International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 2 (2017), 216-230
http://www.etamaths.com

BASIC THEORY FOR DIFFERENTIAL EQUATIONS WITH UNIFIED REIMANN-LIOUVILLE AND HADAMARD TYPE FRACTIONAL DERIVATIVES

BAŞAK KARPUZ[1,*], UMUT M. ÖZKAN[2], TUĞBA YALÇIN[2] AND MUSTAFA K. YILDIZ[2]

Abstract. In this paper, we extend the definition of the fractional integral and derivative introduced in [Appl. Math. Comput. 218 (2011)] by Katugampola, which exhibits nice properties only for numbers whose real parts lie in $[0,1]$. We prove some interesting properties of the fractional integrals and derivatives. Based on these properties, the following concepts for the new type fractional differential equations are explored: Existence and uniqueness of solutions; Solutions of autonomous fractional differential equations; Dependence on the initial conditions; Green's function; Variation of parameters formula.

1. Introduction

The history of fractional calculus was originated in the seventeenth century, when the half-order derivative was discussed by Leibnitz in 1695. Since then, this theory became one of the interesting subjects to mathematicians as well as biologists, chemists, economists, engineers and physicists. There are several books written on this subject, for instance [3,9–11,13]. [13] is one of the most comprehensive main tools of the subject, where several types of derivatives (such as Riemann-Liouville, Hadamard, Grünwald-Letnikov, Riesz and Caputo) were introduced.

Derivatives of fractional order are defined by integrals with a fractional order kernel. Reimann-Liouville ([3,9–11,13]) and Hadamard ([1,2,7,8,12]) type fractional integrals are two of the most studied forms of fractional integrals. The Riemann–Liouville fractional integral of order α for a function f is defined by

$$\int_s^t \frac{[t-\eta]^{\alpha-1}}{\Gamma(\alpha)} f(\eta)\mathrm{d}\eta \quad \text{for } t>s \text{ and } \alpha>0, \tag{1.1}$$

which is motivated by the Cauchy integral formula

$$\int_s^t \int_s^{\eta_1} \cdots \int_s^{\eta_{n-1}} f(\eta_n)\mathrm{d}\eta_n \cdots \mathrm{d}\eta_2 \mathrm{d}\eta_1 = \int_s^t \frac{[t-\eta]^{n-1}}{\Gamma(n)} f(\eta)\mathrm{d}\eta$$

for $t>s$ and $n\in\mathbb{N}$. Another one is the Hadamard fractional integral introduced in [4], which reads as

$$\int_s^t \frac{1}{\eta_1}\int_s^{\eta_1} \frac{1}{\eta_2}\cdots \int_s^{\eta_{n-1}} \frac{f(\eta_n)}{\eta_n}\mathrm{d}\eta_n \cdots \mathrm{d}\eta_2 \mathrm{d}\eta_1 = \frac{1}{\Gamma(n)}\int_s^t \left[\ln\!\left(\frac{t}{\eta}\right)\right]^{n-1} \frac{f(\eta)}{\eta}\mathrm{d}\eta$$

for $t>s$ and $n\in\mathbb{N}$, from which the following fractional integral of f is deduced by

$$\int_s^t \frac{1}{\Gamma(\alpha)}\left[\ln\!\left(\frac{t}{\eta}\right)\right]^{\alpha-1} \frac{f(\eta)}{\eta}\mathrm{d}\eta \quad \text{for } t>s \text{ and } \alpha>0. \tag{1.2}$$

In [5], Katugampola unified the Reimann-Liouville fractional integral and the Hadamard fractional integral by

$$\int_s^t \frac{[t^\rho-\eta^\rho]^{\alpha-1}\eta^{\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)} f(\eta)\mathrm{d}\eta \quad \text{for } t>s \text{ and } \alpha>0, \tag{1.3}$$

Received 25th October, 2016; accepted 10th January, 2017; published 1st March, 2017.
2010 *Mathematics Subject Classification.* Primary 26A33; Secondary 33E12, 34A08, 34K37.
Key words and phrases. Reimann-Liouville fractional calculus; Hadamard fractional calculus; existence and uniqueness; dependence on initial conditions; Green's function; variation of parameters formula.

where $\rho > 0$, which we will call as the Reimann-Liouville-Hadamard (RLH) fractional integral. As $\lim_{\rho\to 0^+} \frac{t^\rho - s^\rho}{\rho} = \ln(\frac{t}{s})$, we see that (1.3) with $\rho = 1$ and $\rho \to 0^+$ contains (1.1) and (1.2), respectively. This fractional integral has also been extended to fractional derivative in [6], which holds "nicely" for α with $\mathrm{Re}(\alpha) \in (0,1)$ (see [6, § 3]). Motivated by the definition of fractional order derivatives given in [6], we will give a new extended the definition for arbitrary positive numbers. Based on this fractional derivative, we will study important properties of the fractional differential equations (FDE) defined with this new type of derivatives. The paper covers the following concepts:

- Existence and uniqueness of solutions to FDEs
- Solutions of autonomous FDEs
- Dependence of solutions on the initial conditions
- Green's function for RLH FDEs
- Variation of parameters formula

The paper is organized as follows. In § 2, we give the basic definitions and related auxiliary results. § 3 includes the fundamental properties of the fractional integral/derivative, which will be required in the latter sections. In § 4, we will provide existence and uniqueness for solutions of differential equations of the new type of fractional derivative. By using direct substitution technique and the Picard iterates, we will consider autonomous type fractional differential equations in § 5. In § 6, we will provide a result on dependence of the initial conditions. § 7 is dedicated to the concept of Green's function and the variation of parameters formula for the new type fractional differential equations. Finally, in § 8, we present some directions for future research and make our final discussion to conclude the paper.

2. Definitions and Auxiliary Results

Let us first introduce the kernel function $\mathcal{K}_\rho^\alpha : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$, where $\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$ and $\rho \in \mathbb{R}^+$, defined by

$$\mathcal{K}_\rho^\alpha(t,s) := \frac{[t^\rho - s^\rho]^{\alpha-1} s^{\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \quad \text{for } s,t\in\mathbb{R}. \tag{2.1}$$

We assume for convenience $\mathcal{K}_\rho^\alpha(t,s) \equiv 0$ for $\alpha \in \mathbb{Z}_0^-$. Also, for $n \in \mathbb{N}$, $k \in \{1,2,\cdots,n\}$ and $\rho \in \mathbb{R}^+$, we let

$$A_{n,k}(\rho) := \begin{cases} [1-(n-1)\rho]A_{n-1,1}(\rho), & k = 1\\ A_{n-1,k-1}(\rho) + [k-(n-1)\rho]A_{n-1,k}(\rho), & k = 2,3,\cdots,n-1\\ 1, & k = n. \end{cases} \tag{2.2}$$

Definition 2.1 (Cf. [5]). *Let $\alpha \in \mathbb{R}$, $\rho \in \mathbb{R}^+$ and $f : (0,\infty) \to \mathbb{R}$. We define the α-order fractional integration of f by*

$$\big[\mathcal{J}_\rho^\alpha f\big](t) := \begin{cases} \displaystyle\int_0^t \mathcal{K}_\rho^\alpha(t,\eta) f(\eta)\mathrm{d}\eta, & \alpha \in \mathbb{R}\backslash\mathbb{Z}_0^-\\ f(t), & \alpha = 0\\ \displaystyle\sum_{i=1}^{(-\alpha)} \frac{A_{(-\alpha),i}(\rho)}{t^{(-\alpha)\rho - i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t), & \alpha \in \mathbb{Z}^- \end{cases} \tag{2.3}$$

for $t > 0$.

Remark 2.1. *One can show that $A_{n,k}(1) = \delta_{n,k}$, where δ is Kronecker's delta. Hence, $\mathcal{J}_\rho^\alpha f = f^{(-\alpha)}$ for $\alpha \in \mathbb{Z}^-$.*

Example 2.1. *For $\alpha \in \mathbb{R}_0^+$, $\nu \in (-1,\infty)$ and $\rho \in \mathbb{R}^+$, we have*

$$\big[\mathcal{J}_\rho^\alpha *^{\rho\nu}\big](t) = \frac{\Gamma(\nu+1)}{\rho^\alpha\Gamma(\nu+\alpha+1)} t^{\rho(\nu+\alpha)} \quad \textit{for } t > 0.$$

The proof is trivial for $\alpha = 0$. We let $\alpha \in \mathbb{R}^+$ and compute for $t > 0$ that

$$\begin{aligned}\big[\mathcal{J}^{\alpha} *^{\rho\nu}\big](t) =& \int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\eta^{\rho\nu}\mathrm{d}\eta = \int_0^t \frac{[t^\rho - \eta^\rho]^{\alpha-1}\eta^{\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)}\eta^{\rho\nu}\mathrm{d}\eta \\ =& \frac{t^{\rho(\alpha+\nu)}}{\rho^\alpha\Gamma(\alpha)}\int_0^1 [1-\zeta]^{\alpha-1}\zeta^\nu \mathrm{d}\zeta = \frac{t^{\rho(\alpha+\nu)}}{\rho^\alpha\Gamma(\alpha)}\mathrm{B}(\alpha,\nu+1) \\ =& \frac{\Gamma(\nu+1)}{\rho^\alpha\Gamma(\alpha+\nu+1)}t^{\rho(\alpha+\nu)}.\end{aligned}$$

We proceed by recalling some important properties of the kernel $\mathcal{K}$.

Lemma 2.1. *The following basic properties of the kernel $\mathcal{K}$ are true.*

(i) $\displaystyle\int_s^t \mathcal{K}_\rho^\alpha(t,\eta)\mathcal{K}_\rho^\beta(\eta,s)\mathrm{d}\eta = \mathcal{K}_\rho^{\alpha+\beta}(t,s)$ *for $t \geq s \geq 0$ and $\alpha, \beta \in \mathbb{R}^+$.*

(ii) $t^{\rho-1}\mathcal{K}_\rho^\alpha(t,s) = (-1)^\alpha s^{\rho-1}\mathcal{K}_\rho^\alpha(s,t)$ *for $s,t \in \mathbb{R}$ and $\alpha \in \mathbb{C}$.*

(iii) $\displaystyle\frac{\partial}{\partial t}\mathcal{K}_\rho^{\alpha+1}(t,s) = t^{\rho-1}\mathcal{K}_\rho^\alpha(t,s)$ *for $s,t \in \mathbb{R}$ and $\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$.*

(iv) $\displaystyle\frac{\partial}{\partial s}\frac{\mathcal{K}_\rho^{\alpha+1}(t,s)}{s^{\rho-1}} = -\mathcal{K}_\rho^\alpha(t,s)$ *for $s \in \mathbb{R}\backslash\{0\}$, $t \in \mathbb{R}$ and $\alpha \in \mathbb{C}\backslash\mathbb{Z}_0^-$.*

Proof. (i) Then, we compute for $t \geq s \geq 0$ that

$$\begin{aligned}\int_s^t \mathcal{K}_\rho^\alpha(t,\eta)\mathcal{K}_\rho^\beta(\eta,s)\mathrm{d}\eta =& \int_s^t \frac{[t^\rho-\eta^\rho]^{\alpha-1}\eta^{\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)}\frac{[\eta^\rho-s^\rho]^{\beta-1}s^{\rho-1}}{\rho^{\beta-1}\Gamma(\beta)}\mathrm{d}\eta \\ =& \frac{1}{\rho^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)}\int_s^t [t^\rho-\eta^\rho]^{\alpha-1}\eta^{\rho-1}[\eta^\rho-s^\rho]^{\beta-1}s^{\rho-1}\mathrm{d}\eta \\ =& \frac{[t^\rho-s^\rho]^{\alpha+\beta-1}s^{\rho-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}\int_0^1 [1-\zeta]^{\alpha-1}\zeta^{\beta-1}\mathrm{d}\zeta \\ =& \frac{[t^\rho-s^\rho]^{\alpha+\beta-1}s^{\rho-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha)\Gamma(\beta)}\mathrm{B}(\alpha,\beta) \\ =& \frac{[t^\rho-s^\rho]^{\alpha+\beta-1}s^{\rho-1}}{\rho^{\alpha+\beta-1}\Gamma(\alpha+\beta)} = \mathcal{K}_\rho^{\alpha+\beta}(t,s).\end{aligned}$$

(ii) The proof is trivial and thus we omit it here.

(iii) For $t \geq s \geq 0$, we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{K}_\rho^{\alpha+1}(t,s) =& \frac{\partial}{\partial t}\frac{[t^\rho-s^\rho]^\alpha s^{\rho-1}}{\rho^\alpha\Gamma(\alpha+1)} = \frac{\alpha\rho t^{\rho-1}[t^\rho-s^\rho]^{\alpha-1}s^{\rho-1}}{\rho^\alpha\Gamma(\alpha+1)} \\ =& \frac{t^{\rho-1}[t^\rho-s^\rho]^{\alpha-1}s^{\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)} = t^{\rho-1}\mathcal{K}_\rho^\alpha(t,s).\end{aligned}$$

(iv) The proof can be given similar to that of (iii). □

Lemma 2.2. *Note that for $\alpha \in \mathbb{R}$, we have*

$$\big[\mathcal{J}_\rho^\alpha f\big](t) = \frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{J}_\rho^{\alpha+1}f](t) \quad \text{for } t > 0. \tag{2.4}$$

Proof. We proceed with the following three distinct cases.

- Let $\alpha \in \mathbb{R}\backslash\mathbb{Z}^-$. Then, we have for $t>0$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{\alpha+1}f\big](t) =&\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \mathcal{K}_\rho^{\alpha+1}(t,\eta)f(\eta)\mathrm{d}\eta = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{[t^\rho-\eta^\rho]^\alpha\eta^{\rho-1}}{\rho^\alpha\Gamma(\alpha+1)}f(\eta)\mathrm{d}\eta\\ =&\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\frac{[t^\rho-\eta^\rho]^\alpha\eta^{\rho-1}}{\rho^\alpha\Gamma(\alpha+1)}f(\eta)\mathrm{d}\eta + \frac{[t^\rho-t^\rho]^\alpha\eta^{\rho-1}}{\rho^\alpha\Gamma(\alpha+1)}f(t)\\ =&\alpha\rho t^{\rho-1}\int_0^t \frac{[t^\rho-\eta^\rho]^{\alpha-1}\eta^{\rho-1}}{\rho^\alpha\Gamma(\alpha+1)}f(\eta)\mathrm{d}\eta\\ =&t^{\rho-1}\int_0^t \frac{[t^\rho-\eta^\rho]^{\alpha-1}\eta^{\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)}f(\eta)\mathrm{d}\eta\\ =&t^{\rho-1}\big[\mathcal{J}_\rho^\alpha f\big](t).\end{aligned}$$

- Let $\alpha=-1$. Then,

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathcal{J}_\rho^0 f](t) = \frac{\mathrm{d}}{\mathrm{d}t}f(t) = t^{\rho-1}\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}f(t) = t^{\rho-1}\big[\mathcal{J}_\rho^{-1}f\big](t) \quad \text{for } t>0.$$

- Let $\alpha \in \{\cdots,-3,-2\}$. Then, putting $n:=-\alpha$ for simplicity, we compute for $t>0$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{\alpha+1}f\big](t) =&\frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n-1}\frac{A_{n-1,i}}{t^{(n-1)\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t) = \sum_{i=1}^{n-1}\frac{\mathrm{d}}{\mathrm{d}t}\frac{A_{n-1,i}}{t^{(n-1)\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)\\ =&\sum_{i=1}^{n-1}\left[\frac{A_{n-1,i}}{t^{(n-1)\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{i+1} - [(n-1)\rho-i]\frac{A_{n-1,i}}{t^{(n-1)\rho-i+1}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\right]f(t)\\ =&t^{\rho-1}\sum_{i=1}^{n-1}\left[\frac{A_{n-1,i}}{t^{n\rho-(i+1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{i+1} - [(n-1)\rho-i]\frac{A_{n-1,i}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\right]f(t)\\ =&t^{\rho-1}\left[\sum_{i=1}^{n-1}\frac{A_{n-1,i}}{t^{n\rho-(i+1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{i+1}f(t) - \sum_{i=1}^{n-1}[(n-1)\rho-i]\frac{A_{n-1,i}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)\right]\\ =&t^{\rho-1}\left[\sum_{i=2}^{n}\frac{A_{n-1,i-1}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{i}f(t) - \sum_{i=1}^{n-1}[(n-1)\rho-i]\frac{A_{n-1,i}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)\right]\\ =&t^{\rho-1}\left[\frac{A_{n-1,n-1}}{t^{n(\rho-1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t) + \sum_{i=2}^{n-1}\frac{A_{n-1,i-1}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)\right.\\ &\left.\quad - \sum_{i=2}^{n-1}\big((n-1)\rho-i\big)\frac{A_{n-1,i}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t) - [(n-1)\rho-1]\frac{A_{n-1,1}}{t^{n\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right]\\ =&t^{\rho-1}\left[\frac{A_{n-1,n-1}}{t^{n(\rho-1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t) + \sum_{i=2}^{n-1}\big[A_{n-1,i-1}-[(n-1)\rho-i]A_{n-1,i}\big]\frac{1}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t)\right.\\ &\left.\quad - [(n-1)\rho-1]\frac{A_{n-1,1}}{t^{n\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right]\\ =&t^{\rho-1}\left[\frac{A_{n,n}}{t^{n(\rho-1)}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t) + \sum_{i=2}^{n-1}\frac{A_{n,i}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t) + \frac{A_{n,1}}{t^{n\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right]\\ =&t^{\rho-1}\sum_{i=1}^{n}\frac{A_{n,i}}{t^{n\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i f(t) = t^{\rho-1}\big[\mathcal{J}_\rho^\alpha f\big](t).\end{aligned}$$

The proof is completed by considering the three cases above. □

Motivated by Lemma 2.2, we suggest the following definition for the fractional derivative of a function.

Definition 2.2. *Let $\alpha\in\mathbb{R}$, $\rho\in\mathbb{R}^+$ and $f:(0,\infty)\to\mathbb{R}$. We define the α-order fractional derivative of f iteratedly by*

$$\big[\mathcal{D}_\rho^\alpha f\big](t):=\begin{cases}\big[\mathcal{J}_\rho^{-\alpha}f\big](t), & \alpha\in\mathbb{R}_0^-\\ \dfrac{1}{t^{\rho-1}}\dfrac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\alpha-1}f\big](t), & \alpha\in\mathbb{R}^+.\end{cases}$$

Example 2.2. *For $\alpha,\nu\in\mathbb{R}_0^+$ and $\rho\in\mathbb{R}^+$, we have*

$$\big[\mathcal{D}_\rho^\alpha *^{\rho\nu}\big](t)=\frac{\rho^\alpha\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\rho(\nu-\alpha)}\quad \text{for } t>0. \tag{2.5}$$

We will prove this by applying induction on $n\in\mathbb{Z}_0^+$ for $\alpha\in[n,n+1)$. First, let $\alpha\in[0,1)$, then we have

$$\begin{aligned}\big[\mathcal{D}_\rho^\alpha *^{\rho\nu}\big](t)=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\alpha-1} *^{\rho\nu}\big](t)=\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{1-\alpha} *^{\rho\nu}\big](t)\\ =&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\Gamma(\nu+1)}{\rho^{1-\alpha}\Gamma\big(\nu+(1-\alpha)+1\big)}t^{\rho(\nu+(1-\alpha))}\\ =&\frac{\rho^\alpha\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\rho(\nu-\alpha)}\end{aligned}$$

for $t>0$, where we have applied Example 2.1 in the second line above. This proves validity of (2.5) for all $\alpha\in[0,1)$. Let $n\in\mathbb{Z}_0^+$, and assume now for all $\alpha\in[n,n+1)$ that (2.5) is true. By Definition 2.2, we have for any $\alpha\in[n+1,n+2)$ that

$$\begin{aligned}\big[\mathcal{D}_\rho^\alpha *^{\rho\nu}\big](t)=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\alpha-1} *^{\rho\nu}\big](t)=\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho^{\alpha-1}\Gamma(\nu+1)}{\Gamma\big(\nu-(\alpha-1)+1\big)}t^{\rho(\nu-(\alpha-1))}\\ =&\frac{\rho^\alpha\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}t^{\rho(\nu-\alpha)}\end{aligned}$$

for $t>0$, which completes the proof.

In the following lemma, we provide a direct form for the definition of the fractional derivative in terms of the coefficients defined in (2.2).

Lemma 2.3. *For $\alpha,\rho\in\mathbb{R}^+$, we have*

$$\big[\mathcal{D}_\rho^\alpha f\big](t)=\sum_{i=1}^{\lceil\alpha\rceil}\frac{A_{\lceil\alpha\rceil,i}(\rho)}{t^{\lceil\alpha\rceil\rho-i}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^i\big[\mathcal{J}_\rho^{\lceil\alpha\rceil-\alpha}f\big](t)\quad \text{for } t>0.$$

Proof. We compute that

$$\begin{aligned}\big[\mathcal{D}_\rho^\alpha f\big](t)=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\alpha-1}f\big](t)=\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\alpha-2}f\big](t)\right]\\ =&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\cdots\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\alpha-\lceil\alpha\rceil}f\big](t)\cdots\right]\right]\\ =&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\cdots\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{\lceil\alpha\rceil-\alpha}f\big](t)\cdots\right]\right],\end{aligned}$$

where we have for a total of $\lceil\alpha\rceil$ usual derivatives above. Let us denote $g:=\mathcal{J}_\rho^{\lceil\alpha\rceil-\alpha}f$ and use (2.4) repeatedly inside to outside, then

$$\begin{aligned}\big[\mathcal{D}_\rho^\alpha f\big](t)=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\cdots\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{0}g\big](t)\cdots\right]\right]\\ =&\cdots=\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{-\lceil\alpha\rceil+1}g\big](t)=\big[\mathcal{J}_\rho^{-\lceil\alpha\rceil}g\big](t),\end{aligned}$$

which completes the proof by using (2.3). ☐

3. Properties of the Operators $\mathcal{J}$ and $\mathcal{D}$

The main result of this section is the following theorem.

Theorem 3.1. *The following properties hold.*

(i) $\mathcal{D}_\rho^\alpha = \mathcal{J}_\rho^{-\alpha}$ *for* $\alpha \in \mathbb{R}$.

(ii) $\mathcal{J}_\rho^\alpha \mathcal{J}_\rho^\beta = \mathcal{J}_\rho^{\alpha+\beta}$ *for* $\alpha, \beta \in \mathbb{R}_0^+$.

(iii) $\mathcal{D}_\rho^\alpha \mathcal{D}_\rho^\beta = \mathcal{D}_\rho^{\alpha+\beta}$ *for* $\alpha \in \mathbb{Z}_0^+$ *and* $\beta \in \mathbb{R}_0^+$ *or for* $\alpha, \beta \in \mathbb{R}^+ \backslash \mathbb{Z}^+$ *with* $(\alpha+\beta) \notin \mathbb{Z}^+$.

(iv) $\mathcal{D}_\rho^\alpha \mathcal{J}_\rho^\alpha = \mathrm{I}$ *for* $\alpha \in \mathbb{R}$, *where* I *is the identity operator.*

(v) $\big[\mathcal{J}_\rho^\alpha \mathcal{D}_\rho^\alpha f\big](t) = f(t) - \sum_{i=1}^{\lceil \alpha \rceil} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)} \big[\mathcal{D}_\rho^{\alpha-i} f\big](0^+)$ *for* $\alpha \in \mathbb{R}^+$.

Proof. (i) For $\alpha \in \mathbb{R}_0^+$, the proof is similar to that of Lemma 2.3, and for $\alpha \in \mathbb{R}^-$, the proof follows from Definition 2.2.

(ii) The proof is trivial for $\alpha = 0$ or $\beta = 0$. Hence, we consider below the case where $\alpha, \beta \in \mathbb{R}^+$. Then,

$$\begin{aligned}
\big[\mathcal{J}_\rho^\alpha \mathcal{J}_\rho^\beta f\big](t) &= \int_0^t \mathcal{K}_\rho^\alpha(t,\eta) \int_0^\eta \mathcal{K}_\rho^\beta(\eta,\zeta) f(\zeta) \mathrm{d}\zeta \mathrm{d}\eta \\
&= \int_0^t \int_0^\eta \mathcal{K}_\rho^\alpha(t,\eta) \mathcal{K}_\rho^\beta(\eta,\zeta) f(\zeta) \mathrm{d}\zeta \mathrm{d}\eta \\
&= \int_0^t \int_\zeta^t \mathcal{K}_\rho^\alpha(t,\eta) \mathcal{K}_\rho^\beta(\eta,\zeta) f(\zeta) \mathrm{d}\eta \mathrm{d}\zeta \\
&= \int_0^t \left[\int_\zeta^t \mathcal{K}_\rho^\alpha(t,\eta) \mathcal{K}_\rho^\beta(\eta,\zeta) \mathrm{d}\eta\right] f(\zeta) \mathrm{d}\zeta \\
&= \int_0^t \mathcal{K}_\rho^{\alpha+\beta}(t,\zeta) f(\zeta) \mathrm{d}\zeta = \big[\mathcal{J}_\rho^{\alpha+\beta} f\big](t),
\end{aligned}$$

where we have applied Lemma 2.1 (i) for the last line.

(iii) The proof is trivial for $\alpha = 0$ or $\beta = 0$. Below, we consider the case where $\alpha \neq 0$ and $\beta \neq 0$.

(a) For $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{R}^+$, then

$$\begin{aligned}
\big[\mathcal{D}_\rho^\alpha \mathcal{D}_\rho^\beta f\big](t) =& \frac{1}{t^{\rho-1}} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{1}{t^{\rho-1}} \frac{\mathrm{d}}{\mathrm{d}t} \left[\cdots \frac{1}{t^{\rho-1}} \frac{\mathrm{d}}{\mathrm{d}t} \big[\mathcal{D}_\rho^\beta f\big](t) \cdots \right]\right] \\
=& \cdots = \frac{1}{t^{\rho-1}} \frac{\mathrm{d}}{\mathrm{d}t} \big[\mathcal{D}_\rho^{\beta+(\alpha-1)} f\big](t) = \big[\mathcal{D}_\rho^{\beta+\alpha} f\big](t).
\end{aligned}$$

(b) For $\alpha, \beta \in \mathbb{R}^+ \backslash \mathbb{Z}^+$ with $(\alpha+\beta) \notin \mathbb{Z}^+$, then $\mathcal{D}_\rho^\alpha \mathcal{D}_\rho^\beta = \mathcal{D}_\rho^{\alpha+\beta}$ as in (ii).

(iv) This follows from (i) by using the steps in the proof of (ii) and (iii).

(v) Performing integration by parts, for $\alpha \in \mathbb{R}^+$, we obtain

$$\begin{aligned}
\big[\mathcal{J}_\rho^{\alpha+1} \mathcal{D}_\rho^\alpha f\big](t) =& \left[\mathcal{J}_\rho^{\alpha+1} \frac{1}{*^{\rho-1}} \big[\mathcal{D}_\rho^{\alpha-1} f\big]'\right](t) = \int_0^t \frac{\mathcal{K}_\rho^{\alpha+1}(t,\eta)}{\eta^{\rho-1}} \big[\mathcal{D}_\rho^{\alpha-1} f\big]'(\eta) \mathrm{d}\eta \\
=& \frac{\mathcal{K}_\rho^{\alpha+1}(t,\eta)}{\eta^{\rho-1}} \big[\mathcal{D}_\rho^{\alpha-1} f\big](\eta) \bigg|_{\eta=0^+}^{\eta=t} - \int_0^t \frac{\partial}{\partial \eta} \left(\frac{\mathcal{K}_\rho^{\alpha+1}(t,\eta)}{\eta^{\rho-1}}\right) \big[\mathcal{D}_\rho^{\alpha-1} f\big](\eta) \mathrm{d}\eta \\
=& - \frac{[t^\rho - \eta^\rho]^\alpha}{\rho^\alpha \Gamma(\alpha+1)} \big[\mathcal{D}_\rho^{\alpha-1} f\big](\eta) \bigg|_{\eta=0^+}^{\eta=t} + \int_0^t \mathcal{K}_\rho^\alpha(t,\eta) \big[\mathcal{D}_\rho^{\alpha-1} f\big](\eta) \mathrm{d}\eta \\
=& \big[\mathcal{J}_\rho^\alpha \mathcal{D}_\rho^{\alpha-1} f\big](t) - \frac{t^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \big[\mathcal{D}_\rho^{\alpha-1} f\big](0^+)
\end{aligned}$$

where $*'$ in the first line stands for the usual derivative. Using (ii), we get

$$\big[\mathcal{J}_\rho^1 \mathcal{J}_\rho^\alpha \mathcal{D}_\rho^\alpha f\big](t) = \big[\mathcal{J}_\rho^1 \mathcal{J}_\rho^{\alpha-1} \mathcal{D}_\rho^{\alpha-1} f\big](t) - \frac{t^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \big[\mathcal{D}_\rho^{\alpha-1} f\big](0^+).$$

An application of $\mathcal{D}_\rho^1$ on both sides yields by using (iv) that

$$\big[\mathcal{J}_\rho^\alpha \mathcal{D}_\rho^\alpha f\big](t) = \big[\mathcal{J}_\rho^{\alpha-1} \mathcal{D}_\rho^{\alpha-1} f\big](t) - \frac{t^{\rho(\alpha-1)}}{\rho^{\alpha-1}\Gamma(\alpha)}\big[\mathcal{D}_\rho^{\alpha-1} f\big](0^+).$$

Repeating this procedure for a total of $\lceil\alpha\rceil$ times, we get

$$\big[\mathcal{J}_\rho^\alpha \mathcal{D}_\rho^\alpha f\big](t) = \big[\mathcal{J}_\rho^{\alpha-\lceil\alpha\rceil} \mathcal{D}_\rho^{\alpha-\lceil\alpha\rceil} f\big](t) - \sum_{i=1}^{\lceil\alpha\rceil} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)}\big[\mathcal{D}_\rho^{\alpha-i} f\big](0^+) \tag{3.1}$$

for all $t > 0$. By Definition 2.1, Definition 2.2 and (ii), we have

$$\mathcal{J}_\rho^{\alpha-\lceil\alpha\rceil} \mathcal{D}_\rho^{\alpha-\lceil\alpha\rceil} = \begin{cases} \mathcal{J}_\rho^0 \mathcal{D}_\rho^0 = \mathrm{I}, & \alpha \in \mathbb{N} \\ \mathcal{J}_\rho^{\alpha-\lceil\alpha\rceil} \mathcal{J}_\rho^{\lceil\alpha\rceil-\alpha} = \mathcal{J}_\rho^0 = \mathrm{I}, & \alpha \in \mathbb{R}^+\backslash\mathbb{N}, \end{cases}$$

which completes the proof by using this in (3.1).

Thus, we have justified the validity of each of the properties above, and completed the proof. $\square$

4. Existence and Uniqueness for RLH FDEs

Let us consider the initial-value problem

$$\begin{cases} \big[\mathcal{D}_\rho^\alpha y\big](t) = f\big(t, y(t)\big) & \text{for } t > 0 \\ \big[\mathcal{D}_\rho^{\alpha-k} y\big](0^+) = y_{\lceil\alpha\rceil-k} & \text{for } k = 1, 2, \cdots, \lceil\alpha\rceil, \end{cases} \tag{4.1}$$

where $\alpha \in \mathbb{R}^+$ and $y_0, y_1, \cdots, y_{\lceil\alpha\rceil-1} \in \mathbb{R}$. Suppose that f is defined in a domain Ω of a plane (t, y), and define a region $R(h, K) \subset \Omega$ as a set of points $(t, y) \in \Omega$, which satisfy the inequality

$$\left| y(t) - \sum_{i=1}^{\lceil\alpha\rceil} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)} \right| \leq K \quad \text{for all } t \in (0, h),$$

where h and K are constants.

Theorem 4.1. *Let $f : \Omega \to \mathbb{R}$ satisfy the Lipschitz condition with respect to its second component, i.e.,*

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \quad \textit{for all } (t, y_1), (t, y_2) \in \Omega, \quad \textit{where} \quad L \in \mathbb{R}^+,$$

and f be bounded on Ω, i.e.,

$$|f(t, y)| \leq M \quad \textit{for all } (t, y) \in \Omega, \quad \textit{where} \quad M \in \mathbb{R}^+.$$

Further, assume that there exist $h, K \in \mathbb{R}^+$ such that

$$\frac{Mh^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \leq K.$$

Then, there exists a unique and continuous solution of the problem (4.1) in the region $R(h, K) \subset \Omega$.

Proof. The method proof based on the ideas in [11, Theorem 3.4]. First, consider Theorem 3.1 (v) and reduce the problem (4.1) to an equivalent fractional integral equation

$$y(t) = \sum_{i=1}^{\lceil\alpha\rceil} \frac{y_{\lceil\alpha\rceil-i}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)} t^{\rho(\alpha-i)} + \int_0^t \mathcal{K}_\rho^\alpha(t, \eta) f\big(\eta, y(\eta)\big) \mathrm{d}\eta \quad \text{for } t \in (0, h]. \tag{4.2}$$

If y satisfies (4.1), then it also satisfies the equation (4.2). On the other hand, if y is a solution of (4.2), then it is satisfies (4.1) initial-value problem. Therefore, the equation (4.2) is equivalent to the initial value problem (4.1). Now, let us define the sequence of functions $\{y_m\}_{m\in\mathbb{N}_0}$ by

$$y_m(t) = \begin{cases} \displaystyle\sum_{i=1}^{\lceil\alpha\rceil} \frac{y_{\lceil\alpha\rceil-i}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)} t^{\rho(\alpha-i)}, & m = 0 \\ y_0(t) + \big[\mathcal{J}_\rho^\alpha f\big(*, y_{m-1}(*)\big)\big](t), & m \in \mathbb{N} \end{cases} \tag{4.3}$$

for $t \in (0, h]$. We will show that $\lim_{m\to\infty} y_m$ exists and gives the required solution y of the integral equation (4.2). First, it can be shown by induction that $y_m(t) \in R(h, K)$ for all $t \in (0, h]$ and $m \in \mathbb{N}_0$. Indeed, for all $t \in (0, h]$ and all $m \in \mathbb{N}_0$, we obtain

$$\begin{aligned}|y_m(t) - y_0(t)| =& \left|\int_0^t \mathcal{K}_\rho^\alpha(t,\eta) f\big(\eta, y_{m-1}(\eta)\big)\mathrm{d}\eta\right| \le \int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\big|f\big(\eta, y_{m-1}(\eta)\big)\big|\mathrm{d}\eta \\ \le& M\int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\mathrm{d}\eta = \frac{Mt^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \le \frac{Mh^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \le K\end{aligned}$$

and thus

$$|y_1(t) - y_0(t)| \le \frac{Mh^{\rho\alpha}}{\rho^\alpha\Gamma(\alpha+1)} \le K \quad \text{for all } t \in (0, h]. \tag{4.4}$$

Let us show by induction that

$$|y_m(t) - y_{m-1}(t)| \le \frac{ML^{m-1}t^{m\rho\alpha}}{\rho^{m\alpha}\Gamma(m\alpha+1)} \quad \text{for all } t \in (0, h] \text{ and all } m \in \mathbb{N}. \tag{4.5}$$

It follows from (4.4) that (4.5) holds for $m = 1$. Suppose for some $m \in \mathbb{N}$ that

$$|y_m(t) - y_{m-1}(t)| \le \frac{ML^{m-1}t^{m\rho\alpha}}{\rho^{m\alpha}\Gamma(m\alpha+1)} \quad \text{for all } t \in (0, h]. \tag{4.6}$$

Then, using (4.3) and (4.6), we have

$$\begin{aligned}|y_{m+1}(t) - y_m(t)| =& \left|\int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\big[f\big(\eta, y_m(\eta)\big) - f\big(\eta, y_{m-1}(\eta)\big)\big]\mathrm{d}\eta\right| \\ \le& \int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\big|f\big(\eta, y_m(\eta)\big) - f\big(\eta, y_{m-1}(\eta)\big)\big|\mathrm{d}\eta \\ \le& L\int_0^t \mathcal{K}_\rho^\alpha(t,\eta)|y_m(\eta) - y_{m-1}(\eta)|\mathrm{d}\eta \\ \le& \frac{ML^m}{\rho^{m\alpha}\Gamma(m\alpha+1)}\int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\eta^{m\rho\alpha}\mathrm{d}\eta \\ =& \frac{ML^m}{\rho^{(m+1)\alpha t^{(m+1)\rho\alpha}}\Gamma(\alpha)\Gamma(m\alpha+1)}\int_0^1 [1-\zeta]^{\alpha-1}\zeta^{m\alpha}\mathrm{d}\zeta \\ =& \frac{ML^m t^{(m+1)\rho\alpha}}{\rho^{(m+1)\alpha}\Gamma(\alpha)\Gamma(m\alpha+1)}\int_0^1 [1-\zeta]^{\alpha-1}\zeta^{m\alpha}\mathrm{d}\zeta \\ =& \frac{ML^m t^{(m+1)\rho\alpha}}{\rho^{(m+1)\alpha}\Gamma(\alpha)\Gamma(m\alpha+1)}\mathrm{B}(\alpha, m\alpha+1) \\ =& \frac{ML^m t^{(m+1)\rho\alpha}}{\rho^{(m+1)\alpha}\Gamma\big((m+1)\alpha+1\big)}\end{aligned}$$

for all $t \in (0, h]$. This means that (4.5) is true.

Let us consider the limiting function

$$y(t) := \lim_{m\to\infty} y_m(t) = y_0(t) + \sum_{j=1}^{\infty}[y_j(t) - y_{j-1}(t)] \quad \text{for } t \in (0, h]. \tag{4.7}$$

According to the estimate (4.5), for $t \in (0, h]$, the absolute value of its terms is less than the corresponding terms of the convergent numeric series

$$\begin{aligned}\sum_{j=1}^{\infty}|y_j(t) - y_{j-1}(t)| \le& \sum_{j=1}^{\infty}\frac{ML^{j-1}h^{j\rho\alpha}}{\rho^{j\alpha}\Gamma(j\alpha+1)} = \frac{M}{L}\sum_{j=1}^{\infty}\frac{L^j h^{j\rho\alpha}}{\rho^{j\alpha}\Gamma(j\alpha+1)} \\ =& \frac{M}{L}\left[E_{\alpha,1}\left(\frac{Lh^{\rho\alpha}}{\rho^\alpha}\right) - 1\right],\end{aligned}$$

where E is the two-parameter Mittag-Leffler function defined by

$$\mathrm{E}_{\alpha,\beta}(z) := \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(\alpha j+\beta)} \quad \text{for } z \in \mathbb{C} \text{ and } \alpha, \beta \in \mathbb{C}, \tag{4.8}$$

which converges for all values of z (i.e., it is an entire function). This means that the series (4.7) converges uniformly. Letting $m \to \infty$ in (4.3) and using (4.7), we get

$$y(t) = y_0(t) + \int_0^t \mathcal{K}_\rho^\alpha(t,\eta) f\big(\eta, y(\eta)\big) \mathrm{d}\eta \quad \text{for all } t \in (0,h].$$

Therefore, y defined by (4.7) is a solution of (4.2), and thus (4.1).

What follows next is to prove the uniqueness of the solution. Let us suppose that z is another solution of the equation (4.2), which is continuous in the interval $(0,h]$. Then $w(t) := y(t) - z(t)$ for $t \in (0,h]$, then satisfies the equation

$$w(t) = \int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\big[f\big(\eta, y(\eta)\big) - f\big(\eta, z(\eta)\big)\big] \mathrm{d}\eta \tag{4.9}$$

from which it follows that $w(0^+) = 0$. Therefore, w extends continuously to $[0,h]$. Then, $|w(t)| \leq C$ for all $t \in (0,h]$, where $C \in \mathbb{R}^+$, and we obtain from (4.9) that

$$|w(t)| \leq \frac{CLt^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} \quad \text{for all } t \in (0,h].$$

Repeating this procedure for a total of $m \in \mathbb{N}$ times, we obtain

$$|w(t)| \leq \frac{CL^m t^{m\rho\alpha}}{\rho^{m\alpha} \Gamma(m\alpha+1)} \quad \text{for all } t \in (0,h].$$

In the right-hand side, we recognize the general term of the series for the Mittag-Leffler function $\mathrm{E}_{\alpha,1}(\frac{Lt^{\rho\alpha}}{\rho^\alpha})$, and therefore

$$\lim_{m\to\infty} \frac{L^m t^{m\rho\alpha}}{\rho^{m\alpha}\Gamma(m\alpha+1)} = 0 \quad \text{for all } t \in (0,h].$$

Then, we have $w(t) \equiv 0$ for all $t \in (0,h]$, and thus $y(t) \equiv z(t)$ for all $t \in (0,h]$. This ends the proof. □

5. The Autonomous Equation of RL Type

Let us consider initial-value problem

$$\begin{cases} \big[\mathcal{D}_\rho^\alpha y\big](t) = \lambda y(t) & \text{for } t > 0 \\ \big[\mathcal{D}_\rho^{\alpha-k} y\big](0^+) = y_{\lceil\alpha\rceil - k} & \text{for } k = 1, 2, \cdots, \lceil\alpha\rceil, \end{cases} \tag{5.1}$$

where $\lambda \in \mathbb{R}$. In this case, when compared to (4.1), we have $f(t,y) = \lambda y$. Now, we will introduce two techniques for obtaining the unique solution of (5.1).

5.1. **Direct Substitution.** Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ and define

$$y_{\alpha,\beta}(t) := \frac{t^{\rho\beta}}{\rho^\beta} \mathrm{E}_{\alpha,\beta+1}\left(\frac{\lambda t^{\rho\alpha}}{\rho^\alpha}\right) \quad \text{for } t > 0,$$

where E is the two-parameter Mittag-Leffler function defined in (4.8).

Then, $\mathcal{D}_\rho^\alpha y = \lambda y$. Indeed, we have

$$\begin{aligned}
\big[\mathcal{D}_\rho^\alpha y_{\alpha,\beta}\big](t) =& \left[\mathcal{D}_\rho^\alpha \frac{*^{\rho\beta}}{\rho^\beta} \mathrm{E}_{\alpha,\beta+1}\left(\frac{\lambda *^{\rho\alpha}}{\rho^\alpha}\right)\right](t) = \left[\mathcal{D}_\rho^\alpha \sum_{j=0}^\infty \frac{\lambda^j *^{\rho(\alpha j+\beta)}}{\rho^{\alpha j+\beta}\Gamma(\alpha j+\beta+1)}\right](t)\\
=&\sum_{j=0}^\infty \frac{\lambda^j \big[\mathcal{D}_\rho^\alpha *^{\rho(\alpha j+\beta)}\big](t)}{\rho^{\alpha j+\beta}\Gamma(\alpha j+\beta+1)} = \sum_{j=0}^\infty \frac{\lambda^j t^{\rho(\alpha(j-1)+\beta)}}{\rho^{\alpha(j-1)+\beta}\Gamma\big(\alpha(j-1)+\beta+1\big)}\\
=&\frac{t^{\rho(\beta-\alpha)}}{\rho^{\beta-\alpha}\Gamma(\beta-\alpha+1)} + \sum_{j=1}^\infty \frac{\lambda^j t^{\rho(\alpha(j-1)+\beta)}}{\rho^{\alpha(j-1)+\beta}\Gamma\big(\alpha(j-1)+\beta+1\big)}\\
=&\frac{t^{\rho(\beta-\alpha)}}{\rho^{\beta-\alpha}\Gamma(\beta-\alpha+1)} + \lambda\frac{t^{\rho\beta}}{\rho^\beta}\sum_{j=0}^\infty \frac{\lambda^j t^{\rho\alpha j}}{\rho^{\alpha j}\Gamma(\alpha j+\beta+1)}\\
=&\frac{t^{\rho(\beta-\alpha)}}{\rho^{\beta-\alpha}\Gamma(\beta-\alpha+1)} + \lambda y_{\alpha,\beta}(t)
\end{aligned}$$

for all $t>0$. Here, we see that $y_{\alpha,\beta}$ solves $\mathcal{D}_\rho^\alpha y = \lambda y$ provided that $(\beta-\alpha)$ is a negative integer. That is, $y_{\alpha,\alpha-i}$, where $i=1,2,\cdots,\lceil\alpha\rceil$, satisfies $\mathcal{D}_\rho^\alpha y = \lambda y$. Moreover, we compute that

$$\begin{aligned}
\big[\mathcal{D}_\rho^{\alpha-k} y_{\alpha,\alpha-i}\big](t) =& \left[\mathcal{D}_\rho^{\alpha-k} \frac{*^{\rho(\alpha-i)}}{\rho^{\alpha-i}} \mathrm{E}_{\alpha,\alpha-i+1}\left(\frac{\lambda *^{\rho\alpha}}{\rho^\alpha}\right)\right](t)\\
=&\left[\mathcal{D}_\rho^{\alpha-k}\sum_{j=0}^\infty \frac{\lambda^j *^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)}\right](t)\\
=&\sum_{j=0}^\infty \frac{\lambda^j \big[\mathcal{D}_\rho^{\alpha-k} *^{\rho(\alpha(j+1)-i)}\big](t)}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)}\\
=&\sum_{j=0}^\infty \frac{\lambda^j t^{\rho(\alpha j-i+k)}}{\rho^{\alpha j-i+k}\Gamma(\alpha j-i+k+1)}
\end{aligned}$$

for $t>0$ and $k=1,2,\cdots,\lceil\alpha\rceil$. Using the properties of the Gamma function and considering the positive powers of t, we find that

$$\big[\mathcal{D}_\rho^{\alpha-k} y_{\alpha,\alpha-i}\big](0^+) = \delta_{i,k} \quad \text{for } k=1,2,\cdots,\lceil\alpha\rceil,$$

where δ is Kronecker's delta. Therefore, $\{y_{\alpha,\alpha-i}\}_{i=1}^{\lceil\alpha\rceil}$ is the set of normalized fundamental solutions of $\mathcal{D}_\rho^\alpha y = \lambda y$. Moreover, the following linear combination of functions

$$y(t) := \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i}\frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}} \mathrm{E}_{\alpha,\alpha-i+1}\left(\frac{\lambda t^{\rho\alpha}}{\rho^\alpha}\right) \quad \text{for } t>0$$

forms the solution desired of (5.1).

5.2. **The Picard Iterates.** In accordance with the proof of Theorem 4.1, let us take

$$y_m(t) = \begin{cases} \displaystyle\sum_{i=1}^{\lceil\alpha\rceil} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)}[\mathcal{D}_\rho^{\alpha-i}y](0^+), & m=0\\ y_0(t)+\lambda\big[\mathcal{J}_\rho^\alpha y_{m-1}\big](t), & m\in\mathbb{N}\end{cases} \tag{5.2}$$

for $t\in(0,h]$.

We will show by induction that

$$y_m(t) = \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i}\sum_{j=0}^m \frac{t^{\rho(\alpha(j+1)-i)}\lambda^j}{\rho^{(\alpha(j+1)-i)}\Gamma\big(\alpha(j+1)-i+1\big)} \tag{5.3}$$

for all $t \in (0,h]$ and $m \in \mathbb{N}_0$. The claim holds for $m=0$ by (5.2). Assume for some $m \in \mathbb{N}_0$ that

$$y_m(t) = \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \sum_{j=0}^{m} \frac{t^{\rho(\alpha(j+1)-i)}\lambda^j}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)} \quad \text{for all } t \in (0,h],$$

which together with Example 2.1 and (5.2) yields

$$\begin{aligned} y_{m+1}(t) =& y_0(t) + \lambda\big[\mathcal{J}_\rho^\alpha y_m\big](t) = y_0(t) + \lambda \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \sum_{j=0}^{m} \frac{\lambda^j\big[\mathcal{J}_\rho^\alpha *^{\rho(\alpha(j+1)-i)}\big](t)}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)} \\ =& \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)} + \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \sum_{j=0}^{m} \frac{\lambda^{j+1}t^{\rho((j+2)\alpha-i)}}{\rho^{(j+2)\alpha-i}\Gamma\big((j+2)\alpha-i+1\big)} \\ =& \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \sum_{j=0}^{m+1} \frac{\lambda^j t^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)} \end{aligned}$$

for all $t \in (0,h]$. This justifies (5.3).

Letting $m \to \infty$ in (5.3), we obtain the solution of the problem (5.2) as

$$\begin{aligned} y(t) = \lim_{m\to\infty} y_m(t) =& \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \sum_{j=0}^{\infty} \frac{\lambda^j t^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)} \\ =& \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}} \mathrm{E}_{\alpha,\alpha-i+1}\left(\frac{\lambda t^{\rho\alpha}}{\rho^\alpha}\right) \end{aligned}$$

for $t > 0$, where E is the two-parameter Mittag-Leffler function defined in (4.8).

6. Dependence on Initial Conditions

Let us introduce small changes in the initial conditions of (4.1) and consider

$$\begin{cases} \big[\mathcal{D}_\rho^\alpha y\big](t) = f\big(t, y(t)\big) & \text{for } t > 0 \\ \big[\mathcal{D}_\rho^{\alpha-k} y\big](0^+) = y_{\lceil\alpha\rceil-k} + \varepsilon_{\lceil\alpha\rceil-k} & \text{for } k = 1, 2, \cdots, \lceil\alpha\rceil, \end{cases} \tag{6.1}$$

where $\varepsilon_{\lceil\alpha\rceil-k}$ are arbitrary constants.

Theorem 6.1. *Assume that conditions of Theorem 4.1 hold. Let y and z be respective solutions of the initial value problems (4.1) and (6.1). Then,*

$$|y(t) - z(t)| \leq \sum_{i=1}^{\lceil\alpha\rceil} \big|\varepsilon_{\lceil\alpha\rceil-i}\big| \frac{\rho^i}{At^{\rho i}} \mathrm{E}_{\alpha,1-i}\left(\frac{At^{\rho\alpha}}{\rho^\alpha}\right) \quad \textit{for } t \in (0,h].$$

Proof. In conformity with Theorem 4.1, we have

$$y(t) = \lim_{m\to\infty} y_m(t) \quad \text{for } t \in (0,h],$$

where the sequence of functions $\{y_m\}_{m\in\mathbb{N}_0}$ is defined by (4.3) for $t \in (0,h]$. Similarly,

$$z(t) = \lim_{m\to\infty} z_m(t) \quad \text{for } t \in (0,h],$$

where

$$z_m(t) = \begin{cases} \displaystyle\sum_{i=1}^{\lceil\alpha\rceil} \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha-i+1)}(y_{\lceil\alpha\rceil-i} + \varepsilon_{\lceil\alpha\rceil-i}), & m = 0 \\ z_0(t) + \big[\mathcal{J}_\rho^\alpha f\big(*, z_{m-1}(*)\big)\big](t), & m \in \mathbb{N} \end{cases} \tag{6.2}$$

for $t \in (0,h]$. Let us prove by induction that

$$|y_m(t) - z_m(t)| \leq \sum_{i=1}^{\lceil\alpha\rceil} \big|\varepsilon_{\lceil\alpha\rceil-i}\big| \sum_{j=0}^{m} \frac{A^j t^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1)-i+1\big)} \tag{6.3}$$

for all $t \in (0, h]$ and all $m \in \mathbb{N}_0$. From (4.3) and (6.2), it directly follows that

$$|y_0(t) - z_0(t)| \le \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha - i + 1)} \quad \text{for all } t \in (0, h].$$

Assume now for some $m \in \mathbb{N}$ that

$$|y_m(t) - z_m(t)| \le \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \sum_{j=0}^{m} \frac{A^j t^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1) - i + 1\big)} \tag{6.4}$$

for all $t \in (0, h]$. Then, using (4.3) and (6.2), the Lipschitz condition for the function f together with the inequality (6.4), we obtain

$$\begin{aligned}
&|y_{m+1}(t) - z_{m+1}(t)| \\
&\quad\le \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha - i + 1)} + A\int_0^t \mathcal{K}_\rho^\alpha(t,\eta)|y_m(\eta) - z_m(\eta)|\mathrm{d}\eta \\
&\quad\le \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha - i + 1)} + A\int_0^t \mathcal{K}_\rho^\alpha(t,\eta)\sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \sum_{j=0}^{m} \frac{A^j \eta^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1) - i + 1\big)}\mathrm{d}\eta \\
&\quad= \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}\Gamma(\alpha - i + 1)} + \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \sum_{j=0}^{m} \frac{A^{j+1} t^{\rho((j+2)\alpha-i)}}{\rho^{(j+2)\alpha-i}\Gamma\big((j+2)\alpha - i + 1\big)} \\
&\quad= \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \sum_{j=0}^{m+1} \frac{A^j t^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1) - i + 1\big)}
\end{aligned}$$

for all $t \in (0, h]$. This proves (6.3).

Taking the limit of (6.4) as $m \to \infty$, we obtain

$$\begin{aligned}
|y(t) - z(t)| &\le \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \sum_{j=0}^{\infty} \frac{A^j t^{\rho(\alpha(j+1)-i)}}{\rho^{\alpha(j+1)-i}\Gamma\big(\alpha(j+1) - i + 1\big)} \\
&= \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \frac{\rho^i}{At^{\rho i}} \sum_{j=0}^{\infty} \frac{A^{j+1} t^{(j+1)\rho\alpha}}{\rho^{\alpha(j+1)}\Gamma\big(\alpha(j+1) - i + 1\big)} \\
&= \sum_{i=1}^{\lceil\alpha\rceil} \left|\varepsilon_{\lceil\alpha\rceil - i}\right| \frac{\rho^i}{At^{\rho i}} \mathrm{E}_{\alpha,1-i}\left(\frac{At^{\rho\alpha}}{\rho^\alpha}\right)
\end{aligned}$$

for all $t \in (0, h]$, which completes the proof. □

7. The Green's Function for Linear Equations

The objective of this section is to define the Green's function notion for the initial value problem

$$\begin{cases} \left[\mathcal{D}_\rho^\alpha y\right](t) = p(t)y(t) + f(t) & \text{for } t > 0 \\ \left[\mathcal{D}_\rho^{\alpha-k} y\right](0^+) = y_{\lceil\alpha\rceil - k} & \text{for } k = 1, 2, \cdots, \lceil\alpha\rceil, \end{cases} \tag{7.1}$$

where $p, f : [0, \infty) \to \mathbb{R}$ are continuous functions, and then present its role in obtaining the solution of the equation.

Let $\Delta := \{(t, s) : \ t > s \ge 0\}$ and denote by ${}_s\mathcal{D}_\rho^\alpha f$ and ${}_s\mathcal{J}_\rho^\alpha f$ the fractional derivative and the fractional integral of a function f centered at $s \in [0, \infty)$, respectively.

Definition 7.1 (Green's function)**.** *Let the continuous function $\mathcal{G}_\rho : \Delta \to \mathbb{R}$ satisfy the following properties.*

(i) $[{}_s\mathcal{D}_\rho^\alpha \mathcal{G}_\rho(*, s)](t) = p(t)\mathcal{G}_\rho(t, s)$ *for all* $(t, s) \in \Delta$.

(ii) $\lim_{s \to t^-} [{}_s\mathcal{D}_\rho^{\alpha-k} \mathcal{G}_\rho(*, s)](t) = \delta_{k,1}$ *for* $t > 0$ *and* $k = 1, 2, \cdots, \lceil\alpha\rceil$.

(iii) $\lim\limits_{\substack{s\to t^-\\ t\to 0^+}}\big[{}_s\mathcal{D}_\rho^{\alpha-k}\mathcal{G}_\rho(*,s)\big](t)=0$ *for* $k=1,2,\cdots,\lceil\alpha\rceil-1$.

Then, $\mathcal{G}_\rho$ *is called the Green's function for the initial value problem* (7.1).

Theorem 7.1. *Let* $\mathcal{G}_\rho$ *be the Green's function for the initial value problem* (7.1)*. Then,*

$$y(t):=\int_0^t \mathcal{G}_\rho(t,\eta)\eta^{\rho-1}f(\eta)\mathrm{d}\eta \quad \textit{for } t>0 \tag{7.2}$$

is the (unique) solution of the initial value problem

$$\begin{cases}\big[\mathcal{D}_\rho^\alpha y\big](t)=p(t)y(t)+f(t) \quad \textit{for } t>0\\ \big[\mathcal{D}_\rho^{\alpha-k}y\big](0^+)=0 \quad \textit{for } k=1,2,\cdots,\lceil\alpha\rceil.\end{cases} \tag{7.3}$$

Proof. First, we will show that y defined by (7.2) solves the fractional differential equation in (7.3). To this end, we let $\beta=\alpha-\lceil\alpha\rceil+1$, then $\beta\in(0,1]$. From (7.2), we have for $t>0$ that

$$\begin{aligned}\big[\mathcal{D}_\rho^\beta y\big](t)=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{D}_\rho^{\beta-1}y\big](t)=\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathcal{J}_\rho^{1-\beta}y\big](t)\\
=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t \mathcal{K}_\rho^{1-\beta}(t,\eta)\int_0^\eta \mathcal{G}_\rho(\eta,\zeta)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta\mathrm{d}\eta\right]\\
=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\int_0^\eta \mathcal{K}_\rho^{1-\beta}(t,\eta)\mathcal{G}_\rho(\eta,\zeta)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta\mathrm{d}\eta\right]\\
=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\int_\zeta^t \mathcal{K}_\rho^{1-\beta}(t,\eta)\mathcal{G}_\rho(\eta,\zeta)\zeta^{\rho-1}f(\zeta)\mathrm{d}\eta\mathrm{d}\zeta\right]\\
=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\left[\int_\zeta^t \mathcal{K}_\rho^{1-\beta}(t,\eta)\mathcal{G}_\rho(\eta,\zeta)\mathrm{d}\eta\right]\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta\right]\\
=&\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^t\big[{}_\zeta\mathcal{J}_\rho^{1-\beta}\mathcal{G}_\rho(*,\zeta)\big](t)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta\right]\\
=&\int_0^t\frac{1}{t^{\rho-1}}\frac{\mathrm{d}}{\mathrm{d}t}\big[{}_\zeta\mathcal{J}_\rho^{1-\beta}\mathcal{G}_\rho(*,\zeta)\big](t)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta+\frac{1}{t^{\rho-1}}\lim_{\zeta\to t^-}\Big[\big[{}_\zeta\mathcal{J}_\rho^{1-\beta}\mathcal{G}_\rho(*,\zeta)(t)\big]\zeta^{\rho-1}f(\zeta)\Big],\end{aligned}$$

where we have applied the well-known Leibnitz rule. Using the definition of the fractional derivative, we obtain

$$\big[\mathcal{D}_\rho^\beta y\big](t)=\int_0^t\big[{}_\zeta\mathcal{D}_\rho^\beta\mathcal{G}_\rho(*,\zeta)\big](t)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta+\lim_{\zeta\to t^-}\big[{}_\zeta\mathcal{D}_\rho^{\beta-1}\mathcal{G}_\rho(*,\zeta)\big](t)f(t)$$

for all $t>0$. Applying $\mathcal{D}^1$ repeatedly for a total of $(\lceil\alpha\rceil-1)$ times and using Definition 7.1 (ii), we find for all $t>0$ that

$$\begin{aligned}\big[\mathcal{D}_\rho^\alpha y\big](t)&=\int_0^t\big[{}_\zeta\mathcal{D}_\rho^\alpha\mathcal{G}_\rho(*,\zeta)\big](t)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta+\lim_{\zeta\to t^-}\big[{}_\zeta\mathcal{D}_\rho^{\alpha-1}\mathcal{G}_\rho(*,\zeta)\big](t)f(t) \qquad (7.4)\\
&=\int_0^t\big[{}_\zeta\mathcal{D}_\rho^\alpha\mathcal{G}(*,\zeta)\big](t)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta+f(t)\\
&=p(t)\int_0^t\mathcal{G}_\rho(t,\zeta)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta+f(t)\\
&=p(t)y(t)+f(t)\end{aligned}$$

for all $t>0$ by Definition 7.1 (i). Thus, y is a solution of the fractional differential equation in (7.3).

Next, we justify the initial conditions. As in (7.4), we compute that

$$\big[\mathcal{D}_\rho^{\alpha-k}y\big](t)=\int_0^t\big[{}_\zeta\mathcal{D}_\rho^{\alpha-k}\mathcal{G}_\rho(*,\zeta)\big](t)\zeta^{\rho-1}f(\zeta)\mathrm{d}\zeta+\lim_{\zeta\to t^-}\big[{}_\zeta\mathcal{D}_\rho^{\alpha-k+1}\mathcal{G}_\rho(*,\zeta)\big](t)f(t)$$

for all $t>0$ and $k=1,2,\cdots,\lceil\alpha\rceil$. By Definition 7.1 (iii) and letting $t\to 0^+$, we obtain $\big[\mathcal{D}_\rho^{\alpha-k}y\big](0^+)=0$ for $k=1,2,\cdots,\lceil\alpha\rceil$.

We have therefore justified that y defined in (7.2) solves the initial value problem (7.3). □

Corollary 7.1. *If $\{H_i\}_{i=1}^{\lceil\alpha\rceil}$ forms the set of normalized fundamental solutions of the homogeneous initial value problem associated with (7.1), i.e.,*

$$\begin{cases} [\mathcal{D}_\rho^\alpha H_i](t) = p(t)H_i(t) & \text{for } t>0 \\ [\mathcal{D}_\rho^{\alpha-k} H_i](0^+) = \delta_{i,k} & \text{for } k=1,2,\cdots,\lceil\alpha\rceil, \end{cases}$$

and $\mathcal{G}$ is the Green's function for the initial value problem (7.1), then the solution of the initial value problem (7.1) is given by

$$y(t) = \sum_{k=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i}H_i(t) + \int_0^t \mathcal{G}_\rho(t,\eta)\eta^{\rho-1}f(\eta)\mathrm{d}\eta \quad \text{for } t>0.$$

7.1. **Representation of Solutions for the Autonomous Equation.** In this section, we confine our attention to the linear autonomous initial value problem

$$\begin{cases} \big[\mathcal{D}_\rho^\alpha y\big](t) = \lambda y(t) + f(t) & \text{for } t>0 \\ \big[\mathcal{D}_\rho^{\alpha-k} y\big](0^+) = y_{\lceil\alpha\rceil-k} & \text{for } k=1,2,\cdots,\lceil\alpha\rceil. \end{cases} \tag{7.5}$$

For linear autonomous equations, we can easily verify that

$$\mathcal{G}_\rho(t,s) = \mathcal{G}_\rho\big(\sqrt[\rho]{t^\rho - s^\rho}, 0\big) \quad \text{for all } (t,s)\in\Delta. \tag{7.6}$$

Moreover, $\mathcal{G}_\rho(*,0)$ is the solution of the associated homogeneous equation

$$\begin{cases} \big[\mathcal{D}_\rho^\alpha y\big](t) = \lambda y(t) & \text{for } t>0 \\ \big[\mathcal{D}_\rho^{\alpha-k} y\big](0^+) = \delta_{k,1} & \text{for } k=1,2,\cdots,\lceil\alpha\rceil. \end{cases}$$

Due to the discussion made in § 5.1, we see that the Green's function of (7.5) is given by

$$\mathcal{G}_\rho(t,s) = \frac{[t^\rho - s^\rho]^{\alpha-1}}{\rho^{\alpha-1}}\,\mathrm{E}_{\alpha,\alpha}\bigg(\frac{\lambda[t^\rho - s^\rho]^\alpha}{\rho^\alpha}\bigg) \quad \text{for } (t,s)\in\Delta.$$

Further, the variation of parameters formula for the initial value problem (7.5) is given by

$$y(t) = \sum_{i=1}^{\lceil\alpha\rceil} y_{\lceil\alpha\rceil-i}\frac{t^{\rho(\alpha-i)}}{\rho^{\alpha-i}}\,\mathrm{E}_{\alpha,\alpha-i+1}\bigg(\frac{\lambda t^{\rho\alpha}}{\rho^\alpha}\bigg) + \int_0^t \frac{[t^\rho-\eta^\rho]^{\alpha-1}}{\rho^{\alpha-1}}\,\mathrm{E}_{\alpha,\alpha}\bigg(\frac{\lambda[t^\rho-\eta^\rho]^\alpha}{\rho^\alpha}\bigg)\eta^{\rho-1}f(\eta)\mathrm{d}\eta \tag{7.7}$$

for $t>0$ (cf. [9, Equation (3.1.11)]).

8. Final Comments

The following two examples can be easily verified.

Example 8.1. *For $\alpha\in\mathbb{R}_0^+$, $\nu\in(-1,\infty)$ and $\rho\in\mathbb{R}^+$, we have*

$$\big[{}_s\mathcal{J}_\rho^\alpha[*^\rho - s^\rho]^\nu\big](t) = \frac{\Gamma(\nu+1)}{\rho^\alpha\Gamma(\nu+\alpha+1)}[t^\rho - s^\rho]^{\nu+\alpha} \quad \text{for } t>s\geq 0.$$

Example 8.2. *For $\alpha,\nu\in\mathbb{R}_0^+$ and $\rho\in\mathbb{R}^+$, we have*

$$\big[{}_s\mathcal{D}_\rho^\alpha[*^\rho - s^\rho]^\nu\big](t) = \frac{\rho^\alpha\Gamma(\nu+1)}{\Gamma(\nu-\alpha+1)}[t^\rho - s^\rho]^{\nu-\alpha} \quad \text{for } t>s\geq 0.$$

As our first remark, we would like to say that one can justify (7.6) similar to that in the third part of [11, § 5.1.2].

As an another note, we would like to emphasis that one can justify the variation of parameters formula given in (7.7) by using Example 8.2 and applying the Picard iterates technique (used in § 5.2) to (7.5).

As to some directions for future research, we would like to mention that the study of numerical solutions to FDEs and numerical integration techniques would very important. Next, we note that the extension of any of the results in this paper to the case FDEs with Miller-Ross type sequential derivatives would be of significant interest too. Finally, obtaining the solutions of FDEs of the type (7.5) by the Laplace transform would also deserve attention for the sake of completeness.

References

[1] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, J. Math. Anal. Appl. 269 (2002), no. 2, 387–400.
[2] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, J. Math. Anal. Appl. 269 (2002), no. 1, 1–27.
[3] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, Berlin, 2010.
[4] J. Hadamard, Essai sur l'étude des fonctions données par leur développement de Taylor, Journal de Mathématiques Pures et Appliquées 4 (1892), no. 8, 101–186.
[5] U. N. Katugampola, New approach to a generalized fractional integral, Appl. Math. Comput. 218 (2011), no. 3, 860–865.
[6] U. N. Katugampola, A new approach to generalized fractional derivatives, Bull. Math. Anal. Appl. 4 (2014), no. 6, 1–15.
[7] A. A. Kilbas, Hadamard-type fractional calculus, J. Korean Math. Soc. 38 (2001), no. 6, 1191–1204.
[8] A. A. Kilbas and J. J. Trujillo, Hadamard-type integrals as G-transforms, Integral Transforms Spec. Funct. 14 (2003), no. 5, 413–427.
[9] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, Theory and Applications of Fractional Differential Equations Elsevier Science B.V., Amsterdam, 2006.
[10] K. B. Oldham and J. Spanier, The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, New York, 1974.
[11] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.
[12] S. Pooseh, P. Almeida and D. F. M. Torres, Expansion formulas in terms of integer-order derivatives for the Hadamard fractional integral and derivative, Numer. Funct. Anal. Optim. 33 (2012), no. 3, 301–319.
[13] S. G. Samko, A. A. Kilbas and O. I. Marichev, Fractional Integrals and Derivatives Theory and Applications, Gordon and Breach Science Publishers, Yverdon, 1993.

[1]Dokuz Eylül University, Tinaztepe Campus, Faculty of Science, Department of Mathematics, Buca, 35160 İzmir, Turkey.

[2]Afyon Kocatepe University, ANS Campus, Faculty of Science and Arts, Department of Mathematics, 03200 Afyonkarahisar, Turkey.

[*]Corresponding author: bkarpuz@gmail.com