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HARMONIC BETA-PREINVEX FUNCTIONS AND INEQUALITIES

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Abstract. In this paper, we introduce and study a new class of harmonic convex functions which is called harmonic beta-preinvex functions. We establish some estimates, involving the Euler Beta function and the Hypergeometric function of the integral $\int_a^{a+\eta(b,a)}(x-a)^p(a+\eta(b,a)-x)^q f(x)\mathrm{d}x$ for the class of functions whose certain powers of the absolute value are generalized harmonic preinvex function. Some special cases are also discussed. Results obtained in this paper can be viewed as significant contribution in this fascinating and dynamic field.

1. Introduction

Theory of convex functions had not only stimulated new and deep results in many branches of mathematical and engineering sciences, but also provided us a unified and general frame work to study a wide class of unrelated problems. For recent applications, generalizations and other aspects of convex functions , see [1, 8, 10, 22, 23, 25–28, 36] and the references therein.

Hanson [6] introduced the concept of invex functions. Note that the invex functions are not convex functions. Ben-Israel and Mond [2] introduced the concept of invex sets and peinvex functions. They proved that a every differentiable preinvex function is an function, but the converse is not true. However, Noor and Noor [14] have established the equivalence between the invex functions and differentiable preinvex functions under some certain conditions. Noor [13] proved that the minimum of the differentiable preinvex function on the invex sets can be characterized by a class of variational inequalities, called the variational-like inequalities. Noor [15–17] also established some Hermite-Hadamard type integral inequalities for preinvex functions and log-preinvex functions. In fact, it has been shown that a function f is a preinvex function, if and only if, it satisfies the Hermite-Hadamard type integral inequalities. This result can be viewed an analogous to the convex functions. These results proved to be the starting point to derive various integral inequalities for different classes of preinvex functions and their variant forms. For recent developments, see [15–21, 29] and the references therein. Tunc et. al. [36] introduced beta-convex functions and established some inequalities. Noor et al [29] introduced and investigated the beta-preinvex functions. They derived several integral estimates for the beta-preinvex functions. For more details, [15–20, 29] and references therein.

The class of harmonic convex function was introduced by Anderson [1] and Iscan [8]. It is natural to unify these different concepts. Motivated by these facts, Noor et al. [23] introduced and investigated another class of harmonic convex functions, which is called harmonic preinvex functions and can be viewed as significant generalization of both the harmonic convex functions and preinvex functions. Noor et al. [30] also considered the harmonic beta-convex functions and derived some estimates for the integrals inequalities.

Motivated and inspired by the on going research in the convexity theory, we introduce and consider a new class of harmonic convex functions. This new of convex functions is called the harmonic beta-preinvex function. It is shown that several new classes of harmonic convex functions and harmonic

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preinvex functions can be obtained as special cases. We obtain some new Hermite-Hadamard type inequalities and estimates for the class of functions whose certain powers of the absolute value are harmonic beta-preinvex involving the Euler beta function and the Hypergeometric functions. Results obtained in this paper continue to hold for the various classes of convex functions. We also mention that the main results of this paper can be derived via the generalized harmonic beta-preinvex functions involving an arbitrary non-negative function. The ideas and techniques of this paper may be starting point for further research.

2. PRELIMINARIES

Let I be a nonempty closed set in $\mathbb{R}\setminus\{0\}$. Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a continuous function and $\eta(\cdot,\cdot) : I \times I \to \mathbb{R}$ be a continuous bifunction. In this section, we recall the following new and known concepts.

Definition 2.1. [23]. *A set $I \subseteq \mathbb{R}\setminus\{0\}$ is said to be a harmonic invex set with respect to the bifunction $\eta(\cdot,\cdot)$, if*

$$\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)} \in I, \quad \forall x, y \in I, t \in [0,1].$$

If $\eta(y,x) = y - x$, then the harmonic invex set reduces to harmonic convex set. Clearly, every harmonic convex set is invex set but the converse is not true.

We now introduce a new class of harmonic convex functions, which is called the harmonic beta-preinvex functions. This class unifies the concept of harmonic beta-convex and beta-preinvex functions.

Definition 2.2. *A function $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic beta-preinvex function, where $p, q > -1$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \le (1-t)^p t^q f(x) + t^p (1-t)^q f(y), \quad \forall x, y \in I, t \in (0,1). \tag{2.1}$$

If $t = \frac{1}{2}$, then, we have

$$f\left(\frac{2x(x+\eta(y,x))}{2x+\eta(y,x)}\right) \le \frac{f(x)+f(y)}{2^{p+q}} \tag{2.2}$$

which is called the harmonic Jensen beta-preinvex function.

We now discuss some important special cases of harmonic beta-preinvex functions, which include some new and known ones.

I). If $p = 1$ and $q = 0$, then Definition 4.1, reduces to the Definition of classical harmonic preinvex functions.

Definition 2.3. [23]. *A function $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonic preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \le (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0,1].$$

II). If $p = -1$ and $q = 0$, then Definition 4.1, reduces to the Definition of Godunova-Levin harmonic preinvex functions.

Definition 2.4. *A function $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be Godunova-Levin harmonic preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \le \frac{f(x)}{1-t} + \frac{f(y)}{t}, \qquad \forall x, y \in I, t \in (0,1).$$

III). If $p = 0$ and $q = 0$, then Definition 4.1, reduces to the Definition of harmonic P-preinvex functions.

Definition 2.5. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic P-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq f(x) + f(y), \qquad \forall x, y \in I, t \in [0,1].$$

IV). If $p = 1$ and $q = 1$, then Definition 4.1, reduces to the Definition of harmonic tgs-preinvex functions.

Definition 2.6. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic tgs-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq t(1-t)[f(x) + f(y)], \qquad \forall x, y \in I, t \in [0,1].$$

V). If $p = s$ and $q = 0$, then Definition 4.1, reduces to the Definition of harmonic s-preinvex functions.

Definition 2.7. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic s-preinvex function with respect to $\eta(\cdot,\cdot)$, where $s \in [-1,1]$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq (1-t)^s f(x) + t^s f(y), \qquad \forall x, y \in I, t \in (0,1).$$

VI). If $p = -s$ and $q = 0$, then Definition 4.1, reduces to the Definition of Godunova-Levin harmonic s-preinvex functions.

Definition 2.8. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be Godunova-Levin harmonic s-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq (1-t)^{-s} f(x) + t^{-s} f(y), \qquad \forall x, y \in I, t \in (0,1).$$

VII). If $p = \frac{1}{2}$ and $q = -\frac{1}{2}$, then Definition 4.1, reduces to the Definition of generalized harmonic MT-preinvex functions.

Definition 2.9. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be generalized harmonic MT-preinvex function with respect to $\eta(\cdot,\cdot)$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq \frac{\sqrt{1-t}}{\sqrt{t}} f(x) + \frac{\sqrt{t}}{\sqrt{1-t}} f(y), \qquad \forall x, y \in I, t \in (0,1).$$

Definition 2.10. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic* log*-beta-preinvex function on I, where $p, q > -1$, if*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq [f(x)]^{(1-t)^p t^q} [f(y)]^{t^p (1-t)^q}, \qquad \forall x, y \in I, t \in (0,1).$$

It follows that

$$\log f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq (1-t)^p t^q \log f(x) + t^p (1-t)^q \log f(y).$$

From definition 2.10, we have

$$\begin{aligned} f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) &\leq [f(x)]^{(1-t)^p t^q} [f(y)]^{t^p (1-t)^q} \\ &\leq (1-t)^p t^q f(x) + t^p (1-t)^q f(y). \end{aligned}$$

This shows that, harmonic log-beta-preinvex function implies harmonic beta-preinvex function, but the converse is not true.

If $\eta(y,x) = y - x$, then Definition 2.2 reduces to:

Definition 2.11. *A function $f : I \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic beta-convex function, where $p, q > -1$, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le (1-t)^p t^q f(x) + t^p (1-t)^q f(y), \quad \forall x, y \in I, t \in (0,1). \tag{2.3}$$

If $t = \frac{1}{2}$, then, we have

$$f\left(\frac{2xy}{x+y}\right) \le \frac{f(x)+f(y)}{2^{p+q}} \tag{2.4}$$

which is called the harmonic Jensen beta-convex function. These classes of harmonic beta-convex functions were introduced and studied by Noor et al. [30].

For appropriate and suitable choices of $p, q > -1$ and the bifunction $\eta(.,.)$, one can obtain several new and known classes of harmonic convex functions from Definition 4.1, and Definition 2.10. This shows that harmonic beta-preinvex functions are quite general and unifying ones.

Definition 2.12. *A function $f : [a, a+\eta(b,a)] \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonic symmetric with respect to $\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}$, if*

$$f(x) = f\left(\frac{a(a+\eta(b,a))x}{(2a+\eta(b,a))x - a(a+\eta(b,a))}\right) \quad \forall x \in [a, a+\eta(b,a)].$$

We recall the following special functions which are known as Beta function and hypergeometric function respectively.

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t, \quad x, y > 0,$$

$${}_2F_1[a, b; c, z] = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, \quad c > b > 0, |z| < 1.$$

We also need the following assumption regarding the bifucntion $\eta(.,.)$, which plays an important role in the derivation of the main results of this paper.

Condition *C* [11]. Let $I \subseteq \mathbb{R}$ be an invex set with respect to bifunction $\eta(\cdot,\cdot) : I \times I \to \mathbb{R}$. For any $x, y \in I$ and any $t \in [0,1]$, we have

$$\begin{aligned} \eta(y, y + t\eta(x,y)) &= -t\eta(x,y) \\ \eta(x, y + t\eta(x,y)) &= (1-t)\eta(x,y). \end{aligned}$$

3. Main results

In this section, we derive Hermite-Hadamard inequalities for harmonic beta-preinvex function.

Theorem 3.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be harmonic beta-preinvex function with $a < a+\eta(b,a)$. If $f \in L[a, a+\eta(b,a)]$, then*

$$\begin{aligned} 2^{p+q-1} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) &\le \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2}\mathrm{d}x \\ &\le \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}[f(a)+f(b)] \end{aligned}$$

Proof. Let f be harmonic beta-preinvex function. Then, taking $x = \frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}$ and $y = \frac{a(a+\eta(b,a))}{a+t\eta(b,a)}$ in (4.4), and using condition C, we have

$$\begin{aligned} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) &\le \frac{1}{2^{p+q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \\ &= \frac{1}{2^{p+q}}\left[\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t + \int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\mathrm{d}t\right] \end{aligned}$$

This implies

$$2^{p+q-1} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \leq \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2} \mathrm{d}x$$

Now consider

$$\begin{aligned}\frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2} \mathrm{d}x &= \int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \mathrm{d}t \\ &\leq f(a) \int_0^1 (1-t)^p t^q \mathrm{d}t + f(b) \int_0^1 t^p (1-t)^q \mathrm{d}t \\ &= [f(a)+f(b)]\beta(p+1,q+1),\end{aligned}$$

which is the required result. □

Theorem 3.2. *Let $f, g : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonic beta-preinvex functions. If $f, g \in L[a, a+\eta(b,a)]$, then*

$$\begin{aligned}&\frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x) g\left(\frac{a(a+\eta(b,a))x}{(2a+\eta(b,a))x - a(a+\eta(b,a))}\right)}{x^2} \mathrm{d}x \\ \leq\ & \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)} M(a,b) + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)} N(a,b),\end{aligned}$$

where

$$M(a,b) = f(a)g(a) + f(b)g(b) \tag{3.1}$$

$$N(a,b) = f(a)g(b) + f(b)g(a) \tag{3.2}$$

Proof. Let f, g be harmonic beta-preinvex functions. Then

$$f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \leq (1-t)^p t^q f(a) + t^p (1-t)^q f(b) \tag{3.3}$$

$$g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) \leq (1-t)^p t^q g(a) + t^p (1-t)^q g(b). \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}&f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) \\ \leq\ & [(1-t)^p t^q f(a) + t^p (1-t)^q f(b)][t^p (1-t)^q g(a) + (1-t)^p t^q g(b)]\end{aligned} \tag{3.5}$$

Integrating both sides of (3.5), we obtain

$$\begin{aligned}&\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) \mathrm{d}t \\ \leq\ & \int_0^1 [(1-t)^p t^q f(a) + t^p (1-t)^q f(b)][t^p (1-t)^q g(a) + (1-t)^p t^q g(b)] \mathrm{d}t \\ =\ & [f(a)g(a) + f(b)g(b)] \int_0^1 t^{p+q} (1-t)^{p+q} \mathrm{d}t + [f(a)g(b) + f(b)g(a)] \int_0^1 t^{2p} (1-t)^{2q} \mathrm{d}t \\ =\ & M(a,b)\beta(p+q+1, p+q+1) + N(a,b)\beta(2p+1, 2q+1)\end{aligned}$$

Thus

$$\begin{aligned}&\frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x) g\left(\frac{a(a+\eta(b,a))x}{(2a+\eta(b,a))x - a(a+\eta(b,a))}\right)}{x^2} \mathrm{d}x \\ \leq\ & M(a,b)\beta(p+q+1, p+q+1) + N(a,b)\beta(2p+1, 2q+1) \\ \leq\ & \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)} M(a,b) + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)} N(a,b),\end{aligned}$$

which is the required result. □

If $g\left(\frac{a(a+\eta(b,a))x}{(2a+\eta(b,a))x-a(a+\eta(b,a))}\right)=g(x)$ in Theorem 3.2, then it reduces to the following result.

Corollary 3.1. *Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic beta-preinvex functions. If $f,g\in L[a,a+\eta(b,a)]$, then*

$$\begin{aligned}&\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x\\ \leq\;&\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}N(a,b),\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) and (3.2) respectively.

Theorem 3.3. *Let $f,g:I\subset\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be harmonic beta-preinvex functions. If $fg\in L[a,a+\eta(b,a)]$, then*

$$\begin{aligned}&2^{2(p+q)-1}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\\ &-\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x\\ \leq\;&\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b),\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) and (3.2) respectively.

Proof. Let f be harmonic beta-preinvex function. Then taking $x=\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}$ and $y=\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}$ in (4.4) and using condition C, we have

$$\begin{aligned}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)&\leq\frac{1}{2^{p+q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right],\\ g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)&\leq\frac{1}{2^{p+q}}\left[g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right].\end{aligned}$$

Consider

$$\begin{aligned}&f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\\ \leq\;&\frac{1}{2^{2p+2q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right]\\ &\left[g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right]\\ \leq\;&\frac{1}{2^{2p+2q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\right.\\ &+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\\ &+[(1-t)^pt^qf(a)+t^p(1-t)^qf(b)][t^p(1-t)^qg(a)+(1-t)^pt^qg(b)]\\ &\left.+[t^p(1-t)^qf(a)+(1-t)^pt^qf(b)][(1-t)^pt^qg(a)+t^p(1-t)^qg(b)]\right].\end{aligned}$$

Integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\mathrm{d}t\\
\leq\ & \frac{1}{2^{2p+2q}}\left[\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\right.\\
&+\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\mathrm{d}t\\
&+2[f(a)g(a)+f(b)g(b)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t\\
&\left.+2[f(a)g(b)+f(b)g(a)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t\right]\\
=\ & \frac{1}{2^{2p+2q}}\left[\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t\right.\\
&+\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\mathrm{d}t\\
&\left.+2M(a,b)\beta(p+q+1,p+q+1)+2N(a,b)\beta(2p+1,2q+1)\right]\\
=\ & \frac{1}{2^{2p+2q-1}}\left[\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}t\right.\\
&\left.+M(a,b)\beta(p+q+1,p+q+1)+N(a,b)\beta(2p+1,2q+1)\right].
\end{aligned}$$

This completes the proof. □

Theorem 3.4. *Let $f,g : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonic beta-preinvex functions. If $fg \in L[a, a+\eta(b,a)]$, then*

$$\begin{aligned}
&\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mu(x)\frac{f(a)g(x)+f(b)g(x)}{x^2}\mathrm{d}x\\
&+\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\mu(x)\frac{g(a)f(x)+g(b)f(x)}{x^2}\mathrm{d}x\\
\leq\ & \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}N(a,b)\\
&+\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) and (3.2) respectively and

$$\mu(x)=\left(\frac{a^p(a+\eta(b,a)^q((a+\eta(b,a))-x)^p(x-a)^q}{x^{p+q}\eta(b,a)^{p+q}}\right)$$

Proof. Let f, g be harmonic beta-preinvex functions. Then, we have

$$\begin{aligned}
f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)&\leq (1-t)^p t^q f(a)+t^p(1-t)^q f(b),\\
g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)&\leq (1-t)^p t^q g(a)+t^p(1-t)^q g(b).
\end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned}
& f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)[(1-t)^p t^q g(a) + t^p(1-t)^q g(b)] \\
& +g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)[(1-t)^p t^q f(a) + t^p(1-t)^q f(b)] \\
\leq\ & [(1-t)^p t^q f(a) + t^p(1-t)^q f(b)][(1-t)^p t^q g(a) + t^p(1-t)^q g(b)] \\
& +f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \\
=\ & [f(a)g(a) + f(b)g(b)]t^{2p}(1-t)^{2q} + [f(a)g(b) + f(b)g(a)]t^{p+q}(1-t)^{p+q} \\
& +f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)
\end{aligned}$$

Integrating over $[0, 1]$, we have

$$\begin{aligned}
& g(a)\int_0^1 (1-t)^p t^q f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t \\
& +g(b)\int_0^1 t^p(1-t)^q f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t
\end{aligned}$$

$$\begin{aligned}
& +f(a)\int_0^1 (1-t)^p t^q g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t \\
& +f(b)\int_0^1 t^p(1-t)^q g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t \\
\leq\ & [f(a)g(a) + f(b)g(b)]\int_0^1 t^{2p}(1-t)^{2q}\mathrm{d}t \\
& +[f(a)g(b) + f(b)g(a)]\int_0^1 t^{p+q}(1-t)^{p+q}\mathrm{d}t \\
& +\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\mathrm{d}t
\end{aligned}$$

This implies

$$\begin{aligned}
& \frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mu(x)\frac{f(a)g(x) + f(b)g(x)}{x^2}\mathrm{d}x \\
& +\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)} \mu(x)\frac{g(a)f(x) + g(b)f(x)}{x^2}\mathrm{d}x \\
\leq\ & M(a,b)\beta(2p+1, 2q+1) + N(a,b)\beta(p+q+1, p+q+1) \\
& +\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2}\mathrm{d}x,
\end{aligned}$$

which is the required result. □

Theorem 3.5. *Let $f, g : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be harmonic beta-preinvex functions. If $fg \in L[a, a+\eta(b,a)]$, then*

$$\begin{aligned} & f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{g(x)}{x^2}\mathrm{d}x \\ & +g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)}{x^2}\mathrm{d}x \\ \leq\ & \frac{1}{2^{p+q}}\left[\frac{a(a+\eta(b,a))}{\eta(b,a)}\int_a^{a+\eta(b,a)}\frac{f(x)g(x)}{x^2}\mathrm{d}x\right. \\ & \left.+\frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)}M(a,b)+\frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)}N(a,b)\right] \\ & +2^{p+q-1}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right), \end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are given by (3.1) and (3.2) respectively.

Proof. Let f, g be harmonic beta-preinvex function. Then taking $x = \frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}$ and $y = \frac{a(a+\eta(b,a))}{a+t\eta(b,a)}$ in (4.4) and using condition C, we have

$$\begin{aligned} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) &\leq \frac{1}{2^{p+q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right], \\ g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) &\leq \frac{1}{2^{p+q}}\left[g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right]. \end{aligned}$$

Now, using $\langle x_1 - x_2, x_3 - x_4\rangle \geq 0, (x_1, x_2, x_3, x_4 \in \mathbb{R})$ and $x_1 < x_2$, $x_3 < x_4$, we have

$$\begin{aligned} & \frac{1}{2^{p+q}}f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\left[g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \\ & +\frac{1}{2^{p+q}}g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \\ \leq\ & \frac{1}{2^{2p+2q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)+f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \\ & \left[g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\right. \\ & \left.+g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right]+f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \\ \leq\ & \frac{1}{2^{2p+2q}}\left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right)\right. \\ & +f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) \\ & +2[f(a)g(a)+f(b)g(b)]t^{p+q}(1-t)^{p+q} \\ & \left.+2[f(a)g(b)+f(b)g(a)]t^{2p}(1-t)^{2q}\right] \\ & +f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \end{aligned}$$

Integrating over [0, 1], we have

$$\begin{aligned}
& \frac{1}{2^{p+q}} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \int_0^1 \left[g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \mathrm{d}t \\
& + \frac{1}{2^{p+q}} g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \int_0^1 \left[f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) + f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right)\right] \mathrm{d}t \\
\leq \; & \frac{1}{2^{2p+2q}} \left[\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \mathrm{d}t \right. \\
& + \int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) \mathrm{d}t \\
& \quad + 2[f(a)g(a) + f(b)g(b)] \int_0^1 t^{p+q}(1-t)^{p+q} \mathrm{d}t \\
& \quad \left. + 2[f(a)g(b) + f(b)g(a)] \int_0^1 t^{2p}(1-t)^{2q} \mathrm{d}t \right] \\
& \quad + \int_0^1 f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \mathrm{d}t \\
= \; & \frac{1}{2^{2p+2q}} \left[\int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+(1-t)\eta(b,a)}\right) \mathrm{d}t \right. \\
& + \int_0^1 f\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) g\left(\frac{a(a+\eta(b,a))}{a+t\eta(b,a)}\right) \mathrm{d}t \\
& \left. + 2M(a,b)\beta(p+q+1, p+q+1) + 2N(a,b)\beta(2p+1, 2q+1) \right] \\
& + f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right)
\end{aligned}$$

From the above inequality, it follows that

$$\begin{aligned}
& f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{g(x)}{x^2} \mathrm{d}x \\
& + g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) \frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)}{x^2} \mathrm{d}x \\
\leq \; & \frac{1}{2^{p+q}} \left[\frac{a(a+\eta(b,a))}{\eta(b,a)} \int_a^{a+\eta(b,a)} \frac{f(x)g(x)}{x^2} \mathrm{d}x \right. \\
& \left. + \frac{\Gamma(p+q+1)\Gamma(p+q+1)}{\Gamma(2p+2q+2)} M(a,b) + \frac{\Gamma(2p+1)\Gamma(2q+1)}{\Gamma(2p+2q+2)} N(a,b) \right] \\
& + 2^{p+q-1} f\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right) g\left(\frac{2a(a+\eta(b,a))}{2a+\eta(b,a)}\right),
\end{aligned}$$

which is the requires result. □

Remark 3.1. *If $\eta(b,a) = b-a$, then we obtain the new integral inequalities for the class of harmonic beta-convex functions.*

4. Integral Inequalities

We need the following Lemma in order to obtain new integral inequalities related to harmonic beta-preinvex function.

Lemma 4.1. *If $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is a function such that $f \in L[a, a+\eta(b,a)]$, then the following equality holds for some fixed $\alpha, \beta > 0$.*

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^{\alpha}(a+\eta(b,a)-x)^{\beta} f(x)\mathrm{d}x \\
= \quad & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}} f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\mathrm{d}t,
\end{aligned}$$

where $A_t = a + (1-t)\eta(b,a)$.

Theorem 4.1. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a, a+\eta(b,a)]$ and $|f|$ is harmonic beta-preinvex function on $[a, a+\eta(b,a)]$ and $\alpha, \beta > 0$, then*

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^{\alpha}(a+\eta(b,a)-x)^{\beta} f(x)\mathrm{d}x \\
\leq \quad & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(|f(a)|\varphi_1(t;a,b) + |f(b)|\varphi_2(t;a,b)\big),
\end{aligned}$$

where

$$\begin{aligned}
\varphi_1(t;a,b) &= \int_0^1 \frac{t^{\alpha+q}(1-t)^{\beta+p}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{\beta(\alpha+q+1, \beta+p+1)}{b^{\alpha+\beta+2}} \\
&\quad {}_2F_1[\alpha+\beta+2, \alpha+q+1; \alpha+\beta+p+q+2; 1-\frac{a}{b}] \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
\varphi_2(t;a,b) &= \int_0^1 \frac{t^{\alpha+p}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{\beta(\alpha+p+1, \beta+q+1)}{b^{\alpha+\beta+2}} \\
&\quad {}_2F_1[\alpha+\beta+2, \alpha+p+1; \alpha+\beta+p+q+2; 1-\frac{a}{b}] \qquad (4.2)
\end{aligned}$$

Proof. Using Lemma 4.1 and harmonic beta-preinvexity of $|f|$, we have

$$\begin{aligned}
&\int_a^{a+\eta(b,a)} (x-a)^{\alpha}(a+\eta(b,a)-x)^{\beta} f(x)\mathrm{d}x \\
= \quad & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}} \left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t \\
\leq \quad & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}} \Big\{(1-t)^p t^q |f(a)| \\
& +t^p(1-t)^q|f(b)|\Big\}\mathrm{d}t \\
= \quad & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(|f(a)|\int_0^1 \frac{t^{\alpha+q}(1-t)^{\beta+p}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \right.\\
& \left. +|f(b)|\int_0^1 \frac{t^{\alpha+p}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}\mathrm{d}t\right) \\
= \quad & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(|f(a)|\varphi_1(t;a,b) + |f(b)|\varphi_2(t;a,b)\big).
\end{aligned}$$

This completes the proof. □

Theorem 4.2. *Let $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ be a differentiable function on the interior I° of I. If $f \in L[a, a+\eta(b,a)]$ and $|f|^{\lambda}$ is harmonic beta-preinvex function on $[a, a+\eta(b,a)]$ and $\alpha, \beta > 0$,*

$\lambda \geq 1$, *then*

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^{\alpha}(a+\eta(b,a)-x)^{\beta} f(x)\mathrm{d}x \\ \leq\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(\varphi_3(t;a,b)\big)^{1-\frac{1}{\lambda}} \\ & \big(|f(a)|^{\lambda}\varphi_1(t;a,b)+|f(b)|^{\lambda}\varphi_2(t;a,b)\big)^{\frac{1}{\lambda}},\end{aligned}$$

where $\varphi_1(t;a,b)$, $\varphi_2(t;a,b)$ *are given by* (4.1) *and* (4.2) *respectively, and*

$$\begin{aligned}\varphi_3(t;a,b) &= \int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\ &= \frac{\beta(\alpha+1,\beta+1)}{b^{\alpha+\beta+2}}\,{}_2F_1[\alpha+\beta+2,\alpha+1;\alpha+\beta+2;1-\frac{a}{b}]\end{aligned}$$

Proof. Using Lemma 4.1, harmonic beta-preinvexity of $|f|^{\lambda}$ and power mean inequality, we have

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^{\alpha}(a+\eta(b,a)-x)^{\beta} f(x)\mathrm{d}x \\ =\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t \\ \leq\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\mathrm{d}t\right)^{1-\frac{1}{\lambda}} \\ & \left(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^{\lambda}\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ \leq\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\mathrm{d}t\right)^{1-\frac{1}{\lambda}} \\ & \left(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\left\{(1-t)^p t^q|f(a)|^{\lambda}+t^p(1-t)^q|f(b)|^{\lambda}\right\}\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ =\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\mathrm{d}t\right)^{1-\frac{1}{\lambda}} \\ & \left(|f(a)|^{\lambda}\int_0^1 \frac{t^{\alpha+q}(1-t)^{\beta+p}}{A_t^{\alpha+\beta+2}}\mathrm{d}t+|f(b)|^{\lambda}\int_0^1 \frac{t^{\alpha+p}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ =\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(\varphi_3(t;a,b)\big)^{1-\frac{1}{\lambda}} \\ & \big(|f(a)|^{\lambda}\varphi_1(t;a,b)+|f(b)|^{\lambda}\varphi_2(t;a,b)\big)^{\frac{1}{\lambda}},\end{aligned}$$

which the the required result. □

Theorem 4.3. *Let* $f : I = [a, a+\eta(b,a)] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ *be a differentiable function on the interior* I° *of* I. *If* $f \in L[a, a+\eta(b,a)]$ *and* $|f|^{\lambda}$ *is harmonic beta-preinvex function on* $[a, a+\eta(b,a)]$ *and* $\alpha, \beta > 0$, *then*

$$\begin{aligned}&\int_a^{a+\eta(b,a)} (x-a)^{\alpha}(a+\eta(b,a)-x)^{\beta} f(x)\mathrm{d}x \\ \leq\ & a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(\varphi_4(t;a,b)\big)^{\frac{1}{\mu}} \\ & \times\big(|f(a)|^{\lambda}+|f(b)|^{\lambda}\beta(p+1,q+1)\big)^{\frac{1}{\lambda}},\end{aligned}$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$ *and*

$$\begin{aligned}\varphi_4(t;a,b) &= \int_0^1 \frac{t^{\alpha\mu}(1-t)^{\beta\mu}}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t \\ &= \frac{\beta(\alpha\mu+1,\beta\mu+1)}{b^{(\alpha+\beta+2)\mu}} \\ &\quad {}_2F_1[(\alpha+\beta+2)\mu,\alpha\mu+1;(\alpha+\beta)\mu+2;1-\frac{a}{b}].\end{aligned}$$

Proof. Using Lemma 4.1, harmonic beta-preinvexity of $|f|^\lambda$ and the Holder's integral inequality, we have

$$\begin{aligned}&\int_a^{a+\eta(b,a)}(x-a)^\alpha(a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\ = \;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t \\ \leq \;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{t^{\alpha\mu}(1-t)^{\beta\mu}}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}} \\ & \left(\int_0^1 \left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}} \\ \leq \;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{t^{\alpha\mu}(1-t)^{\beta\mu}}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}} \\ & \left(\int_0^1 [(1-t)^p t^q|f(a)|^\lambda + t^p(1-t)^q|f(b)|^\lambda]\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ = \;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(\varphi_4(t;a,b)\big)^{\frac{1}{\mu}} \\ & \times\big(|f(a)|^\lambda+|f(b)|^\lambda\beta(p+1,q+1)\big)^{\frac{1}{\lambda}}.\end{aligned}$$

This completes the proof. □

Theorem 4.4. *Let* $f: I=[a,a+\eta(b,a)]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ *be a differentiable function on the interior* I° *of* I*. If* $f\in L[a,a+\eta(b,a)]$ *and* $|f|^\lambda$ *is harmonic beta-preinvex function on* $[a,a+\eta(b,a)]$ *and* $\alpha,\beta>0$*, then*

$$\begin{aligned}&\int_a^{a+\eta(b,a)}(x-a)^\alpha(a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x \\ \leq \;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\beta^{\frac{1}{\mu}}(p\mu+1,q\mu+1) \\ & \big(|f(a)|^\lambda\varphi_5(t;a,b)+|f(b)|^\lambda\varphi_6(t;a,b)\big)^{\frac{1}{\lambda}},\end{aligned}$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$ *and*

$$\begin{aligned}\varphi_5(t;a,b) &= \int_0^1 \frac{t^q(1-t)^p}{A_t^{(\alpha+\beta+2)\lambda}}\mathrm{d}t \\ &= \frac{\beta(q+1,p+1)}{b^{(\alpha+\beta+2)\lambda}} \\ &\quad {}_2F_1[(\alpha+\beta+2)\lambda,q+1;p+q+2;1-\frac{a}{b}] \\ \varphi_6(t;a,b) &= \int_0^1 \frac{t^p(1-t)^q}{A_t^{(\alpha+\beta+2)\lambda}}\mathrm{d}t \\ &= \frac{\beta(p+1,q+1)}{b^{(\alpha+\beta+2)\lambda}} \\ &\quad {}_2F_1[(\alpha+\beta+2)\lambda,p+1;p+q+2;1-\frac{a}{b}].\end{aligned}$$

Proof. Using Lemma 4.1, harmonic beta-preinvexity of $|f|^\lambda$ and the Holder integral inequality, we have

$$\begin{aligned}
&\int_a^{a+\eta(b,a)}(x-a)^\alpha(a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x\\
=\;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1\frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t\\
\le\;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}\mathrm{d}t\right)^{\frac{1}{\mu}}\\
&\left(\int_0^1\frac{1}{A_t^{(\alpha+\beta+2)\lambda}}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^\lambda\mathrm{d}t\right)^{\frac{1}{\lambda}}\\
\le\;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}\mathrm{d}t\right)^{\frac{1}{\mu}}\\
&\left(\int_0^1\frac{1}{A_t^{(\alpha+\beta+2)\lambda}}\left\{(1-t)^pt^q|f(a)|^\lambda+t^p(1-t)^q|f(b)|^\lambda\right\}\mathrm{d}t\right)^{\frac{1}{\lambda}}\\
=\;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 t^{\alpha\mu}(1-t)^{\beta\mu}\mathrm{d}t\right)^{\frac{1}{\mu}}
\end{aligned}$$

$$\begin{aligned}
&\left(|f(a)|^\lambda\int_0^1\frac{t^q(1-t)^p}{A_t^{(\alpha+\beta+2)\lambda}}\mathrm{d}t+|f(b)|^\lambda\int_0^1\frac{t^p(1-t)^q}{A_t^{(\alpha+\beta+2)\lambda}}\mathrm{d}t\right)^{\frac{1}{\lambda}}\\
=\;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\beta^{\frac{1}{\mu}}(p\mu+1,q\mu+1)\\
&\left(|f(a)|^\lambda\varphi_5(t;a,b)+|f(b)|^\lambda\varphi_6(t;a,b)\right)^{\frac{1}{\lambda}}.
\end{aligned}$$

This completes the proof. □

Theorem 4.5. *Let $f:I=[a,a+\eta(b,a)]\subseteq\mathbb{R}\setminus\{0\}\to\mathbb{R}$ be a differentiable function on the interior I° of I. If $f\in L[a,a+\eta(b,a)]$ and $|f|^\lambda$ is harmonic beta-preinvex function on $[a,a+\eta(b,a)]$ and $\alpha,\beta>0$, then*

$$\begin{aligned}
&\int_a^{a+\eta(b,a)}(x-a)^\alpha(a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x\\
\le\;& a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)(\varphi_7(t;a,b))^{\frac{1}{\mu}}\\
&\Big(|f(a)|^\lambda\beta(\alpha\lambda+q+1,\beta\lambda+p+1)\\
&+|f(b)|^\lambda\beta(\alpha\lambda+p+1,\beta\lambda+q+1)\Big)^{\frac{1}{\lambda}},
\end{aligned}$$

where $\frac{1}{\lambda}+\frac{1}{\mu}=1$ and

$$\begin{aligned}
\varphi_7(t;a,b)&=\int_0^1\frac{1}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t\\
&=\frac{{}_2F_1[(\alpha+\beta+2)\mu,1;2;1-\frac{a}{b}]}{b^{(\alpha+\beta+2)\mu}}.
\end{aligned}$$

Proof. Using Lemma 4.1, harmonic beta-preinvexity of $|f|^\lambda$ and the Holder's integral inequality, we have

$$\int_a^{a+\eta(b,a)} (x-a)^\alpha (a+\eta(b,a)-x)^\beta f(x)\mathrm{d}x$$

$$= a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{\alpha+\beta+2}}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|\mathrm{d}t$$

$$\leq a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}$$

$$\left(\int_0^1 t^{\alpha\lambda}(1-t)^{\beta\lambda}\left|f\left(\frac{a(a+\eta(b,a))}{A_t}\right)\right|^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}}$$

$$\leq a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}$$

$$\left(\int_0^1 t^{\alpha\lambda}(1-t)^{\beta\lambda}[(1-t)^p t^q|f(a)|^\lambda + t^p(1-t)^q|f(b)|^\lambda]\mathrm{d}t\right)^{\frac{1}{\lambda}}$$

$$= a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\left(\int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)\mu}}\mathrm{d}t\right)^{\frac{1}{\mu}}$$

$$\left(|f(a)|^\lambda \int_0^1 t^{\alpha\lambda+q}(1-t)^{\beta\lambda+p}\mathrm{d}t + |f(b)|^\lambda \int_0^1 t^{\alpha\lambda+p}(1-t)^{\beta\lambda+q}\mathrm{d}t\right)^{\frac{1}{\lambda}}$$

$$= a^{\alpha+1}(a+\eta(b,a))^{\beta+1}\eta^{\alpha+\beta+1}(b,a)\big(\varphi_7(t;a,b)\big)^{\frac{1}{\mu}}$$

$$\left(|f(a)|^\lambda\beta(\alpha\lambda+q+1,\beta\lambda+p+1) + |f(b)|^\lambda\beta(\alpha\lambda+p+1,\beta\lambda+q+1)\right)^{\frac{1}{\lambda}},$$

This completes the proof. □

Remarks

Results obtained in this paper can be the extended for generalized harmonic beta-preinvex functions with appropriate and suitable modifications. To be more precise, we introduce the generalized harmonic beta-preinvex functions as:

Definition 4.1. *A function $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be generalized harmonic beta-preinvex function, if there exists a non-negative arbitrary function h such that*

$$f\left(\frac{x(x+\eta(y,x))}{x+(1-t)\eta(y,x)}\right) \leq h(1-t)f(x) + h(t)f(y), \quad \forall x,y \in I, t \in (0,1). \tag{4.3}$$

If $t=\frac{1}{2}$, then, we have

$$f\left(\frac{2x(x+\eta(y,x))}{2x+\eta(y,x)}\right) \leq h(\frac{1}{2})\{f(x)+f(y)\}, \quad \forall x,y \in I, \tag{4.4}$$

which is called the generalized harmonic Jensen beta-preinvex function.

We note that, if $h(t)=(1-t)^p t^q, \quad p,q>-1$, then the generalized harmonic beta-preinvex functions reduce to the harmonic beta-preinvex functions as defined in Definition 2.2.

For appropriate choice of the bifunctions $\eta(.,.)$ and the non-negative function $h(.)$, one can obtain several new classes of convex, harmonic convex, harmonic preinvex functions and their variant forms. These class of convex functions are the subject of our future work.

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