators play an important role in the mathematical modeling of several real world problems. It has been mainly due to the fact that such operators can describe the memory and hereditary properties of various materials and processes involved in the problem at hand. Examples include physics, chemical technology, population dynamics, biotechnology, and economics [1–3]. In recent years, the study of initial and boundary value problems of fractional differential equations involving a variety of conditions have been investigated by several researchers, and the literature on the topic is now much enriched. For examples and details, see [4]- [18] and the references cited therein.

Nonlocal conditions are found to be more plausible than the standard initial conditions for the formulation of some physical phenomena in certain problems of thermodynamics, elasticity and wave propagation. As a matter of fact, such conditions become inevitable in a situation when the controllers at the boundary positions dissipate or add energy according to censors located at intermediate positions. Further details can be found in the work by Byszewski [19, 20].

Integral boundary conditions also find decent applications in blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. In particular, the assumption of 'circular cross-section' throughout the vessels in the study of fluid flow problems is not always justifiable. In this situation, integral boundary conditions provide a more realistic (practical) approach. Also, integral boundary conditions are found to be useful in regularizing ill-posed parabolic backward problems in time partial differential equations, see for example, mathematical models for bacterial self-regularization [21]. Integral boundary conditions involve classical, Riemann-Liouville or Hadamard or Erdélyi-Kober type integral operators. In [22], it has been discussed that Riemann-Liouville and Hadamard fractional integrals can jointly be represented by a single integral, which is called generalized Riemann-Liouville fractional integral.

In this paper, we introduce a new class of boundary value problems of fractional differential equations and inclusions supplemented with nonlocal and generalized Riemann-Liouville fractional integral

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boundary conditions. In precise terms, we consider the following nonlocal problems:

$$\begin{cases} D^\alpha x(t) = f(t, x(t)), \quad t \in [0, T], \\ x(0) = g(x), \\ x(T) = \beta \dfrac{\rho^{1-q}}{\Gamma(q)} \displaystyle\int_0^\xi \frac{s^{\rho-1}}{(\xi^\rho - s^\rho)^{1-q}} x(s) ds := \beta\ {}^\rho I^q x(\xi), \quad 0 < \xi < T, \end{cases} \tag{1.1}$$

and

$$\begin{cases} D^\alpha x(t) \in F(t, x(t)), \quad t \in [0, T], \\ x(0) = g(x), \quad x(T) = \beta\ {}^\rho I^q x(\xi), \quad 0 < \xi < T, \end{cases} \tag{1.2}$$

where D^α is the Caputo fractional derivative of order $1 < \alpha \leq 2$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $g : C([0, T], \mathbb{R}) \to \mathbb{R}$ is a continuous function, ${}^\rho I^q$ is the generalized Riemann-Liouville fractional integral of order $q > 0$, $\rho > 0$ (see Definition 2.4) and $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi valued function, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subjects of $\mathbb{R}$).

We remark that $g(x)$ in (1.1) and (1.2) may be represented as $g(x) = \sum_{j=1}^p \alpha_j x(t_j)$, where $\alpha_j, j = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leq 1$.

The rest of the paper is organized as follows: In Section 2 we present some useful preliminaries and lemmas. Section 3 deals with the existence and uniqueness results for problem (1.1) which are established via contraction mapping principle and fixed point theorems due to Sadovski, a Krasnoselskii-Schaefer fixed point theorem due to Burton and Kirk, and a fixed point theorem due O'Regan. In Section 4, we discuss the existence of solutions for problem (1.2) by means of Krasnoselskii fixed point theorem for multivalued maps and nonlinear alternative for contractive maps. Examples illustrating the main work are also presented.

2. Preliminaries

In this section, we recall some basic concepts of fractional calculus [1,2] and present known results needed in our forthcoming analysis.

Definition 2.1. *The Riemann-Liouville fractional integral of order $q > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is defined by*

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Definition 2.2. *The Riemann-Liouville fractional derivative of order $q > 0$, $n - 1 < q < n$, $n \in \mathbb{N}$, is defined as*

$$D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} f(s) ds,$$

where the function $f(t)$ has absolutely continuous derivative up to order $(n-1)$.

Definition 2.3. *The Caputo derivative of order q for a function $f : [0, \infty) \to \mathbb{R}$ can be written as*

$${}^c D^q f(t) = D_{0+}^q \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right), \quad t > 0, \quad n - 1 < q < n.$$

Remark 2.1. *If $f(t) \in C^n[0, \infty)$, then*

$${}^c D^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{q+1-n}} ds = I^{n-q} f^{(n)}(t), \ t > 0, \ n - 1 < q < n.$$

Definition 2.4. *[22] The generalized Riemann-Liouville fractional integral of order $q > 0$ and $\rho > 0$, of a function $f(t)$, for all $0 < t < \infty$, is defined as*

$${}^\rho I^q f(t) = \frac{\rho^{1-q}}{\Gamma(q)} \int_0^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-q}} ds,$$

provided the right-hand side is point-wise defined on $(0, \infty)$.

Remark 2.2. *We notice that the above definition corresponds to the one for Riemann-Liouville fractional integral of order $q > 0$ when $\rho = 1$, while the famous Hadamard fractional integral follows for $\rho \to 0$, that is,*

$$\lim_{\rho \to 0} {}^{\rho}I^{q} f(t) = \frac{1}{\Gamma(q)} \int_0^t \left(\log \frac{t}{s} \right)^{q-1} \frac{f(s)}{s} ds.$$

The following lemma is obvious via Definition 2.4.

Lemma 2.1. *Let $q > 0$ and $p > 0$ be the given constants. Then*

$$ {}^{\rho}I^{q} t^{p} = \frac{\Gamma\left(\frac{p+\rho}{\rho}\right)}{\Gamma\left(\frac{p+\rho q+\rho}{\rho}\right)} \frac{t^{p+\rho q}}{\rho^{q}}. \tag{2.1}$$

Lemma 2.2. *[2] For $q > 0$, the general solution of the fractional differential equation ${}^{c}D^{q}x(t) = 0$ is given by*

$$x(t) = c_0 + c_1 t + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

In view of Lemma 2.2, it follows that

$$I^{q}\ {}^{c}D^{q} x(t) = x(t) + c_0 + c_1 t + \ldots + c_{n-1} t^{n-1}, \tag{2.2}$$

for some $c_i \in \mathbb{R},\ i = 1, 2, \ldots, n-1$ $(n = [q] + 1)$.

Lemma 2.3. *For any $y \in C([0, T], \mathbb{R})$, the following linear fractional boundary value problem*

$$\begin{cases} {}^{c}D^{\alpha} x(t) = y(t),\ 1 < \alpha \leq 2, \\ x(0) = g(x),\ x(T) = \beta\ {}^{\rho}I^{q} x(\xi),\ 0 < \xi < T, \end{cases} \tag{2.3}$$

is equivalent to fractional integral equation:

$$x(t) = J^{\alpha} y(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^{q} J^{\alpha} y(\xi) - J^{\alpha} y(T)\Big\} + \Big[1 + \beta \frac{t}{\Lambda} \frac{\xi^{\rho q}}{\rho^{q}} \frac{1}{\Gamma(q+1)}\Big] g(x), \tag{2.4}$$

where

$$\Lambda = T - \beta \frac{\xi^{\rho q+1}}{\rho^{q}} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})} \neq 0. \tag{2.5}$$

Proof. It is well known that the general solution of the fractional differential equation in (2.3) can be written as

$$x(t) = c_0 + c_1 t + J^{\alpha} y(t), \tag{2.6}$$

where $c_0, c_1 \in \mathbb{R}$ are arbitrary constants. Using the first condition $(x(0) = g(x))$ given by (2.3) in (2.6), we get $c_0 = g(x)$. Applying the generalized Riemann-Liouville fractional integral operator on (2.6) and using Lemma 2.1, we obtain

$${}^{\rho}I^{q} x(t) = {}^{\rho}I^{q} J^{\alpha} y(t) + c_0 \frac{t^{\rho q}}{\rho^{q}} \frac{1}{\Gamma(q+1)} + c_1 \frac{t^{\rho q+1}}{\rho^{q}} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}, \tag{2.7}$$

which together with the second condition of (2.3) yields

$$J^{\alpha} y(T) + c_1 T + c_0 = \beta\ {}^{\rho}I^{q} J^{\alpha} y(\xi) + \beta c_0 \frac{\xi^{\rho q}}{\rho^{q}} \frac{1}{\Gamma(q+1)} + \beta c_1 \frac{\xi^{\rho q+1}}{\rho^{q}} \frac{\Gamma(\frac{1+\rho}{\rho})}{\Gamma(\frac{1+\rho q+\rho}{\rho})}. \tag{2.8}$$

Using $c_0 = g(x)$ in (2.8), we find that

$$c_1 = \frac{1}{\Lambda} \left\{ \beta\ {}^{\rho}I^{q} J^{\alpha} y(\xi) - J^{\alpha} y(T) + \beta g(x) \frac{\xi^{\rho q}}{\rho^{q}} \frac{1}{\Gamma(q+1)} \right\}.$$

Substituting the values of c_0, c_1 in (2.6), we get (2.4). Conversely, it follows by direct computation that the integral equation (2.4) satisfies the problem (2.3). This completes the proof. ∎

For computational convenience, we introduce the notations:

$$J^z f(s,x(s))(y) = \frac{1}{\Gamma(z)}\int_0^y (y-s)^{z-1} f(s,x(s))ds,$$
$$^\rho I^z f(s,x(s))(y) = \frac{\rho^{1-z}}{\Gamma(z)}\int_0^y \frac{s^{\rho-1} f(s,x(s))}{(y^\rho - s^\rho)^{1-z}}ds,$$

where $z>0$ and $y\in[0,T]$.

3. Existence and uniqueness results for problem (1.1)

We denote by $\mathcal{C} = C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]\to\mathbb{R}$ endowed with a topology of uniform convergence with the norm defined by $\|x\| = \sup\{|x(t)| : t\in[0,T]\}$. Also by $L^1([0,T],\mathbb{R})$ we denote the Banach space of measurable functions $x:[0,T]\to\mathbb{R}$ which are Lebesgue integrable and normed by $\|x\|_{L^1} = \int_0^T |x(t)|dt$.

In view of Lemma 2.3, we define an operator $\mathcal{P}:\mathcal{C}\to\mathcal{C}$ associated with problem (1.1) as

$$\begin{aligned}(\mathcal{P}x)(t) &= J^\alpha f(s,x(s))(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f(s,x(s))(\xi) - J^\alpha f(s,x(s))(T)\Big\}\\ &\quad+\Big[1+\beta\frac{t}{\Lambda}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\Big]g(x),\ \ t\in[0,T].\end{aligned}\tag{3.1}$$

Let us define $\mathcal{P}_{1,2}:\mathcal{C}\to\mathcal{C}$ by

$$(\mathcal{P}_1x)(t) = J^\alpha f(s,x(s))(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f(s,x(s))(\xi) - J^\alpha f(s,x(s))(T)\Big\},\tag{3.2}$$

and

$$(\mathcal{P}_2x)(t) = \Big[1+\beta\frac{t}{\Lambda}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\Big]g(x).\tag{3.3}$$

Clearly

$$(\mathcal{P}x)(t) = (\mathcal{P}_1x)(t) + (\mathcal{P}_2x)(t),\ \ t\in[0,T].\tag{3.4}$$

In the sequel, we use the notations:

$$p_0 := \frac{T^\alpha}{\Gamma(\alpha+1)}\left(1+\frac{T}{|\Lambda|}\right) + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})},\tag{3.5}$$

and

$$k_0 := 1 + \frac{T}{|\Lambda|}|\beta|\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}.\tag{3.6}$$

Theorem 3.1. *Let $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Assume that*

(A_1) $|f(t,x)-f(t,y)| \le L\|x-y\|, \forall t\in[0,T],\ L>0,\ x,y\in\mathbb{R}$;
(A_2) $g: C([0,1],\mathbb{R})\to\mathbb{R}$ *is a continuous function satisfying the condition:*

$$|g(u)-g(v)| \le \ell\|u-v\|,\ \ \ell < k_0^{-1},\ \forall\ u,v\in C([0,1],\mathbb{R}),\ \ \ell>0;$$

(A_3) $\gamma := Lp_0 + k_0\ell < 1$.

Then the boundary value problem (1.1) has a unique solution on $[0,T]$.

Proof. For $x, y \in \mathcal{C}$ and for each $t \in [0,T]$, from the definition of $\mathcal{P}$ and assumptions (A_1) and (A_2), we obtain

$$\begin{aligned}
|(\mathcal{P}x)(t) - (\mathcal{P}y)(t)| \le\ & \sup_{t\in[0,T]} \Bigg\{ J^\alpha |f(s,x(s)) - f(s,y(s))|(t) + \frac{t}{|\Lambda|} J^\alpha |f(s,x(s)) - f(s,y(s))|(T) \\
& + \frac{|\beta| t}{|\Lambda|}\, {}^\rho I^q J^\alpha |f(s,x(s)) - f(s,y(s))|(\xi) \\
& + \left| 1 + \beta \frac{t}{\Lambda} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \right| |g(x) - g(y)| \Bigg\} \\
\le\ & L\|x-y\| J^\alpha(1)(T) + L\|x-y\| \frac{T}{|\Lambda|} J^\alpha(1)(T) + L\|x-y\| \frac{|\beta| T}{|\Lambda|}\, {}^\rho I^q J^\alpha(1)(\xi) \\
& + \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \right) |g(x) - g(y)| \\
\le\ & L \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|} \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|} \frac{|\beta|}{\Gamma(\alpha+1)} \frac{\xi^{\alpha+\rho q}}{\rho^q} \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \right\} \|x-y\| \\
& + \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \right) \ell \|x-y\| \\
=\ & (Lp_0 + k_0 \ell) \|x-y\|.
\end{aligned}$$

Hence

$$\|(\mathcal{P}x) - (\mathcal{P}y)\| \le \gamma \|x-y\|.$$

As $\gamma < 1$ by (A_3), the operator $\mathcal{P}$ is a contraction map from the Banach space $\mathcal{C}$ into itself. Hence the conclusion of the theorem follows by the contraction mapping principle (Banach fixed point theorem). □

Example 3.1. *Consider the following fractional boundary value problem*

$$\begin{cases} {}^cD^{3/2} x(t) = \dfrac{\sin^2(\pi t)}{2(e^t+9)} \left(\dfrac{|x(t)|}{|x(t)|+1} + 1 \right) |x(t)| + \dfrac{\sqrt{3}}{4}, \ t \in [0,1], \\ x(0) = \dfrac{1}{2} + \dfrac{1}{12} \tan^{-1}(x(1/8)), \quad x(1) = \dfrac{1}{2}{}^{2/3} I^{3/2} x(3/4). \end{cases} \tag{3.7}$$

Here, $\alpha = 3/2$, $T = 1$, $\beta = 1/2$, $\xi = 3/4$, $\rho = 2/3$, $q = 3/2$,

$$f(t,x) = \frac{\sin^2(\pi t)}{2(e^t+9)} \Big(\frac{|x|}{|x|+1} + 1 \Big) |x| + \frac{\sqrt{3}}{4},\ g(x) = (1/12)\tan^{-1}(x(1/8)).$$

Since $|f(t,x) - f(t,y)| \le \frac{1}{10}\|x-y\|$, $|g(x) - g(y)| \le \frac{1}{12}\|x-y\|$, therefore, (A_1) and (A_2) are respectively satisfied with $L = 1/10$ and $\ell = 1/12$. Using the given values, it is found that $\Lambda \approx 0.8851733$, $p_0 \approx 1.6599468$, $k_0 \approx 1.563258$. Clearly $\gamma = Lp_0 + k_0\ell \approx 0.2962661 < 1$. Thus, the conclusion of Theorem 3.1 applies and the boundary value problem (3.7) has a solution on $[0,1]$.

Our second existence result is based on Sadovskii's fixed point theorem. Before proceeding further, let us recall some auxiliary material.

Definition 3.1. *Let M be a bounded set in metric space (X,d), then Kuratowskii measure of noncompactness, $\alpha(M)$ is defined as* $\inf\{\epsilon : M$ *covered by a finitely many sets such that the diameter of each set* $\le \epsilon\}$.

Definition 3.2. *[23] Let $\Phi : \mathcal{D}(\Phi) \subseteq X \to X$ be a bounded and continuous operator on Banach space X. Then Φ is called a condensing map if $\alpha(\Phi(B)) < \alpha(B)$ for all bounded sets $B \subset \mathcal{D}(\Phi)$, where α denotes the Kuratowski measure of noncompactness.*

Lemma 3.1. *[24, Example 11.7] The map $K + C$ is a k-set contraction with $0 \le k < 1$, and thus also condensing, if*

(i) $K, C : \mathcal{D} \subseteq X \to X$ *are operators on the Banach space* X;
(ii) K *is* k*-contractive, i.e.,*

$$\|Kx - Ky\| \le k\|x - y\|$$

for all $x, y \in \mathcal{D}$ *and fixed* $k \in [0,1)$;
(iii) C *is compact.*

Lemma 3.2. *[25] Let* B *be a convex, bounded and closed subset of a Banach space* X *and* $\Phi : B \to B$ *be a condensing map. Then* Φ *has a fixed point.*

Theorem 3.2. *Let* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *be a continuous function and condition* (A_2) *holds. In addition we assume that:*

(A_4) $g(0) = 0$;
(A_5) *there exists a nonnegative function* $p \in C([0,T],\mathbb{R})$ *and a nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *such that*

$$|f(t,u)| \le p(t)\psi(\|u\|) \text{ for any } (t,u) \in [0,T] \times \mathbb{R};$$

Then the problem (1.2) has at least one solution on $[0,T]$.

Proof. Let $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ be a closed bounded and convex subset of $\mathcal{C}$, where r will be fixed later. We define a map $\mathcal{P} : B_r \to \mathcal{C}$ as

$$(\mathcal{P}x)(t) = (\mathcal{P}_1 x)(t) + (\mathcal{P}_2 x)(t), \;\; t \in [0,T],$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are defined by (3.2) and (3.3) respectively. Notice that the problem (1.2) is equivalent to a fixed point problem $\mathcal{P}(x) = x$.

Step 1. $(\mathcal{P}x)B_r \subset B_r$.

Let us select $r \ge \dfrac{\psi(r)\|p\|p_0}{1 - \ell k_0}$ where p_0 and k_0 are defined by (3.5) and (3.6). For any $x \in B_r$, we have

$$\begin{aligned}
\|\mathcal{P}x\| &\le J^{\alpha}|f(s,x(s))|(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^{\rho}I^{q}J^{\alpha}|f(s,x(s))|(\xi) + J^{\alpha}|f(s,x(s))|(T)\Big\} \\
&\quad + \left(1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^{q}}\frac{1}{\Gamma(q+1)}\right)|g(x) - g(y)| \\
&\le \psi(\|x\|)J^{\alpha}p(s)(T) + \psi(\|x\|)\frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \psi(\|x\|)\frac{T}{|\Lambda|}J^{\alpha}p(s)(T) \\
&\quad + \left(1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^{q}}\frac{1}{\Gamma(q+1)}\right)\ell\|x\| \\
&\le \|p\|\psi(r)\left\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^{q}}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\} \\
&\quad + \left(1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^{q}}\frac{1}{\Gamma(q+1)}\right)\ell r \\
&= \psi(r)\|p\|p_0 + k_0\ell r < r,
\end{aligned}$$

which implies that $(\mathcal{P}x)B_r \subset B_r$,

Step 2. $\mathcal{P}_1$ is compact.

Observe that the operator $\mathcal{P}_1$ is uniformly bounded in view of Step 1. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then we obtain

$$\begin{aligned}
|(\mathcal{P}_1 x)(\tau_2) - (\mathcal{P}_1 x)(\tau_1)| &\leq |J^\alpha f(s,x(s))(\tau_2) - J^\alpha f(s,x(s))(\tau_1)| + \frac{|\tau_2-\tau_1|}{|\Lambda|} J^\alpha |f(s,x(s))|(T) \\
&\quad + \frac{|\beta||\tau_2-\tau_1|}{|\Lambda|}\, {}^\rho I^q J^\alpha |f(s,x(s))|(\xi) \\
&\leq \frac{\psi(r)}{\Gamma(\alpha)} \left| \int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}] p(s) ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} p(s) ds \right| \\
&\quad + \frac{\psi(r)|\tau_2-\tau_1|}{|\Lambda|} \left(J^\alpha p(s)(T) + |\beta|\, {}^\rho I^q J^\alpha p(s)(\xi) \right),
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{P}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}_1(B_r)$ is a relatively compact set.

Step 3. $\mathcal{P}_2$ is continuous and γ-contractive.

To show the continuity of $\mathcal{P}_2$ for $t \in [0,T]$, let us consider a sequence x_n converging to x. Then, we have

$$\begin{aligned}
\|\mathcal{P}_2 x_n - \mathcal{P}_2 x\| &\leq \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)}\right) |g(x_n) - g(x)| \\
&\leq \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)}\right) \ell \|x_n - x\|,
\end{aligned}$$

which, in view of (A_2), implies that $\mathcal{P}_2$ is continuous. Also $\mathcal{P}_2$ is γ-contractive, since

$$\gamma = \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)}\right) \ell = k_0 \ell < 1.$$

Step 4. $\mathcal{P}$ is condensing.

Since $\mathcal{P}_2$ is continuous, γ-contraction and $\mathcal{P}_1$ is compact, therefore, by Lemma 3.1, $\mathcal{P} : B_r \to B_r$ with $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$ is a condensing map on B_r.

From the above four steps, we conclude by Lemma 3.2 that the map $\mathcal{P}$ has a fixed point which, in turn, implies that the problem (1.2) has a solution. □

Example 3.2. *Consider the following boundary value problem*

$$\begin{cases} {}^c D^{3/2} x(t) = \dfrac{e^{-2t}}{\pi\sqrt{9+t^2}} \left(x \tan^{-1} x + \pi/2 \right), \ 0 < t < 1, \\ x(0) = \dfrac{1}{4}\left(1 - \cos x\right), \quad x(1) = \dfrac{1}{2}{}^{2/3} I^{3/2} x(3/4). \end{cases} \tag{3.8}$$

Observe that $|f(t,x)| \leq p(t)\psi(|x|)$ with $p(t) = \dfrac{e^{-2t}}{2\sqrt{9+t^2}}$, $\psi(|x|) = 1 + |x|$, and $g(0) = 0$, $\ell = 1/4$ as $|g(u) - g(v)| \leq (1/4)|u - v|$. Thus, all the conditions of Theorem 3.2 are satisfied and hence by its conclusion, the problem (3.8) has at least one solution on $[0,1]$.

Our next result relies on the following fixed point theorem due to Burton and Kirk [26].

Theorem 3.3. *Let X be a Banach space, and $A, B : X \to X$ be two operators such that A is a contraction and B is completely continuous. Then either*

(a) *the operator equation $y = A(y) + B(y)$ has a solution, or*
(b) *the set $\mathcal{E} = \left\{ u \in X : \lambda A\left(\frac{u}{\lambda}\right) + \lambda B(u) = u \right\}$ is unbounded for $\lambda \in (0,1)$.*

Theorem 3.4. *Assume that $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and conditions (A_2) and (A_4) hold. In addition we suppose that:*

(A_6) *there exists a function* $p \in L^1(J, \mathbb{R}_+)$ *such that*

$$|f(t,u)| \leq p(t), \quad \text{for a.e. } t \in J, \ \text{and each } u \in \mathbb{R}.$$

Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. To transform the problem (1.1) into a fixed point problem, we consider the map $\mathcal{P} : \mathcal{C} \to \mathcal{C}$ given by

$$(\mathcal{P}x)(t) = (\mathcal{P}_1 x)(t) + (\mathcal{P}_2 x)(t), \ t \in [0,T],$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are defined by (3.2) and (3.3) respectively.

We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all the conditions of Theorem 3.3.

Step 1. The operator $\mathcal{P}_1$ defined by (3.2) is continuous.

Let $x_n \subset B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,T] \times [-r,r]$, it follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ uniformly on $[0,T]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$ which implies that the operator $\mathcal{P}_1$ is continuous.

Step 2. The operator $\mathcal{P}_1$ maps bounded sets into bounded sets in $\mathcal{C}$.

It is indeed enough to show that for any $r > 0$ there exists a positive constant L such that for each $x \in B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$, we have $\|\mathcal{P}_1 x\| \leq L$. Let $x \in B_r$. Then

$$\begin{aligned}
\|\mathcal{P}_1 x\| &\leq J^\alpha |f(s,x(s))|(T) + \frac{T}{|\Lambda|}\Big\{|\beta| \ {}^\rho I^q J^\alpha |f(s,x(s))|(\xi) + J^\alpha |f(s,x(s))|(T)\Big\} \\
&\leq J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|} \ {}^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T) := L.
\end{aligned}$$

Step 3. The operator $\mathcal{P}_1$ maps bounded sets into equicontinuous sets in $\mathcal{C}$.

Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then, for each $x \in B_r$, we obtain

$$\begin{aligned}
|(\mathcal{P}_1 x)(\tau_2) - (\mathcal{P}_1 x)(\tau_1)| &\leq |J^\alpha f(s,x(s))(\tau_2) - J^\alpha f(s,x(s))(\tau_1)| + \frac{|\tau_2 - \tau_1|}{|\Lambda|} J^\alpha |f(s,x(s))|(T) \\
&\quad + \frac{|\beta||\tau_2 - \tau_1|}{|\Lambda|} |{}^\rho I^q J^\alpha |f(s,x(s))|(\xi) \\
&\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^{\tau_1} [(\tau_2 - s)^{\alpha-1} - (\tau_1 - s)^{\alpha-1}] p(s) ds + \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1} p(s) ds \right| \\
&\quad + \frac{|\tau_2 - \tau_1|}{|\Lambda|} \left(J^\alpha p(s)(T) + |\beta| \ {}^\rho I^q J^\alpha p(s)(\xi) \right),
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{P}_1$ is equicontinuous.

Step 4. The operator $\mathcal{P}_2$ defined by (3.3) is a contraction.

This was established in Step 3 of Theorem 3.2.

Step 5. A priori bounds on solutions.

Now it remains to show that the set $\mathcal{E} = \left\{u \in \mathcal{C} : \lambda \mathcal{P}_2\left(\frac{u}{\lambda}\right) + \lambda \mathcal{P}_1(u) = u\right\}$ is unbounded for some $\lambda \in (0,1)$.

Let $\lambda \in (0,1)$ and $x \in \mathcal{E}$ be a solution of the integral equation

$$\begin{aligned}
x(t) &= \lambda J^\alpha f(s,x(s))(t) + \lambda \frac{t}{\Lambda}\Big\{\beta \ {}^\rho I^q J^\alpha f(s,x(s))(\xi) - J^\alpha f(s,x(s))(T)\Big\} \\
&\quad + \lambda \Big[1 + \beta \frac{t}{\Lambda} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)}\Big] g(x), \ t \in [0,T].
\end{aligned}$$

Then, for each $t \in [0,T]$, we have

$$\begin{aligned}|x(t)| &\leq J^{\alpha}p(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}p(s)(T) \\ &\quad + \lambda\Big[1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^{q}}\frac{1}{\Gamma(q+1)}\Big]\Bigg[\Big|g\Big(\frac{x(s)}{\lambda}\Big) - g(0)\Big| + |g(0)|\Bigg] \\ &\leq J^{\alpha}p(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}p(s)(T) + \Big[1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^{q}}\frac{1}{\Gamma(q+1)}\Big]\ell\|x\|,\end{aligned}$$

or

$$(1 - k_0\ell)\|x\| \leq J^{\alpha}p(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}p(s)(T).$$

Consequently we have

$$\|x\| \leq M := \frac{1}{(1 - k_0\ell)}\Bigg[J^{\alpha}p(s)(T) + \frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + \frac{T}{|\Lambda|}J^{\alpha}p(s)(T)\Bigg],$$

which shows that the set $\mathcal{E}$ is bounded, since $k_0\ell < 1$. Hence, $\mathcal{P}$ has a fixed point in $[0,T]$ by Theorem 3.3, and consequently the problem (1.1) has a solution. This completes the proof. □

Finally, we show the existence of solutions for the boundary value problem (1.1) by applying a fixed point theorem due to O'Regan in [27].

Lemma 3.3. *Denote by U an open set in a closed, convex set C of a Banach space E. Assume $0 \in U$. Also assume that $F(\bar{U})$ is bounded and that $F : \bar{U} \to C$ is given by $F = F_1 + F_2$, in which $F_1 : \bar{U} \to E$ is continuous and completely continuous and $F_2 : \bar{U} \to E$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|F_2(x) - F_2(y)\| \leq \phi(\|x - y\|)$ for all $x, y \in \bar{U}$). Then, either*

(C1) *F has a fixed point $u \in \bar{U}$; or*

(C2) *there exist a point $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F(u)$, where $\bar{U}$ and ∂U, respectively, represent the closure and boundary of U.*

In the next result, we use the terminology:

$$\Omega_r = \{x \in \mathcal{C} : \|x\| < r\},\ M_r = \max\{|f(t,x)| : (t,x) \in [0,T] \times [-r,r]\}.$$

Theorem 3.5. *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function and conditions (A_1), (A_2), (A_4) and (A_5) hold. In addition we assume that:*

(A_7) $\displaystyle\sup_{r\in(0,\infty)} \frac{r}{p_0\psi(r)\|p\|} > \frac{1}{1 - k_0\ell}$, *where p_0 and k_0 are defined in (3.5) and (3.6) respectively.*

Then the boundary value problem (1.1) has at least one solution on $[0,T]$.

Proof. By the assumption (A_7), there exists a number $r_0 > 0$ such that

$$\frac{r_0}{p_0\psi(r_0)\|p\|} > \frac{1}{1 - k_0\ell}. \tag{3.9}$$

We shall show that the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ defined by (3.2) and (3.3) respectively, satisfy all the conditions of Lemma 3.3.

Step 1. *The operator $\mathcal{P}_1$ is continuous and completely continuous.* We first show that $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is bounded. For any $x \in \bar{\Omega}_{r_0}$, we have

$$\begin{aligned}\|\mathcal{P}_1 x\| &\leq J^{\alpha}|f(s,x(s))|(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^{\rho}I^{q}J^{\alpha}|f(s,x(s))|(\xi) + J^{\alpha}|f(s,x(s))|(T)\Big\} \\ &\leq M_r J^{\alpha}p(s)(T) + M_r\frac{|\beta|T}{|\Lambda|}\ {}^{\rho}I^{q}J^{\alpha}p(s)(\xi) + M_r\frac{T}{|\Lambda|}J^{\alpha}p(s)(T) \\ &\leq \|p\|M_r\Bigg\{\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^{\alpha}}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^{q}}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\Bigg\} \\ &= M_r\|p\|p_0.\end{aligned}$$

Thus the operator $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is uniformly bounded. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then

$$\begin{aligned}
|(\mathcal{P}_1 x)(\tau_2) - (\mathcal{P}_1 x)(\tau_1)| &\leq |J^\alpha f(s,x(s))(\tau_2) - J^\alpha f(s,x(s))(\tau_1)| + \frac{|\tau_2-\tau_1|}{|\Lambda|} J^\alpha |f(s,x(s))|(T) \\
&\quad + \frac{|\beta||\tau_2-\tau_1|}{|\Lambda|} |{}^\rho I^q J^\alpha |f(s,x(s))|(\xi) \\
&\leq \frac{M_r}{\Gamma(\alpha)} \left| \int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}] p(s) ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} p(s) ds \right| \\
&\quad + \frac{M_r|\tau_2-\tau_1|}{|\Lambda|} \left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|\beta|}{\Gamma(\alpha+1)} \frac{\xi^{\alpha+\rho q}}{\rho^q} \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \right),
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 - \tau_1 \to 0$. Thus, $\mathcal{P}_1$ is equicontinuous. Hence, by the Arzelá-Ascoli Theorem, $\mathcal{P}_1(\bar{\Omega}_{r_0})$ is a relatively compact set. Now, let $x_n \subset \bar{\Omega}_{r_0}$ with $\|x_n - x\| \to 0$. Then the limit $\|x_n(t) - x(t)\| \to 0$ is uniformly valid on $[0,T]$. From the uniform continuity of $f(t,x)$ on the compact set $[0,T] \times [-r_0, r_0]$, it follows that $\|f(t,x_n(t)) - f(t,x(t))\| \to 0$ uniformly on $[0,T]$. Hence $\|\mathcal{P}_1 x_n - \mathcal{P}_1 x\| \to 0$ as $n \to \infty$ which proves the continuity of $\mathcal{P}_1$. This completes the proof of Step 1.

Step 2. *The operator $\mathcal{P}_2 : \bar{\Omega}_{r_0} \to C([0,T],\mathbb{R})$ is contractive.* This is a consequence of (A_2).

Step 3. *The set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.* The assumptions (A_2) and (A_4) imply that

$$\|\mathcal{P}_2(x)\| \leq k_0 \ell r_0,$$

for any $x \in \bar{\Omega}_{r_0}$. This, with the boundedness of the set $\mathcal{P}_1(\bar{\Omega}_{r_0})$ implies that the set $\mathcal{P}(\bar{\Omega}_{r_0})$ is bounded.

Step 4. *Finally, it will be shown that the case (C2) in Lemma 3.3 does not hold.* On the contrary, we suppose that (C2) holds. Then, we have that there exist $\lambda \in (0,1)$ and $x \in \partial\Omega_{r_0}$ such that $x = \lambda \mathcal{P} x$. So, we have $\|x\| = r_0$ and

$$\begin{aligned}
x(t) &= \lambda J^\alpha f(s,x(s))(t) + \lambda \frac{t}{\Lambda} \Big\{ \beta\, {}^\rho I^q J^\alpha f(s,x(s))(\xi) - J^\alpha f(s,x(s))(T) \Big\} \\
&\quad + \lambda \Big[1 + \beta \frac{t}{\Lambda} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \Big] g(x), \ t \in [0,T].
\end{aligned}$$

Using the assumptions $(A_4) - (A_6)$, we get

$$\begin{aligned}
r_0 &\leq \|p\| \psi(r_0) \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|} \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|} \frac{|\beta|}{\Gamma(\alpha+1)} \frac{\xi^{\alpha+\rho q}}{\rho^q} \frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})} \right\} \\
&\quad + \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \right) \ell r_0,
\end{aligned}$$

which yields

$$r_0 \leq p_0 \psi(r_0) \|p\| + k_0 \ell r_0.$$

Thus, we get a contradiction:

$$\frac{r_0}{p_0 \psi(r_0) \|p\|} \leq \frac{1}{1 - k_0 \ell}.$$

Thus the operators $\mathcal{P}_1$ and $\mathcal{P}_2$ satisfy all the conditions of Lemma 3.3. Hence, the operator $\mathcal{P}$ has at least one fixed point $x \in \bar{\Omega}_{r_0}$, which is a solution of the problem (1.2). This completes the proof. □

Example 3.3. *Consider the following fractional order boundary value problem*

$$\begin{cases} {}^c D^{3/2} x(t) = \dfrac{e^{-2t}}{2\pi\sqrt{9+t^2}} \Big(x \tan^{-1} x + \pi/2 \Big), \ 0 < t < 1, \\[2ex] x(0) = \dfrac{1}{4} \left(1 - \cos x \right), \quad x(1) = \dfrac{1}{2} {}^{2/3} I^{3/2} x(3/4). \end{cases} \tag{3.10}$$

Observe that $|f(t,x)| \leq p(t) \psi(|x|)$ with $p(t) = \dfrac{e^{-2t}}{4\sqrt{9+t^2}}$, $\psi(|x|) = 1 + |x|$, and $g(0) = 0$, $\ell = 1/4$ as $|g(u) - g(v)| \leq (1/4)|u - v|$. With $\psi(r) = 1 + r$, $\|p\| = 1/12$, $\Lambda \approx 0.8851733$, $p_0 \approx 1.6599468$, $k_0 \approx$

1.563258 (as found in Example 3.1), we have that (A_7) holds, since $\sup_{r\in(0,\infty)} \frac{r}{p_0\psi(r)\|p\|} \approx 7.2291473 > 1.6415361 \approx \frac{1}{1-k_0\ell}$. Thus, all the conditions of Theorem 3.5 are satisfied and hence by its conclusion, the problem (3.10) has at least one solution on $[0,1]$.

4. Existence results for problem (1.2)

First of all, we introduce notions and recall some basic material on multivalued maps related to our work [28–30].

For a normed space $(X,\|\cdot\|)$, let $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$ and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$.

A multivalued map $G : X \to \mathcal{P}(X)$:

(i) is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$;
(ii) is *bounded* on bounded sets if $G(\mathbb{B}) = \cup_{x\in\mathbb{B}} G(x)$ is bounded in X for all $\mathbb{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathbb{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$);
(iii) is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{N}_0$ of x_0 such that $G(\mathcal{N}_0) \subseteq N$;
(iv) G is *lower semi-continuous (l.s.c.)* if the set $\{y \in X : G(y) \cap B \neq \emptyset\}$ is open for any open set B in E;
(v) is said to be *completely continuous* if $G(\mathbb{B})$ is relatively compact for every $\mathbb{B} \in \mathcal{P}_b(X)$;
(vi) is said to be *measurable* if for every $y \in \mathbb{R}$, the function
$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$
is measurable;
(vii) *has a fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

Definition 4.1. *A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if*

(i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$;
(ii) $x \longmapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$;

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that
$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \le \varphi_\alpha(t)$$
for all $\|x\| \le \alpha$ and for a. e. $t \in [0,T]$.

For each $x \in C([0,T],\mathbb{R})$, define the set of selections of F by
$$S_{F,x} := \{v \in L^1([0,T],\mathbb{R}) : v(t) \in F(t,x(t)) \text{ for a.e. } t \in [0,T]\}.$$

We define the graph of G to be the set $Gr(G) = \{(x,y) \in X \times Y, y \in G(x)\}$ and recall two useful results on closed graphs and upper-semicontinuity.

Lemma 4.1. *([28, Proposition 1.2]) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$; i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 4.2. *([31]) Let X be a Banach space. Let $F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1- Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([0,T],X)$ to $C([0,T],X)$. Then the operator*
$$\Theta \circ S_F : C([0,T],X) \to \mathcal{P}_{cp,c}(C([0,T],X)),\ x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$
is a closed graph operator in $C([0,T],X) \times C([0,T],X)$.

Lemma 4.3. *([32], Krasnoselskii's fixed point theorem). Let X be a Banach space, $Y \in \mathcal{P}_{b,cl,c}(X)$ and $A, B : Y \to \mathcal{P}_{cp,c}(X)$ two multivalued operators. If the following conditions are satisfied*

(i) *$Ay + By \subset Y$ for all $y \in Y$;*
(ii) *A is contraction;*
(iii) *B is u.s.c and compact,*

then, there exists $y \in Y$ such that $y \in Ay + By$.

Definition 4.2. *A function $x \in C^2([0,T],\mathbb{R})$ is a solution of the problem (1.2) if $x(0) = g(x)$, $x(T) = \beta\ {}^{\rho}I^q x(\xi)$, and there exists a function $f \in L^1([0,T],\mathbb{R})$ such that $f(t) \in F(t, x(t))$ a.e. on $[0,T]$ and*

$$x(t) = J^{\alpha} f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^q J^{\alpha} f(s)(\xi) - J^{\alpha} f(s)(T)\Big\} + \Big[1 + \beta\frac{t}{\Lambda}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\Big] g(x).$$

Theorem 4.1. *Assume that (A_2) holds. In addition we suppose that:*

(H_1) *$F : [0,T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory multivalued map;*
(H_2) *there exists a function $p \in C([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t), \text{ for each } (t,x) \in [0,T] \times \mathbb{R}.$$

Then the boundary value problem (1.2) has at least one solution on $[0,T]$.

Proof. To transform the problem (1.2) to a fixed point problem, we define an operator $\mathcal{N} : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C})$ by

$$\mathcal{N}(x) = \left\{ \begin{array}{l} h \in \mathcal{C} : \\ h(t) = \left\{ \begin{array}{l} J^{\alpha} f(s)(t) + \dfrac{t}{\Lambda}\Big\{\alpha^{\rho} I^q J^{\alpha} f(s)(\xi) - J^{\alpha} f(s)(T)\Big\} \\ +\Big[1 + \beta\dfrac{t}{\Lambda}\dfrac{\xi^{\rho q}}{\rho^q}\dfrac{1}{\Gamma(q+1)}\Big] g(x), \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$.

Next we introduce operators $\mathcal{A} : \mathcal{C} \longrightarrow \mathcal{C}$ and $\mathcal{B} : \mathcal{C} \longrightarrow \mathcal{P}(\mathcal{C})$ by

$$\mathcal{A}x(t) = \left[1 + \beta\frac{t}{\Lambda}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right] g(x), \tag{4.1}$$

$$\mathcal{B}(x) = \left\{h \in \mathcal{C} : h(t) = J^{\alpha} f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^q J^{\alpha} f(s)(\xi) - J^{\alpha} f(s)(T)\Big\}.\right\} \tag{4.2}$$

Observe that $\mathcal{N} = \mathcal{A} + \mathcal{B}$. We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of lemma 4.3 on $[0,T]$. First, we show that the operators $\mathcal{A}$ and $\mathcal{B}$ define the multivalued operators $\mathcal{A}, \mathcal{B} : B_r \to \mathcal{P}_{cp,c}(\mathcal{C})$ where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ is a bounded set in $\mathcal{C}$. First we prove that $\mathcal{B}$ is compact-valued on B_r. Note that the operator $\mathcal{B}$ is equivalent to the composition $\mathcal{L} \circ S_F$, where $\mathcal{L}$ is the continuous linear operator on $L^1([0,T],\mathbb{R})$ into $\mathcal{C}$, defined by

$$\mathcal{L}(v)(t) = J^{\alpha} v(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^{\rho}I^q J^{\alpha} v(s)(\xi) - J^{\alpha} v(s)(T)\Big\}.$$

Suppose that $x \in B_r$ is arbitrary and let $\{v_n\}$ be a sequence in $S_{F,x}$. Then, by definition of $S_{F,x}$, we have $v_n(t) \in F(t, x(t))$ for almost all $t \in [0,T]$. Since $F(t, x(t))$ is compact for all $t \in J$, there is a convergent subsequence of $\{v_n(t)\}$ (we denote it by $\{v_n(t)\}$ again) that converges in measure to some $v(t) \in S_{F,x}$ for almost all $t \in J$. On the other hand, $\mathcal{L}$ is continuous, so $\mathcal{L}(v_n)(t) \to \mathcal{L}(v)(t)$ pointwise on $[0,T]$.

In order to show that the convergence is uniform, we have to show that $\{\mathcal{L}(v_n)\}$ is an equicontinuous sequence. Let $t_1, t_2 \in [0,T]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
& |\mathcal{L}(v_n)(t_2) - \mathcal{L}(v_n)(t_1)| \\
\leq\ & |J^\alpha v_n(s)(t_2) - J^\alpha v_n(s)(t_1)| + \frac{|t_2-t_1|}{|\Lambda|} J^\alpha |v_n(s)|(T) + \frac{|\beta||t_2-t_1|}{|\Lambda|} |{}^\rho I^q J^\alpha |v_n(s)|(\xi) \\
\leq\ & \frac{1}{\Gamma(\alpha)} \left| \int_0^{t_1} [(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}] p(s) ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} p(s) ds \right| \\
& + \frac{|t_2-t_1|}{|\Lambda|} \left(J^q p(s)(T) + |\beta|\ {}^\rho I^q J^q p(s)(\xi) \right).
\end{aligned}$$

We see that the right hand of the above inequality tends to zero as $t_2 \to t_1$. Thus, the sequence $\{\mathcal{L}(v_n)\}$ is equicontinuous and hence, by the Arzelá-Ascoli theorem, we get that there is a uniformly convergent subsequence. So, there is a subsequence of $\{v_n\}$ (we denote it again by $\{v_n\}$) such that $\mathcal{L}(v_n) \to \mathcal{L}(v)$. Note that $\mathcal{L}(v) \in \mathcal{L}(S_{F,x})$. Hence, $\mathcal{B}(x) = \mathcal{L}(S_{F,x})$ is compact for all $x \in B_r$. So $\mathcal{B}(x)$ is compact.

Now, we show that $\mathcal{B}(x)$ is convex for all $x \in \mathcal{C}$. Let $z_1, z_2 \in \mathcal{B}(x)$. We select $f_1, f_2 \in S_{F,x}$ such that

$$z_i(t) = J^\alpha f_i(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f_i(s)(\xi) - J^\alpha f_i(s)(T)\Big\},\ i = 1, 2,$$

for almost all $t \in [0,T]$. Let $0 \leq \lambda \leq 1$. Then, we have

$$\begin{aligned}
[\lambda z_1 + (1-\lambda) z_2](t) =\ & J^\alpha[\lambda f_1(s) + (1-\lambda) f_2(s)](t) \\
& + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha [\lambda f_1(s) + (1-\lambda) f_2(s)](\xi) - J^\alpha[\lambda f_1(s) + (1-\lambda) f_2(s)](s)(T)\Big\}.
\end{aligned}$$

Since F has convex values, so $S_{F,u}$ is convex and $\lambda f_1(s) + (1-\lambda) f_2(s) \in S_{F,x}$. Thus

$$\lambda z_1 + (1-\lambda) z_2 \in \mathcal{B}(x).$$

Consequently, $\mathcal{B}$ is convex-valued. Obviously, $\mathcal{A}$ is compact and convex-valued.

The rest of the proof consists of several steps and claims.

Step 1: We show that $\mathcal{A}$ *is a contraction on* $\mathcal{C}$. For $x, y \in \mathcal{C}$, we have

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &= \left| 1 + \beta \frac{t}{\Lambda} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \right| |g(x) - g(y)| \\
&\leq \left(1 + |\beta| \frac{T}{|\Lambda|} \frac{\xi^{\rho q}}{\rho^q} \frac{1}{\Gamma(q+1)} \right) |g(x) - g(y)|, \\
&\leq k_0 \ell \|x - y\|,
\end{aligned}$$

which, on taking supremum over $t \in [0,T]$, yields

$$\|\mathcal{A}x - \mathcal{A}y\| \leq L_0 \|x - y\|, \quad L_0 = k_0 \ell.$$

This shows that $\mathcal{A}$ is a contraction as $L_0 < 1$.

Step 2: *$\mathcal{B}$ is compact and upper semicontinuous.* This will be established in several claims.

Claim I: *$\mathcal{B}$ maps bounded sets into bounded sets in $\mathcal{C}$.*

Let $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$ be a bounded set in $\mathcal{C}$. Then, for each $h \in \mathcal{B}(x), x \in B_r$, there exists $f \in S_{F,x}$ such that

$$h(t) = J^\alpha f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f(s)(\xi) - J^\alpha f(s)(T)\Big\}.$$

Then, for $t \in [0,T]$, we have

$$\begin{aligned} |h(t)| &\le J^\alpha |f(s)|(T) + \frac{T}{|\Lambda|}\{|\beta|^\rho I^q J^\alpha |f(s)|(\xi) - J^\alpha |f(s)|(T)\Big\} \\ &\le J^\alpha p(s)(T) + \frac{|\beta| T}{|\Lambda|}\ {}^\rho I^q J^\alpha p(s)(\xi) + \frac{T}{|\Lambda|} J^\alpha p(s)(T) \\ &\le \|p\| \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\}. \end{aligned}$$

Thus,

$$\|h\| \le \|p\| \left\{ \frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\}.$$

CLAIM II: *$\mathcal{B}$ maps bounded sets into equicontinuous sets.*

Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $x \in B_r$. Then, for each $h \in \mathcal{B}(x)$, we obtain

$$\begin{aligned} & |h(\tau_2) - h(\tau_1)| \\ \le\ & |J^\alpha f(s)(\tau_2) - J^\alpha f(s)(\tau_1)| + \frac{|\tau_2-\tau_1|}{|\Lambda|} J^\alpha |f(s)|(T) + \frac{|\beta||\tau_2-\tau_1|}{|\Lambda|}\ {}^\rho I^q J^\alpha |f(s)|(\xi) \\ \le\ & \frac{1}{\Gamma(\alpha)}\left| \int_0^{\tau_1} [(\tau_2-s)^{\alpha-1} - (\tau_1-s)^{\alpha-1}] p(s)ds + \int_{\tau_1}^{\tau_2} (\tau_2-s)^{\alpha-1} p(s) ds \right| \\ & + \|p\| \frac{|\tau_2-\tau_1|}{|\Lambda|}\left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right) \\ \le\ & \frac{\|p\|}{\Gamma(\alpha+1)}[\tau_2^\alpha - \tau_1^\alpha + 2(\tau_2-\tau_1)^\alpha] + \|p\|\frac{|\tau_2-\tau_1|}{|\Lambda|}\left(\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right). \end{aligned}$$

Obviously the right hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_2 - \tau_1 \to 0$. Therefore it follows by the Ascoli-Arzelá theorem that $\mathcal{B} : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is completely continuous.

By Claims I and II, $\mathcal{B}$ is completely continuous. By Lemma 4.1, $\mathcal{B}$ will be upper semicontinuous (since it is completely continuous) if we prove that it has a closed graph.

CLAIM III: *$\mathcal{B}$ has a closed graph.*

Let $x_n \to x_*, h_n \in \mathcal{B}(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in \mathcal{B}(x_*)$. Associated with $h_n \in \mathcal{B}(x_n)$, there exists $f_n \in S_{F,x_n}$ such that for each $t \in [0,T]$,

$$h(t) = J^\alpha f_n(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f_n(s)(\xi) - J^\alpha f_n(s)(T)\Big\}.$$

Thus it suffices to show that there exists $f_* \in S_{F,x_*}$ such that for each $t \in [0,T]$,

$$h_*(t) = J^\alpha f_*(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f_*(s)(\xi) - J^\alpha f_*(s)(T)\Big\}.$$

Let us consider the linear operator $\Theta : L^1([0,T],\mathbb{R}) \to \mathcal{C}$ given by

$$f \mapsto \Theta(f)(t) = J^\alpha f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f(s)(\xi) - J^\alpha f(s)(T)\Big\}.$$

Observe that

$$\begin{aligned} \|h_n(t) - h_*(t)\| &= \Big\| J^\alpha (f_n(s) - f_*(s))(t) \\ &\quad + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha (f_n(s)-f_*(s))(\xi) - J^\alpha (f_n(s) - f_*(s))(T)\Big\}\Big\| \to 0, \end{aligned}$$

as $n \to \infty$.

Thus, it follows by Lemma 4.2 that $\Theta \circ S_F$ is a closed graph operator. Further, we have $h_n(t) \in \Theta(S_{F,x_n})$. Since $x_n \to x_*$, we have that

$$h_*(t) = J^\alpha f_*(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f_*(s)(\xi) - J^\alpha f_*(s)(T)\Big\},$$

for some $f_* \in S_{F,x_*}$. Hence $\mathcal{B}$ has a closed graph (and therefore has closed values). In consequence, the operator $\mathcal{B}$ is compact and upper semicontinuous.

Step 3: Here, we show that $\mathcal{A}(x)+\mathcal{B}(x) \subset B_r$ for all $x \in B_r$. Suppose $x \in B_r$, with $r > \dfrac{p_0\|p\|}{1-k_0\ell}$ and $h \in \mathcal{B}$ are arbitrary elements. Choose $f \in S_{F,x}$ such that

$$h(t) = J^\alpha f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f(s)(\xi) - J^\alpha f(s)(T)\Big\} + \left[1+\beta\frac{t}{\Lambda}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right]g(x), \quad t \in [0,T].$$

Following the method for proof for Claim I, we can obtain

$$\begin{aligned} |h(t)| &\le \|p\|\left\{\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{T^\alpha}{\Gamma(\alpha+1)} + \frac{T}{|\Lambda|}\frac{|\beta|}{\Gamma(\alpha+1)}\frac{\xi^{\alpha+\rho q}}{\rho^q}\frac{\Gamma(\frac{\alpha+\rho}{\rho})}{\Gamma(\frac{\alpha+\rho q+\rho}{\rho})}\right\} \\ &\quad + \left(1+|\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)\ell\|x\|. \end{aligned}$$

Thus

$$\|h\| \le p_0\|p\| + k_0\ell r < r. \tag{4.3}$$

Hence $\|h\| \le r$, which means that $\mathcal{A}(x)+\mathcal{B}(x) \subset B_r$ for all $x \in B_r$.

Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Lemma 4.3 and hence its conclusion implies that $x \in \mathcal{A}(x)+\mathcal{B}(x)$ in B_r. Therefore the boundary value problem (1.2) has a solution in B_r and the proof is completed. □

To prove our next result, we make use of the following form of the nonlinear alternative for contractive maps [33, Corollary 3.8].

Theorem 4.2. *Let X be a Banach space, and D a bounded neighborhood of $0 \in X$. Let $Z_1 : X \to \mathcal{P}_{cp,c}(X)$ and $Z_2 : \bar{D} \to \mathcal{P}_{cp,c}(X)$ two multivalued operators satisfying*

(a) *Z_1 is contraction, and*
(b) *Z_2 is u.s.c and compact.*

Then, if $G = Z_1 + Z_2$, either

(i) *G has a fixed point in $\bar{D}$ or*
(ii) *there is a point $u \in \partial D$ and $\lambda \in (0,1)$ with $u \in \lambda G(u)$.*

Theorem 4.3. *Assume that $(A_2)$and (H_1) are satisfied. In addition we suppose that:*

(H_3) *there exists a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}^+)$ such that*

$$\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \text{ for each } (t,x) \in [0,T]\times\mathbb{R};$$

(H_4) *there exists a number $M > 0$ such that*

$$\frac{M}{\psi(M)} > \frac{1}{(1-k_0\ell)}\left[J^\alpha p(s)(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^\rho I^q J^\alpha p(s)(\xi) + J^\alpha p(s)(T)\Big\}\right], \tag{4.4}$$

where k_0 is defined in (3.6).

Then the boundary value problem (1.2) has at least one solution on $[0,T]$.

Proof. To transform the problem (1.2) to a fixed point, we define an operator $\mathcal{N} : C([0,T],\mathbb{R}) \longrightarrow \mathcal{P}(\mathcal{C})$ and consider the operators $\mathcal{A}$ and $\mathcal{B}$ defined in the beginning of the proof of Theorem 4.1. As in Theorem 4.1, one can show that the operators $\mathcal{A}$ and $\mathcal{B}$ are indeed the multivalued operators $\mathcal{A},\mathcal{B} : B_r \to \mathcal{P}_{cp,c}(\mathcal{C})$ where $B_r = \{x \in \mathcal{C} : \|x\| \le r\}$ is a bounded set in $\mathcal{C}$, $\mathcal{A}$ is a contraction on $\mathcal{C}$ and $\mathcal{B}$ is u.s.c. and compact.

Thus the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 4.2 and hence its conclusion implies either condition (i) or condition (ii) holds. We show that the conclusion (ii) is not possible. If $x \in \lambda\mathcal{A}(x) + \lambda\mathcal{B}(x)$ for $\lambda \in (0,1)$, then there exists $f \in S_{F,x}$ such that

$$x(t) = J^\alpha f(s)(t) + \frac{t}{\Lambda}\Big\{\beta\ {}^\rho I^q J^\alpha f(s)(\xi) - J^\alpha f(s)(T)\Big\} + \left[1 + \beta\frac{t}{\Lambda}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right]g(x), \quad t \in [0,T],$$

and

$$\begin{aligned} |x(t)| &\le J^\alpha p(s)|\psi(\|x\|)(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^\rho I^q J^\alpha p(s)\psi(\|x\|)(\xi) + J^\alpha p(s)\psi(\|x\|)(T)\Big\} \\ &\quad + \left(1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)\ell\|x\| \\ &\le \psi(\|x\|)\left[J^\alpha p(s)(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^\rho I^q J^\alpha p(s)(\xi) + J^\alpha p(s)(T)\Big\}\right] \\ &\quad + \left(1 + |\beta|\frac{T}{|\Lambda|}\frac{\xi^{\rho q}}{\rho^q}\frac{1}{\Gamma(q+1)}\right)\ell\|x\|. \end{aligned}$$

Thus

$$(1 - k_0\ell)\|x\| \le \psi(\|x\|)\left[J^\alpha p(s)(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^\rho I^q J^\alpha p(s)(\xi) + J^\alpha p(s)(T)\Big\}\right]. \tag{4.5}$$

If condition (ii) of Theorem 4.2 holds, then there exists $\lambda \in (0,1)$ and $x \in \partial B_M$ with $x = \lambda\mathcal{N}(x)$. Then, x is a solution of (1.2) with $\|x\| = M$. Now, by the inequality (4.5), we get

$$\frac{M}{\psi(M)} \le \frac{1}{(1-k_0\ell)}\left[J^\alpha p(s)(T) + \frac{T}{|\Lambda|}\Big\{|\beta|\ {}^\rho I^q J^\alpha p(s)(\xi) + J^\alpha p(s)(T)\Big\}\right],$$

which contradicts (4.4). Hence, $\mathcal{N}$ has a fixed point in $[0,T]$ by Theorem 4.2, and consequently the problem (1.2) has a solution. This completes the proof. □

Example 4.1. *Consider the following boundary value problem of fractional differential inclusions*

$$\begin{cases} D^{3/2}x(t) \in F(t, x(t)),\ t \in [0,1],\ 0 < t < 1, \\ x(0) = \dfrac{1}{8}x(1/4), \quad x(1) = \dfrac{1}{2}{}^{2/3}I^{3/2}x(3/4), \end{cases} \tag{4.6}$$

where

$$F(t, x(t)) = \left[\frac{2}{\sqrt{t^2+64}}\left(\frac{|x(t)|}{2}\left(\frac{|x(t)|}{|x(t)|+1} + 1\right) + \frac{1}{5}\right), \frac{e^{-t}}{(10+t)}\left(\sin x(t) + \frac{1}{15}\right)\right].$$

Clearly $|F(t,x)| \le p(t)\psi(|x|)$, where $p(t) = 2/\sqrt{t^2+64}$, $\psi_1(|x|) = |x| + 1/5$ and $\ell = 1/8$. Using the values: $\Lambda \approx 0.8851733$, $p_0 \approx 1.6599468$, $k_0 \approx 1.563258$ (see Example 3.1) and the condition (H_4), we find that $M > M_1 \simeq 0.2130287$. Since the hypotheses of Theorem 4.3 are satisfied, the problem (4.6) has a solution on $[0,1]$.

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[1]Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

[2]Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece

[*]Corresponding author: bashirahmad_qau@yahoo.com