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NEW INEQUALITIES OF HERMITE-HADAMARD TYPE FOR n-TIMES DIFFERENTIABLE s-CONVEX FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, some new inequalities Hermite-Hadamard type are obtained for functions whose nth derivatives in absolute value are s-convex functions. From our results, several inequalities of Hermite-Hadamard type can be derived in terms of functions whose first and second derivatives in absolute value are s-convex functions as special cases. Our results may provide refinements of some results already exist in literature. Applications to trapezoidal rule and to special means of established results are given.

1. Introduction

A function $f : I \to \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The double inequality (1.1) is known as the Hermite-Hadamard inequality (see [4]). The inequalities (1.1) hold in reversed direction if f is concave.

For recent results, refinements, counterparts, generalizations and new Hermite-Hadamard-type inequalities see [3, 7, 10–12, 14] and the references therein.

In [4], Hudzik and Maligranda considered among others the class of functions which are s-convex in the second sense and is defined as follows.

Definition 1.1. *[4] Let $s \in (0, 1]$ be a fixed real number. A function $f : [0,\infty) \to [0,\infty)$ is said to be s-convex in the second sense, if*

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y) \tag{1.2}$$

holds for all $x, y \in [0,\infty)$ and $t \in [0, 1]$. The class of s-convex functions in the second sense is denoted by K_s^2. If the inequality (1.2) holds in reversed direction, then f is to be an s-concave function in the second sense.

It is clear that the definition of s-convexity (s-concavity) coincides with the definition of convexity (concavity) when $s = 1$.

In [2], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

Theorem 1.1. *[2] Suppose that $f : [0,\infty) \to [0,\infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, $a, b \in [0,\infty)$ with $a < b$. If $a, b \in L([a, b])$, then the following inequalities hold*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

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For more recent results on Hermite-Hadamard type inequalities for functions whose derivatives in absolute value are s-convex functions, we refer the interested reader to [1, 8, 9, 13] and the references therein.

The main purpose of the present paper is to establish new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are s-convex. We believe that the results presented in this paper are better than those already exist in the literature concerning the inequalities of Hermite-Hadamard type for s-convex functions. Applications of our results to trapezoidal formula and to special means are given in Section 3 and Section 4.

2. Main Results

We will use the following Lemmas to establish our main results in this section.

Lemma 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a, b])$, where a, $b \in I^{\circ}$ with $a < b$, $n \in \mathbb{N}$, we have the identity*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)$$
$$= \frac{(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt$$
$$+ \frac{(-1)^n (b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right) dt, \quad (2.1)$$

where an empty sum is understood to be nil.

Proof. Suppose

$$I_n = \frac{(b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt$$

and

$$J_n = \frac{(-1)^n (b-a)^n}{2^{n+1}n!}\int_0^1 (1-t)^{n-1}(n-1+t) f^{(n)}\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right) dt.$$

For $n = 1$, we have

$$I_1 = \frac{b-a}{4}\int_0^1 t f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt$$

and

$$J_1 = \frac{(-1)(b-a)}{4}\int_0^1 t f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt.$$

By integration by parts and using the substitution $x = \frac{1-t}{2}a + \frac{1+t}{2}b$ for I_1 and $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ for J_1, we obtain

$$I_1 = \frac{1}{2}f(b) - \frac{1}{b-a}\int_{\frac{a+b}{2}}^b f(x)$$

and

$$J_1 = \frac{1}{2}f(a) - \frac{1}{b-a}\int_a^{\frac{a+b}{2}} f(x).$$

Hence

$$I_1 + J_1 = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx,$$

which coincides with the L.H.S of (2.1) for $n = 1$.
Similarly for $n = 2$, and using similar arguments as above, we have

$$I_2 + J_2 = \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx$$

which coincides with the L.H.S of (2.1) for $n = 2$.
Suppose (2.1) holds for $n = m - 1 \geq 3$.
Now for $n = m$, we have

$$\begin{aligned}
&\frac{(b-a)^m}{2^{m+1}m!}\int_0^1 (1-t)^{m-1}(m-1+t)f^{(m)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
&\quad+\frac{(-1)^m(b-a)^m}{2^{m+1}m!}\int_0^1 (1-t)^{m-1}(m-1+t)f^{(m)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt \\
&= -\frac{(b-a)^{m-1}(m-1)\left[1+(-1)^{m-1}\right]}{2^m m!}f^{(m-1)}\left(\frac{a+b}{2}\right) \\
&\quad+\frac{(b-a)^{m-1}}{2^m(m-1)!}\int_0^1 (1-t)^{m-2}(m-2+t)f^{(m)}\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \\
&\quad+\frac{(-1)^{m-1}(b-a)^{m-1}}{2^m(m-1)!}\int_0^1 (1-t)^{m-2}(m-2+t)f^{(m)}\left(\frac{1-t}{2}b+\frac{1+t}{2}a\right)dt \\
&= \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{m-2}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right) \\
&\quad-\frac{(b-a)^{m-1}(m-1)\left[1+(-1)^{m-1}\right]}{2^m m!}f^{(m-1)}\left(\frac{a+b}{2}\right) \\
&= \frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{m-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right).
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 2.2. *[15] Let $x \geq 0$, $y \geq 0$, the inequality*

$$(x+y)^\theta \leq x^\theta + y^\theta$$

holds for $0 < \theta \leq 1$ and the inequality

$$(x-y)^\theta \leq x^\theta - y^\theta$$

holds for $\theta \geq 1$.

Now we state and prove some new Hermite-Hadamard type inequalities for functions whose nth derivatives in absolute value are s-convex and s-concave in the second sense.

Theorem 2.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$, where $a,\ b \in I^\circ$ with $a < b$, $n \in \mathbb{N}$. If $\left|f^{(n)}\right|^q$ is s-convex on $[a,b]$ for some fixed $s \in (0,1]$ and $q \in [1,\infty)$, we have the inequality*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right. \\
&\qquad\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^n}{2^{n+s/q+1}n!}\left(\frac{n}{n+1}\right)^{1-1/q} \\
&\times\left\{\left(\frac{\left(n^2+s(n-1)\right)\left|f^{(n)}(a)\right|^q}{(n+s)(n+s+1)}+\left(\frac{n}{n+1}+\frac{n(n+s)}{n+s+1}B(s+1,n)\right)\left|f^{(n)}(b)\right|^q\right)^{1/q}\right. \\
&\left.+\left(\frac{\left(n^2+s(n-1)\right)\left|f^{(n)}(b)\right|^q}{(n+s)(n+s+1)}+\left(\frac{n}{n+1}+\frac{n(n+s)}{n+s+1}B(s+1,n)\right)\left|f^{(n)}(a)\right|^q\right)^{1/q}\right\}, \qquad (2.2)
\end{aligned}$$

where

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt,\alpha,\beta>0$$

is the Euler Beta function.

Proof. From Lemma 2.1, the Hölder inequality and s-convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{(b-a)^n}{2^{n+s/q+1}n!}\left(\int_0^1(1-t)^{n-1}(n-1+t)\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1(1-t)^{n-1}(n-1+t)\left[(1-t)^s\left|f^{(n)}(a)\right|^q+(1+t)^s\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1(1-t)^{n-1}(n-1+t)\left[(1-t)^s\left|f^{(n)}(b)\right|^q+(1+t)^s\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}. \quad (2.3)$$

Since

$$\int_0^1(1-t)^{n-1}(n-1+t)\,dt=\frac{n}{n+1},$$
$$\int_0^1(1-t)^{n-1}(n-1+t)(1-t)^s\,dt=\frac{n^2+s(n-1)}{(n+s)(n+s+1)},$$

by using the property

$$B(x,y+1)=\frac{y}{x+y}B(x,y)$$

of the Euler Beta function and Lemma 2.2, we have

$$\int_0^1(1-t)^{n-1}(n-1+t)(1+t)^s\,dt\le\int_0^1(1-t)^{n-1}(n-1+t)(1+t^s)\,dt$$
$$=\frac{n}{n+1}+nB(s+1,n)-B(s+1,n+1)$$
$$=\frac{n}{n+1}+nB(s+1,n)-\frac{n}{n+s+1}B(s+1,n)$$
$$=\frac{n}{n+1}+\frac{n(n+s)}{n+s+1}B(s+1,n).$$

From the above facts and the inequality (2.3), we get the required inequality (2.2). This completes the proof of the Theorem. □

The following corollaries are direct consequences of Theorem 2.1.

Corollary 2.1. *Under the assumptions of Theorem 2.1, if $q=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right.$$
$$\left.-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\le\frac{(b-a)^n}{2^{n+s+1}n!}$$
$$\times\left(\frac{n^2+s(n-1)}{(n+s)(n+s+1)}+\frac{n}{n+1}+\frac{n(n+s)}{n+s+1}B(s+1,n)\right)\left[\left|f^{(n)}(a)\right|+\left|f^{(n)}(b)\right|\right], \quad (2.4)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1.

Corollary 2.2. *Under the assumptions of Theorem 2.1, if $n = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq (b-a)\left(\frac{1}{2}\right)^{3+s/q-1/q}\left\{\left[\frac{\left|f'(a)\right|^q}{(1+s)(2+s)}+\left(\frac{1}{2}+\frac{1}{2+s}\right)\left|f'(b)\right|^q\right]^{1/q} + \left[\left(\frac{1}{2}+\frac{1}{2+s}\right)\left|f'(a)\right|^q+\frac{\left|f'(b)\right|^q}{(1+s)(2+s)}\right]^{1/q}\right\}. \quad (2.5)$$

Corollary 2.3. *If we take $q = 1$ in Corollary 2.2, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)(s+3)}{s+1}\left(\frac{1}{2}\right)^{3+s}\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \quad (2.6)$$

Corollary 2.4. *Suppose the assumptions of Theorem 2.1 are fulfilled and if $n = 2$, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2^{4+s/q}}\left(\frac{2}{3}\right)^{1-1/q} \times\left\{\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(a)\right|^q+\left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(b)\right|^q\right]^{1/q} + \left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(b)\right|^q+\left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(a)\right|^q\right]^{1/q}\right\}. \quad (2.7)$$

Corollary 2.5. *If $q = 1$ in Corollary 2.4, we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{2^{4+s}}\left(\frac{2}{3}+\frac{s^2+7s+8}{(s+1)(s+2)(s+3)}\right)\left[\left|f''(a)\right|+\left|f''(b)\right|\right]. \quad (2.8)$$

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function on I°, $n \in \mathbb{N}$. If $f^{(n)} \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \in (1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^n\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right]^{1-1/q}}{2^{n+s/q+1}n!}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \times\left\{\left[\frac{\left|f^{(n)}(a)\right|^q}{nq-q+s+1}+\left(\frac{1}{nq-q+1}+B(s+1,nq-q+1)\right)\left|f^{(n)}(b)\right|^q\right]^{1/q} + \left[\frac{\left|f^{(n)}(b)\right|^q}{nq-q+s+1}+\left(\frac{1}{nq-q+1}+B(s+1,nq-q+1)\right)\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\}. \quad (2.9)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1.

Proof. Using Lemma 2.1, by using the first inequality in Lemma 2.2, the Hölder inequality and convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\le\frac{(b-a)^n}{2^{n+s/q+1}n!}\left(\int_0^1(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1(1-t)^{q(n-1)}\left[(1-t)^s\left|f^{(n)}(a)\right|^q+(1+t^s)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.$$
$$\left.+\left(\int_0^1(1-t)^{q(n-1)}\left[(1-t)^s\left|f^{(n)}(b)\right|^q+(1+t^s)\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}. \quad (2.10)$$

By simple computations, we observe that

$$\int_0^1(n-1+t)^{q/(q-1)}\,dt=\left(\frac{q-1}{2q-1}\right)\left[n^{(2q-1)/(q-1)}-(n-1)^{(2q-1)/(q-1)}\right],$$

$$\int_0^1(1-t)^{q(n-1)+s}\,dt=\frac{1}{nq-q+s+1}$$

and

$$\int_0^1(1-t)^{q(n-1)}(1+t^s)\,dt=\frac{1}{nq-q+1}+B(s+1,nq-q+1).$$

Using the above results in (2.10), we obtain the required result. This completes the proof of the theorem. □

Corollary 2.6. *Suppose the assumptions of Theorem 2.2 are satisfied and if $n=1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\le\frac{(b-a)}{2^{2+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left(\frac{1}{s+1}\right)^{1/q}$$
$$\times\left\{\left[\left|f'(a)\right|^q+(s+2)\left|f'(b)\right|^q\right]^{1/q}+\left[\left|f'(b)\right|^q+(s+2)\left|f'(a)\right|^q\right]^{1/q}\right\}. \quad (2.11)$$

Corollary 2.7. *Under the assumptions of Theorem 2.2, if $n=2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le\frac{(b-a)^2\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{2^{4+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}$$
$$\times\left\{\left[\frac{1}{q+s+1}\left|f''(a)\right|^q+\left(\frac{1}{q+1}+B(s+1,q+1)\right)\left|f''(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{q+s+1}\left|f''(b)\right|^q+\left(\frac{1}{q+1}+B(s+1,q+1)\right)\left|f''(a)\right|^q\right]^{1/q}\right\}, \quad (2.12)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1.

Theorem 2.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function on I°, $n \in \mathbb{N}$. If $f^{(n)} \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \in (1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+\frac{s}{q}+1} n!}\left(\frac{q-1}{nq-1}\right)^{1-1/q}\left\{\left[P\left|f^{(n)}(a)\right|^q + Q\left|f^{(n)}(b)\right|^q\right]^{1/q} + \left[P\left|f^{(n)}(b)\right|^q + Q\left|f^{(n)}(a)\right|^q\right]^{1/q}\right\}, \quad (2.13)$$

where

$$P = n^{q+s+1} B\left(\frac{1}{n}; s+1, q+1\right),$$
$$Q = n^q\left(\frac{s+2}{s+1}\right) - \frac{1}{q+1} - B(s+1, q+1),$$

$B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1 and

$$B(x;\alpha,\beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\,dt, \alpha,\beta > 0, 0 \leq x \leq 1$$

is the incomplete Beta function.

Proof. Using Lemma 2.1, the first inequality in Lemma 2.2, the Hölder inequality and convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx - \sum_{k=1}^{n-1} \frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!} f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^n}{2^{n+s/q+1} n!}\left(\int_0^1 (1-t)^{q(n-1)/(q-1)}\,dt\right)^{1-1/q}$$
$$\times\left\{\left(\int_0^1 (n-1+t)^q\left[(1-t)^s\left|f^{(n)}(a)\right|^q + (1+t^s)\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q} + \left(\int_0^1 (n-1+t)^q\left[(1-t)^s\left|f^{(n)}(b)\right|^q + (1+t^s)\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}. \quad (2.14)$$

By using the second inequality of Lemma 2.2 and simple computation, it is easy to observe that

$$\int_0^1 (1-t)^{q(n-1)/(q-1)}\,dt = \frac{q-1}{nq-1},$$

$$\int_0^1 (n-1+t)^q(1-t)^s\,dt = n^{q+s+1}\int_0^{\frac{1}{n}} t^s(1-t)^q\,dt = n^{q+s+1}B\left(\frac{1}{n}; s+1, q+1\right) = P$$

and

$$\int_0^1 (n-1+t)^q(1+t^s)\,dt \leq \int_0^1 \left(n^q - (1-t)^q\right)(1+t^s)\,dt = n^q\left(\frac{s+2}{s+1}\right) - \frac{1}{q+1} - B(s+1,q+1) = Q.$$

Hence (2.13) follows from (2.14) and using the above results. This completes the proof of the theorem. □

Corollary 2.8. *Suppose the assumptions of Theorem 2.3 are satisfied and if $n = 1$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \le \frac{(b-a)}{2^{2+s/q}}$$
$$\times\left\{\left[B(s+1,q+1)\left|f'(a)\right|^q+\left(\frac{s+2}{s+1}-\frac{1}{q+1}-B(s+1,q+1)\right)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[B(s+1,q+1)\left|f'(b)\right|^q+\left(\frac{s+2}{s+1}-\frac{1}{q+1}-B(s+1,q+1)\right)\left|f'(a)\right|^q\right]^{1/q}\right\}, \quad (2.15)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1.

Corollary 2.9. *Suppose the assumptions of Theorem 2.3 are satisfied and if $n = 2$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$
$$\le \frac{(b-a)^2}{2^{4+\frac{s}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''(a)\right|^q\right.\right.$$
$$\left.+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-B(s+1,q+1)\right)\left|f''(b)\right|^q\right]^{1/q}$$
$$+\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''(b)\right|^q\right.$$
$$\left.\left.+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-2^{q+s+1}B(s+1,q+1)\right)\left|f''(a)\right|^q\right]^{1/q}\right\}, \quad (2.16)$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1 and $B(x;\alpha,\beta)$ is the incomplete Beta function defined as in Theorem 2.3.

A different approach results in the following theorem.

Theorem 2.4. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f^{(n)}$ exists on I° and $f^{(n)} \in L([a,b])$ for $n \in \mathbb{N}$, where $a, b \in I^\circ$ with $a < b$. If $\left|f^{(n)}\right|^q$ is convex on $[a,b]$ for $q \in (1,\infty)$, we have the inequality*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{n^{n+1-1/q}(b-a)^n}{2^{n+s/q+1}n!}\left(\frac{1}{s+1}\right)^{1/q}\left[B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q}$$
$$\times\left\{\left[\left|f^{(n)}(a)\right|^q+\left(2^{s+1}-1\right)\left|f^{(n)}(b)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\left(2^{s+1}-1\right)\left|f^{(n)}(a)\right|^q+\left|f^{(n)}(b)\right|^q\right]^{1/q}\right\}, \quad (2.17)$$

where

$$B(x;\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{1-\beta}\,dt, 0\le x\le 1, \alpha>0, \beta>0$$

is the incomplete beta function.

Proof. Using Lemma 2.1, the Hölder inequality and the convexity of $\left|f^{(n)}\right|^q$ on $[a,b]$, we have

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx-\sum_{k=1}^{n-1}\frac{k\left[1+(-1)^k\right](b-a)^k}{2^{k+1}(k+1)!}f^{(k)}\left(\frac{a+b}{2}\right)\right|\\
&\leq \frac{(b-a)^n}{2^{n+s/q+1}n!}\left(\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt\right)^{1-1/q}\\
&\times\left\{\left(\int_0^1\left[(1-t)^s\left|f^{(n)}(a)\right|^q+(1+t)^s\left|f^{(n)}(b)\right|^q\right]dt\right)^{1/q}\right.\\
&\left.+\left(\int_0^1\left[(1-t)^s\left|f^{(n)}(b)\right|^q+(1-t)^s\left|f^{(n)}(a)\right|^q\right]dt\right)^{1/q}\right\}. \quad (2.18)\end{aligned}$$

By simple computations of integrals, the inequality (2.17) follows from the inequality (2.18) and using the fact that

$$\int_0^1(1-t)^{q(n-1)/(q-1)}(n-1+t)^{q/(q-1)}\,dt=n^{\frac{nq+q-1}{q-1}}B\left(\frac{1}{n};\frac{nq-1}{q-1},\frac{2q-1}{q-1}\right).$$

This completes the proof of the theorem. □

Corollary 2.10. *Suppose the assumptions of Theorem 2.4 are satisfied and if $n=1$, we have*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{(b-a)}{2^{2+s/q}}\left(\frac{1}{s+1}\right)^{1/q}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\\
&\times\left\{\left[\left|f'(a)\right|^q+\left(2^{s+1}-1\right)\left|f'(b)\right|^q\right]^{1/q}+\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}. \quad (2.19)\end{aligned}$$

Proof. Proof follows from the fact that

$$B\left(1;1,\frac{2q-1}{q-1}\right)=B\left(1,\frac{2q-1}{q-1}\right)=\int_0^1(1-t)^{\frac{2q-1}{q-1}-1}\,dt=\frac{q-1}{2q-1}.$$

□

Corollary 2.11. *Under the assumptions of Theorem 2.4 and $n=2$, we have the following inequality*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\\
&\leq\frac{(b-a)^2}{2^{1+s/q+1/q}}\left(\frac{1}{s+1}\right)^{1/q}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q}\\
&\times\left\{\left[\left|f''(a)\right|^q+\left(2^{s+1}-1\right)\left|f''(b)\right|^q\right]^{1/q}+\left[\left(2^{s+1}-1\right)\left|f''(a)\right|^q+\left|f''(b)\right|^q\right]^{1/q}\right\}, \quad (2.20)\end{aligned}$$

where $B(x;\alpha,\beta)$ is defined as in Theorem 2.4.

3. Applications to the Trapezoidal Formula

Let d be a division of the interval $[a,b]$, i.e. $a=x_0<x_1<...<x_{n-1}<x_n=b$, and consider the quadrature formula

$$\int_a^b f(x)dx=T(f,d)+E(f,d),$$

where

$$T(f,d)=\sum_{i=0}^{n-1}(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2}$$

is the trapezoidal versions and $E(f,d)$ is the associated error. Here, we derive some error estimates for the trapezoidal formula in terms of absolute values of the second derivative of f which may be better than those already exist in the literature.

Theorem 3.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f'' \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f''\right|^q$ is s-convex on $[a,b]$ for $q \geq 1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned}|E(f,d)| \leq{}& \frac{1}{2^{4+s/q}}\left(\frac{2}{3}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\\ &\times\left\{\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_i)\right|^q+\left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right.\\ &\left.+\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_{i+1})\right|^q+\left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}. \quad (3.1)\end{aligned}$$

Proof. By applying Corollary 2.4 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the division d, we have

$$\begin{aligned}|E(f,d)| &= \left|\sum_{i=0}^{n-1}\left\{(x_{i+1}-x_i)\frac{f(x_i)+f(x_{i+1})}{2}-\int_{x_i}^{x_{i+1}}f(x)\,dx\right\}\right|\\ &\leq \sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2}-\frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)\,dx\right|\\ &\leq \frac{1}{2^{4+s/q}}\left(\frac{2}{3}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\\ &\times\left\{\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_i)\right|^q+\left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right.\\ &\left.+\left[\frac{(s+4)}{(s+2)(s+3)}\left|f''(x_{i+1})\right|^q+\left(\frac{2}{3}+\frac{2}{(s+1)(s+3)}\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}. \quad (3.2)\end{aligned}$$

□

Corollary 3.1. *Under the assumptions of Theorem 3.1, for $q = 1$, we have*

$$\begin{aligned}|E(f,d)| \leq{}& \frac{1}{2^{4+s}}\left(\frac{2}{3}+\frac{s^2+7s+8}{(s+1)(s+2)(s+3)}\right)\\ &\times\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\left[\left|f''(x_i)\right|+\left|f''(x_{i+1})\right|\right]. \quad (3.3)\end{aligned}$$

Theorem 3.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f'' \in L([a,b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f''\right|^q$ is s-convex on $[a,b]$ for $q > 1$, then for every division d of $[a,b]$, we have*

$$\begin{aligned}|E(f,d)| \leq{}& \frac{\left[2^{(2q-1)/(q-1)}-1\right]^{1-1/q}}{2^{4+s/q}}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\\ &\times\left\{\left[\frac{1}{q+s+1}\left|f''(x_i)\right|^q+\left(\frac{1}{q+1}+B(s+1,q+1)\right)\left|f''(x_{i+1})\right|^q\right]^{1/q}\right.\\ &\left.+\left[\frac{1}{q+s+1}\left|f''(x_{i+1})\right|^q+\left(\frac{1}{q+1}+B(s+1,q+1)\right)\left|f''(x_i)\right|^q\right]^{1/q}\right\}, \quad (3.4)\end{aligned}$$

where $B(\alpha,\beta)$ is the Euler Beta function defined as in Theorem 2.1.

Proof. It follows from Corollary 2.7. □

Theorem 3.3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f'' \in L([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f''\right|^q$ is s-convex on $[a, b]$ for $q > 1$, then for every division d of $[a, b]$, we have*

$$
\begin{aligned}
|E(f,d)| \leq \frac{1}{2^{4+\frac{1}{q}}}\left(\frac{q-1}{2q-1}\right)^{1-1/q} & \sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3 \\
& \times\left\{\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''\left(x_i\right)\right|^q\right.\right. \\
+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-B\left(s+1,q+1\right)\right) & \left.\left|f''\left(x_{i+1}\right)\right|^q\right]^{1/q} \\
& +\left[2^{q+s+1}B\left(\frac{1}{2};s+1,q+1\right)\left|f''\left(x_{i+1}\right)\right|^q\right. \\
+\left(2^q\left(\frac{s+2}{s+1}\right)-\frac{1}{q+1}-2^{q+s+1}B\left(s+1,q+1\right)\right) & \left.\left.\left|f''\left(x_i\right)\right|^q\right]^{1/q}\right\}, \quad (3.5)
\end{aligned}
$$

where $B(\alpha, \beta)$ is the Euler Beta functions and $B(x; \alpha, \beta)$ is the incomplete beta function defined as in Theorem 2.3.

Proof. It is a direct consequence of Corollary 2.9. □

Theorem 3.4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° such that $f'' \in L([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $\left|f''\right|^q$ is s-convex on $[a, b]$ for $q > 1$, then for every division d of $[a, b]$, we have*

$$
\begin{aligned}
|E(f,d)| \leq & \frac{1}{2^{1+s/q+1/q}}\left(\frac{1}{s+1}\right)^{1/q}\left[B\left(\frac{1}{2};\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q} \\
& \times\sum_{i=0}^{n-1}\left(x_{i+1}-x_i\right)^3\left\{\left[\left|f''\left(x_i\right)\right|^q+\left(2^{s+1}-1\right)\left|f''\left(x_{i+1}\right)\right|^q\right]^{1/q}\right. \\
& \left.+\left[\left(2^{s+1}-1\right)\left|f''\left(x_i\right)\right|^q+\left|f''\left(x_{i+1}\right)\right|^q\right]^{1/q}\right\}, \quad (3.6)
\end{aligned}
$$

where $B(x; \alpha, \beta)$ is the incomplete beta function defined as in Theorem 2.3.

Proof. The proof follows by using Corollary 2.9. □

4. Applications to the Special Means

Now, we consider applications of our results to special means. We consider the means for positive real numbers a, $b \in \mathbb{R}_+$. We consider

(1) The arithmetic mean:
$$A(a,b) = \frac{a+b}{2}; a, b > 0.$$
(2) Generalized log-mean:
$$L_p(a,b) = \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}; \ a,\ b > 0,\ p \in \mathbb{R}\setminus\{-1,0\},\ a \neq b.$$

Now we apply our results from Section 2 to give some inequalities for special means.

It is shown in [4] that the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0, \end{cases}$$

where $s \in (0, 1)$, a, b, $c \in \mathbb{R}$ with $0 \leq c \leq a$, $b \geq 0$, is s-convex on $[0, \infty)$. Hence, for $a = c = 0$, $b \geq 0$, the function $f(t) = bt^s$ is an s-convex function on $[0, \infty)$, where $s \in (0, 1)$.

Theorem 4.1. *For a, $b \in \mathbb{R}_+$, $a < b$, $0 < s < 1$ and $q \in [1, \infty)$, we have*

$$\left| A\left(a^{s+2}, b^{s+2}\right) - L_{s+2}^{s+2}\left(a^{s+2}, b^{s+2}\right) \right| \leq \frac{(b-a)^2}{2^{3+s}} \left(\frac{2}{3}(s+2)(s+1) + \frac{s^2+7s+8}{s+3} \right) A\left(a^s, b^s\right). \quad (4.1)$$

Proof. Let $f(x) = x^{s+2}$, $x \in \mathbb{R}_+$. Then $\left| f''(x) \right| = (s+2)(s+1) x^s$ is an s-convex function on $\mathbb{R}_+$. Applying Corollary 2.5, we obtain the required result. □

Theorem 4.2. *For a, $b \in \mathbb{R}_+$, $a < b$, $0 < s < 1$ and $q \in (1, \infty)$, we have*

$$\left| A\left(a^{s/q+1}, b^{s/q+1}\right) - L_{s/q+1}^{s/q+1}\left(a^{s/q+1}, b^{s/q+1}\right) \right| \leq \frac{(b-a)}{2^{2+s/q}} \left(\frac{q-1}{2q-1} \right)^{1-1/q} \left(\frac{1}{s+1} \right)^{1/q} (s/q+1) \times \left\{ \left[2A\left(a^s, b^s\right) + (s+1) b^s\right]^{1/q} + \left[2A\left(a^s, b^s\right) + (s+1) a^s\right]^{1/q} \right\}. \quad (4.2)$$

Proof. Let $f(x) = x^{s/q+1}$, $x \in \mathbb{R}_+$. Then $\left| f'(x) \right|^q = [(s/q+1)]^q x^s$ is an s-convex function on $\mathbb{R}_+$. Applying Corollary 2.6, we obtain the required result. □

Theorem 4.3. *For a, $b \in \mathbb{R}_+$, $a < b$, $0 < s < 1$ and n, $q \in \mathbb{N}$, $n, q > 1$, we have*

$$\left| A\left(a^{s/q+1}, b^{s/q+1}\right) - L_{s/q+1}^{s/q+1}\left(a^{s/q+1}, b^{s/q+1}\right) \right| \leq \frac{(b-a)}{2^{2+s/q-1/q}} \left(\frac{1}{s+1} \right)^{1/q} \left(\frac{q-1}{2q-1} \right)^{1-1/q} (s/q+1) \times \left\{ \left[A\left(a^s, b^s\right) + (2^s-1) b^s\right]^{1/q} + \left[(2^s-1) a^s + A\left(a^s, b^s\right)\right]^{1/q} \right\}. \quad (4.3)$$

Proof. Let $f(x) = x^{s/q+1}$, $x \in \mathbb{R}_+$. Then $\left| f'(x) \right|^q = [(s/q+1)]^q x^s$ is an s-convex function on $\mathbb{R}_+$. Applying Corollary 2.10, we obtain the required result. □

Remark 4.1. *Many other interesting inequalities for means can be obtained by applying the other results to some suitable s-convex functions, however the details are left to the interested reader.*

Remark 4.2. *For $s = 1$, we get some of the results from [12].*

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