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ON STABILITY OF CONVOLUTION OF JANOWSKI FUNCTIONS

KHALIDA INAYAT NOOR AND HUMAYOUN SHAHID*

ABSTRACT. In this paper, the classes $S^*[A,B]$ and $C[A,B]$ are discussed in terms of dual sets. Using duality, various geometric properties of mentioned class are analyzed. Problem of neighborhood as well as stability of convolution of $S^*[A,B]$ and $C[A,B]$ are studied. Some of our results generalize previously known results.

1. INTRODUCTION

Let $\mathcal{A}$ be the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$.

Let $S \subset \mathcal{A}$ be the class of functions which are univalent and also $S^*(\alpha)$ and $C(\alpha)$ be the well known subclasses of S which, respectively consist of starlike and convex functions of order α. If $f(z)$ and $g(z)$ are analytic in E, we say that $f(z)$ is subordinate to $g(z)$, written as $f \prec g$ or $f(z) \prec g(z)$ if there exist a Schwarz function $w(z)$ which is analytic in E with $w(0) = 0$ and $|w(z)| < 1$ where $z \in E$, such that $f(z) = g(w(z))$, $z \in E$. Also, if $g \in S$, then

$$f(z) \prec g(z) \text{ if and only if } f(0) = g(0) \text{ and } f(E) \subset g(E).$$

A number of subclasses of analytic functions were introduced using subordination. In 1973, Janowski [2] introduced the class $P[A,B]$ which is defined as

$$P[A,B] = \left\{ p(z) : p(z) \prec \frac{1+Az}{1+Bz} \right\},$$

where $-1 \le B < A \le 1$. Geometrically, $p(E)$ is contained in the open disc centered on the real axis having diameter end points $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ with centered at $\frac{1-AB}{1-B^2}$. For specific values of A and B we obtain many known subclasses of $P[A,B]$. Some specific cases include

(i). $P[1,-1] = P$, the class of Caratheodory functions.
(ii). $P[1-2\alpha,-1] = P(\alpha)$, the class of Caratheodory functions of order α.
(iii). $p(z) \in P[\alpha, 0]$ satisfies the condition $|p(z) - 1| < \alpha$, see [4].

Using $P[A,B]$, Janowski [2] introduced $S^*[A,B]$ and $C[A,B]$ which are defined as

$$S^*[A,B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz},\ z \in E \right\}$$

and

$$C[A,B] = \left\{ f \in \mathcal{A} : \frac{(zf'(z))'}{f'(z)} \prec \frac{1+Az}{1+Bz},\ z \in E \right\},$$

where $-1 \le B < A \le 1$. We note that Alexander relation holds between $S^*[A,B]$ and $C[A,B]$.

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The convolution (Hadamard) of two functions $f(z)$ given by (1.1) and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ is defined as

$$(f*g)(z)=(g*f)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n.$$

Let $V\subset\mathcal{A}$ the dual set V^* (see [6]) is defined as following

$$V^*=\left\{g\in\mathcal{A}:\frac{(f*g)(z)}{z}\neq 0,\ \forall f\in A,\ z\in E\right\}. \tag{1.2}$$

Silverman et al. [8] proved that $S^*[A,B]=G^*$, where G^* represents the dual set of G defined in (1.2) and G is given by

$$G=\left\{g\in\mathcal{A}:g(z)=\frac{z-Lz^2}{(1-z)^2}\right\}, \tag{1.3}$$

where $L=\frac{e^{-i\theta}+A}{A-B}$ and $\theta\in[0,2\pi]$. Using the Alexander type relation, $C[A,B]=H^*$ where

$$H=\left\{h\in\mathcal{A}:h(z)=\frac{z+(1-2L)z^2}{(1-z)^3}\right\}, \tag{1.4}$$

where L is same as given in (1.3) and $-1\leq B<A\leq 1$.

For $f\in\mathcal{A}$ and is of form (1.1) and $\delta\geq 0$, the N_δ neighborhood of function f is defined as following (see [7]).

$$N_\delta(f)=\left\{g(z)=z+\sum_{n=2}^{\infty}b_nz^n\in A:\sum_{n=2}^{\infty}n|b_n-a_n|\leq\delta\right\}.$$

Ruscheweyh proved many inclusion results of $N_\delta(f)$ especially $N_{\frac{1}{4}}(f)\subset S^*$ for all $f\in C$.

For X, $Y\subset\mathcal{A}$. The convolution is called stable univalent if there exist $\delta>0$ such that $N_\delta(f)*N_\delta(g)\subset S$, where $f\in X$ and $g\in Y$. The constant δ is defined as

$$\delta(X*Y,Z)=\sup\{\delta:N_\delta(f)*N_\delta(g)\subset Z\}. \tag{1.5}$$

In the current paper, we estimate the coefficient bounds of functions given in (1.3) and (1.4). Using these estimates we discuss some interesting properties of $N_\delta(f)$ for different classes and inclusion properties of $N_\delta(f)$.

2. Preliminaries

To prove our main results, we need the following Lemmas.

Lemma 2.1. *[8]. Let $-1\leq B<A\leq 1$ and $\theta\in[0,2\pi]$. Then $G^*=S^*[A,B]$, where*

$$G=\left\{g\in A:g(z)=\frac{z-\frac{e^{-i\theta}+A}{A-B}z^2}{(1-z)^2}\right\}.$$

Lemma 2.2. *[8]. Let $-1\leq B<A\leq 1$ and $\theta\in[0,2\pi]$. Then $H^*=C[A,B]$, where*

$$H=\left\{h\in A:h(z)=\frac{z+\left(1-2\frac{e^{-i\theta}+A}{A-B}\right)z^2}{(1-z)^3}\right\}. \tag{2.1}$$

Lemma 2.3. *[5]. Let Ψ be convex and g be starlike in E. Then, for F analytic in E with $F(0)=1$, $\frac{\Psi*Fg}{\Psi*g}$ is contained in the convex hull of $F(E)$.*

3. Main Results

Theorem 3.1. *Let* $-1 \leq B < A \leq 1$, *then for* $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in G$,

$$\left|\frac{n(1+B)-(A+1)}{A-B}\right| \leq |c_n| \leq \frac{n(1-B)+A-1}{A-B},$$

Proof. For $h \in G$, the coefficients can be written as

$$c_n = n(1-L)+L,$$

where $L = \frac{e^{-i\theta}+A}{A-B}$ and $\theta \in [0, 2\pi]$. To find the maximum value of $|c_n(\theta)|$ where θ varies from 0 to 2π, consider

$$|c_n(\theta)|^2 = \frac{(nB-A)^2 + (n-1)^2 + 2(n-1)(nB-A)\cos\theta}{(A-B)^2} = \phi(\theta),$$

$\phi(\theta)$ attains its maximum value at $\theta = \pi$ as $\phi'(\pi) = -\frac{2(n-1)(nB-A)}{(A-B)^2}\sin\pi = 0$ and $\phi''(\pi) = -\frac{2(n-1)(nB-A)}{(A-B)^2}\cos\pi < 0$ as $nB - A < 0$.The maximum value of $\phi(\theta)$ is $\phi(\pi) = \left(\frac{n(B-1)+1-A}{A-B}\right)^2$, we note that $\phi(\theta) \leq \phi(\pi)$ for all $\theta \in [0, 2\pi]$. Substituting the value of $\phi(\pi)$ we obtain

$$|c_n| \leq \frac{n(1-B)+A-1}{A-B}.$$

Now again consider $\phi(\theta)$ and we note that $\phi(z)$ has its minimum at $\theta = 0$ and $\phi(0) = \left(\frac{n(B+1)-(A+1)}{A-B}\right)^2$. Thus we obtain

$$|c_n| \geq \left|\frac{n(B+1)-(A+1)}{A-B}\right|.$$

This completes the proof. □

Applying the Alexander type relation between set G and H we obtain following

Corollary 3.1. *Let* $-1 \leq B < A \leq 1$, *then for* $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in H$,

$$\left|\frac{n[n(B+1)-(A+1)]}{A-B}\right| \leq |c_n| \leq \frac{n[n(1-B)+A-1]}{A-B}.$$

Corollary 3.2. *Let* $-1 \leq B < A \leq 1$ *and let* $f(z) = z + \lambda z^n$, $n \geq 2$. *Then* $f \in S^*[A, B]$ *if and only if*

$$|\lambda| \leq \frac{A-B}{n(1-B)+A-1}. \tag{3.1}$$

Proof. Let $f(z) = z + \lambda z^n$ where λ is given in inequality (3.1) and then for $g \in G$, consider

$$\left|\frac{(f*g)(z)}{z}\right| \geq 1 - |\lambda||c_n| z^{n-1}, \ z \in E.$$

Now using Theorem 3.1 and value of λ given in (3.1), we obtain

$$\left|\frac{(f*g)(z)}{z}\right| > 0, \ z \in E.$$

Hence $f \in S^*[A, B]$. Conversely, now consider $f(z) = z + \lambda z^n \in S^*[A, B]$ and let $g(z) = z + \sum_{n=2}^{\infty} \frac{n(1-B)+A-1}{A-B} z^n$ and

$$\frac{(f*g)(z)}{z} = 1 + \lambda \frac{n(1-B)+A-1}{A-B} z^{n-1} \neq 0.$$

If $|\lambda| > \frac{A-B}{n(1-B)+A-1}$, then there exist $\xi \in E$ such that

$$\frac{(f*g)(\xi)}{\xi} = 0.$$

which is a contradiction, hence $|\lambda| \leq \frac{A-B}{n(1-B)+A-1}$. □

Corollary 3.3. *Let* $-1 \leq B < A \leq 1$ *and let* $f(z) = z + \lambda z^n$, $n \geq 2$. *Then* $f \in C[A,B]$ *if and only if*

$$|\lambda| \leq \frac{A-B}{n\left[n(1-B)+A-1\right]}.$$

Using the coefficient bounds of functions in set G, we now give alternate method to prove the Theorem given in [1].

Corollary 3.4. *Let* $-1 \leq B < A \leq -1$ *and let* f *is of the form (1.1) and satisfy*

$$\sum_{n=2}^{\infty}\left[n(1-B)+A-1\right]|a_n| \leq A-B,$$

then $f \in S^*[A,B]$.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} \frac{n(1-B)+A-1}{A-B} z^n$, consider

$$\frac{(f*g)(z)}{z} = 1 + \sum_{n=2}^{\infty} \frac{n(1-B)+A-1}{A-B} a_n z^{n-1},\ z \in E.$$

it is known from Lemma 2.1 that $f \in S^*[A,B]$ if and only if $\frac{(f*g)(z)}{z} \neq 0$. Now

$$\left|\frac{(f*g)(z)}{z}\right| \geq 1 - \sum_{n=2}^{\infty} \frac{n(1-B)+A-1}{A-B}|a_n||z|^{n-1} > 0,$$

which gives us the required condition. □

We now consider two specific functions

$$F_\alpha(z) = \frac{f(z)+\alpha z}{1+\alpha} \tag{3.2}$$

and

$$F_{n,\alpha}(z) = f(z) + \frac{\alpha}{n} z^n\ \ (n \geq 2). \tag{3.3}$$

Here α is a non zero complex number also we note that if $f(z) \in \mathcal{A}$, then both $F_\alpha(z)$ and $F_{n,\alpha}(z) \in \mathcal{A}$. The geometric properties of these functions are studied by various authors (see [3]). Using these two functions, we study the geometric properties of $N_\delta(f)$ for classes of $S^*[A,B]$ and $C[A,B]$. We first discuss the relation between $f(z)$ and $F_\alpha(z)$ in the following Lemma.

Lemma 3.1. *Let* $-1 \leq B < A \leq 1$, $f \in \mathcal{A}$ *and* $\delta > 0$ *and let for for all* $\alpha \in \mathbb{C}$, $F_\alpha \in S^*[A,B]$ *(or* $C[A,B]$*), then* $f \in S^*[A,B]$ *(or* $C[A,B]$*) furthermore for all* $g \in G$ *(or* H*)*

$$\left|\frac{(f*g)(z)}{z}\right| > \delta,$$

where $|\alpha| < \delta$ *and* $z \in E$.

Proof. Since $F_\alpha \in S^*[A,B]$ then by Lemma 2.1, we know that for all $g \in G$,

$$\frac{(F_\alpha * g)(z)}{z} \neq 0,\ z \in E.$$

Using (3.2) and simplifying, we obtain

$$\frac{(f*g)(z)}{z} \neq -\alpha,$$

for all α. Thus we obtain

$$\left|\frac{(f*g)(z)}{z}\right| > \delta.$$

Using Lemma 2.1, we obtain that $f \in S^*[A,B]$. This completes the proof. □

Applying the similar method, we have the following result.

Lemma 3.2. *Let $-1 \leq B < A \leq 1$, $f \in A$ and $\delta > 0$ and let for for all α, $F_{n,\alpha} \in S^*[A,B]$, then for all $h \in G$*

$$\left|\frac{(f*h)(z)}{zc_n}\right| > \frac{\delta}{n},$$

where $|\alpha| < \delta$ and $z \in E$.

Using Theorem 3.1 in Lemma 3.1, we obtain the following.

Corollary 3.5. *Let $-1 \leq B < A \leq 1$, $f \in A$ and $\delta > 0$ and let for for all α, $F_{n,\alpha} \in S^*[A,B]$, then for all $h \in G$*

$$\left|\frac{(f*h)(z)}{z}\right| > \frac{\delta}{n}\left|\frac{n(B+1)-(A+1)}{A-B}\right|,$$

where $|\alpha| < \delta$ and $z \in E$.

We now prove the following

Theorem 3.2. *Let $-1 \leq B < A \leq 1$ and $\delta > 0$ if for all α, $F_\alpha \in S^*[A,B]$ then $N_{\delta_1}(f) \subset S^*[A,B]$ where*

$$\delta_1 = \frac{\delta(A-B)}{1-B}.$$

Proof. Let $g \in N_{\delta_1}(f)$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. To prove that $g \in S^*[A,B]$, it is enough to show that

$$\frac{(g*h)(z)}{z} \neq 0,$$

where $h \in G$ and $z \in E$.

Consider

$$\begin{aligned}\left|\frac{(g*h)(z)}{z}\right| &= \left|\frac{(f*h)(z)}{z} + \frac{((g-f)*h)(z)}{z}\right| \\ &\geq \left|\frac{(f*h)(z)}{z}\right| - \left|\frac{((g-f)*h)(z)}{z}\right|.\end{aligned}$$

Using Lemma 3.1 and series representations of $f(z)$, $g(z)$ and $h(z)$, we obtain

$$\left|\frac{(g*h)(z)}{z}\right| > \delta - \sum_{n=2}^{\infty}\frac{(n(1-B)-(1-A))|b_n-a_n|}{A-B}. \tag{3.4}$$

Since

$$\sum_{n=2}^{\infty}\frac{(n(1-B)-(1-A))|b_n-a_n|}{A-B} \leq \frac{1-B}{A-B}\sum_{n=2}^{\infty} n|b_n-a_n| \leq \frac{1-B}{A-B}\delta_1. \tag{3.5}$$

Using (3.4) in (3.5), we obtain

$$\left|\frac{(g*h)(z)}{z}\right| > \delta - \frac{1-B}{A-B}\delta_1 > 0.$$

Hence

$$\delta_1 = \frac{\delta(A-B)}{1-B}.$$

This completes the proof. □

Theorem 3.3. *Let $-1 \leq B < A \leq 1$. $f \in C[A,B]$, then $F_\alpha \in S^*[A,B]$ for $|\alpha| < \frac{1}{4}$.*

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

$$\begin{aligned}F_\alpha(z) &= \frac{f(z)+\alpha z}{(1+\alpha)} \\ &= (f(z)*\psi(z)),\ z \in E.\end{aligned}$$

Here

$$\psi(z)=\frac{z-\frac{\alpha}{1+\alpha}z^2}{1-z}.$$

Using the properties of convolution we obtain

$$f(z)*\psi(z)=zf'(z)*\left(\psi(z)*\log\left(\frac{1}{1-z}\right)\right).$$

Since $f\in C[A,B]$, $zf'\in S^*[A,B]$, also if $|\alpha|<\frac{1}{4}$, $\psi\in S^*$. Applying the convolution we obtain

$$\psi(z)*\log\left(\frac{1}{1-z}\right)=\int_0^z\frac{\psi(t)}{t}dt. \tag{3.6}$$

Using the Alexander relation in (3.6), we obtain $\psi(z)*\log\left(\frac{1}{1-z}\right)\in C$. Using Lemma 2.3 one can prove that $C*S^*[A,B]\subset S^*[A,B]$, hence

$$F_\alpha(z)=zf'(z)*\left(\psi(z)*\log\left(\frac{1}{1-z}\right)\right)\in S^*[A,B],\ \ |\alpha|<\frac{1}{4}.$$

This completes the proof. □

We now prove the following.

Theorem 3.4. *Let $-1\le B<A\le -1$. If $f\in C[A,B]$, then $N_\delta(f)\subset S^*[A,B]$ where $\delta=\frac{A-B}{4(1-B)}$.*

Proof. If $f\in C[A,B]$, then by Theorem 3.3 $F_\alpha\in S^*[A,B]$ for $|\alpha|<\frac{1}{4}$. choosing $\delta=\frac{1}{4}$ and applying Theorem 3.2, we obtain our required result. □

For specific values of A and B we have the following

Corollary 3.6. *[7]. If $f\in C[1,-1]=C$, then $N_\delta(f)\subset S^*$ where $\delta=\frac{1}{4}$.*

Corollary 3.7. *If $f\in C[1-2\beta,-1]=C(\beta)$, then $N_\delta(f)\subset S^*(\beta)$ where $\delta=\frac{1-\beta}{4}$ and $0\le\beta<1$.*

We now prove the stability of convolution given in (1.5) for different classes of $N_\delta(f)$. In the next Theorem I represent the identity function $I(z)=z$.

Theorem 3.5. *Let $-1\le B<A\le 1$. The following relation holds*

$$\delta(I*I,C[A,B]) \ \ge\ \sqrt{\frac{A-B}{1-B}} \tag{3.7}$$

$$\delta(I*I,S^*[A,B]) \ \ge\ \sqrt{\frac{2(A-B)}{1-B}} \tag{3.8}$$

$$\delta(C[A,B]*C,C[A,B]) \ =\ 0 \tag{3.9}$$

$$\delta(S^*[A,B]*C,C[A,B]) \ =\ 0 \tag{3.10}$$

$$\delta(C[A,B]*C,S^*[A,B]) \ \ge\ \sqrt{4+\frac{(A-B)^2}{2(1-B)^2}}-2=\delta_0. \tag{3.11}$$

Proof. Let $f,g\in N_\delta(I)$, then applying definition of $N_\delta(f)$, we obtain $\sum\limits_{n=2}^{\infty}n|a_n|\le\delta$ and $\sum\limits_{n=2}^{\infty}n|b_n|\le\delta$. Consider

$$\sum_{n=2}^{\infty}\frac{n(n(1-B)-1+A)|a_n||b_n|}{A-B}\le\frac{1-B}{A-B}\sum_{n=2}^{\infty}n^2|a_n||b_n|\le\frac{1-B}{A-B}\delta^2.$$

Now for $h\in H$,

$$\left|\frac{((f*g)*h)(z)}{z}\right|\ge\sum_{n=2}^{\infty}\frac{n(n(1-B)-1+A)|a_n||b_n|}{A-B}-1\ge\frac{1-B}{A-B}\delta^2-1>0.$$

Using value of δ given in (3.7), we obtain our first inequality.

Similarly consider f, $g \in N_\delta(I)$ and consider

$$\sum_{n=2}^{\infty} \frac{(n(1-B)-1+A)|a_n||b_n|}{A-B} \leq \frac{1-B}{A-B} \sum_{n=2}^{\infty} n^2 |a_n||b_n| \leq \frac{1-B}{2(A-B)}\delta^2,$$

Thus we obtain

$$\left|\frac{((f*g)*h)(z)}{z}\right| \geq \sum_{n=2}^{\infty} \frac{(n(1-B)-1+A)|a_n||b_n|}{A-B} - 1 \geq \frac{1-B}{2(A-B)}\delta^2 - 1 > 0.$$

Which gives us inequality in (3.8).

To prove (3.9), consider $f(z) = z + \left(\frac{A-B}{2(2(1-B)-1+A)}\right) z^2 \in C[A,B]$and $g(z) = g_0(z) + \frac{\delta}{2} z^2 \in C$, where $g_0(z) = \frac{z}{1-z}$. Taking the convolution of f and g, we get

$$(f*g)(z) = z + \left(\frac{A-B}{2(2(1-B)-1+A)} + \frac{\delta(A-B)}{4(2(1-B)-1+A)}\right) z^2,$$

Applying Corollary 3.3 with $n=2$, $(f*g)(z) \in C[A,B]$ if and only if, $\delta = 0$.

To prove (3.10), we are applying the same method with $f(z) = z + \left(\frac{A-B}{(2(1-B)-1+A)}\right) z^2$ and $g(z) = g_0(z) + \frac{\delta}{2} z^2$.

For relation given in (3.11), consider $f_0 \in C[A,B]$ and $g_0 \in C$ and $f \in N_\delta(f_0)$ and $g \in N_\delta(g_0)$, then for $h \in G$

$$\begin{aligned}\left|\frac{(f*g*h)(z)}{z}\right| &\geq \left|\frac{(f_0*g_0*h)(z)}{z}\right| - \left|\frac{(f_0*(g-g_0)*h)(z)}{z}\right| \\ &\quad - \left|\frac{(g_0*(f-f_0)*h)(z)}{z}\right| \\ &\quad - \left|\frac{((f-f_0)*(g-g_0)*h)(z)}{z}\right|.\end{aligned} \tag{3.12}$$

Applying Lemma 2.3 one can prove $f_0 * g_0 \in S^*[A,B]$ and using Theorem 3.4, we obtain

$$\left|\frac{(f*g*h)(z)}{z}\right| > \frac{A-B}{4(1-B)}. \tag{3.13}$$

If $f_0(z) = z + \sum_{n=2}^{\infty} a_{0n} z^n$ and $g_0(z) = z + \sum_{n=2}^{\infty} b_{0n} z^n$ and we know that $f_0(z) \in C[A,B] \subset C$ therefore $|a_{0n}| \leq 1$ and $|b_{0n}| \leq 1$. Now

$$\left|\frac{(f_0*(g-g_0)*h)(z)}{z}\right| \leq \sum_{n=2}^{\infty} \frac{|a_{0n}||b_n - b_{0n}||n(1-B)-1+A|}{A-B} \leq \frac{1-B}{A-B}\delta. \tag{3.14}$$

Using definition of $N_\delta(f)$, we know that $n|a_n - a_{0n}| \leq \delta$ or $|a_n - a_{0n}| \leq \frac{\delta}{2}$ as $n \geq 2$. Now consider

$$\begin{aligned}\left|\frac{((f-f_0)*(g-g_0)*h)(z)}{z}\right| &\leq \sum_{n=2}^{\infty} \frac{|a_n - a_{0n}||b_n - b_{0n}||n(1-B)-1+A|}{A-B} \\ &\leq \frac{(1-B)\delta^2}{2(A-B)}.\end{aligned} \tag{3.15}$$

Using (3.13), (3.14) and (3.15) in (3.12), we obtain

$$\left|\frac{(f*g*h)(z)}{z}\right| \geq \frac{A-B}{4(1-B)} - \frac{2(1-B)}{A-B}\delta - \frac{(1-B)\delta^2}{2(A-B)} > 0.$$

Solving for δ we obtain relation given in (3.11) which is non negative when $\delta \leq \delta_0$. This completes the proof. □

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Department of Mathematics COMSATS Institute of Information Technology Park Road, Islamabad, Pakistan

*Corresponding author: shahid_humayoun@yahoo.com