International Journal of Analysis and Applications
ISSN 2291-8639
Volume 13, Number 2 (2017), 198-205
http://www.etamaths.com

BEST PROXIMITY POINTS FOR A NEW CLASS OF GENERALIZED PROXIMAL MAPPINGS

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Abstract. The best proximity points are usually used to find the optimal approximate solution of the operator equation $Tx = x$, when T has no fixed point. In this paper, we prove some best proximity point theorems for nonself multivalued operators, following the foot steps of Basha and Shahzad [Best proximity point theorems for generalized proximal contractions, Fixed Point Theory Appl., 2012, 2012:42].

1. Introduction

Fixed point theory have an important role in many branches of mathematics such as differential and integral equations, optimization and variational analysis. This theory mainly concerns with the fixed point equation $Tx = x$, where $T\colon A \to B$ is some nonlinear operator. The solution of this equation is called a fixed point of the operator T. It is not necessary that the equation has a solution for every nonlinear operator T. For example this one has no solution when $A \cap B = \emptyset$. In this case we may find a point $x \in A$ which is closest to Tx, that is, the distance between Tx and x is least as compare to other elements of A. Such a point is called the best proximity point of T. The notion of best proximity point was initiated by Fan [1] for normed spaces. Eldred and Veeramani [2] generalized this notion in the context of metric spaces. In literature there are many important best proximity point theorems in different settings: Jleli *et al.* and Ali *et al.* [3,4], for α-ψ-proximal mappings; Akbar and Gabeleh [5,6], Derafshpour *et al.* [7], Di Bari *et al.* [8], Rezapour *et al.* [9], Vetro [10], for cyclic mappings; Alghamdi *et al.* [11] for mappings in geodesic metric spaces; Al-Thagafi and Shahzad [12], for Kakutani multimaps; Markin and Shahzad [13], for relatively u-continuous mappings; Nashine *et al.* [14], for rational proximal contractions; Akbar and Gabeleh [15], for multivalued non-self mappings; Choudhury *et al.* [16] for best proximity point and coupled best proximity point in partially ordered metric spaces; Shatanawi and Pitea [17], for best proximity points and best proximity coupled points in complete metric spaces with (P)-property; Jamali and Vaespour [18], for best proximity point for nonlinear contractions in Menger probabilistic metric spaces; Bejenaru and Pitea [19], for fixed point and best proximity point theorems in partial metric spaces.

Motivated and inspired by the research introduced above, in this paper we introduce our best proximity point theorems for nonself multivalued operators, following the foot steps method of Basha and Shahzad [20].

2. Previous results

Now, we recollect some basic notions, definitions and results which we require subsequently. Let (X, d) be a metric space. For $A, B \subseteq X$, $\operatorname{dist}(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$, $d(x, B) = \inf\{d(x, b) : \ b \in B\}$, $A_0 = \{a \in A : \ d(a, b) = \operatorname{dist}(A, B) \text{ for some } b \in B\}$, $B_0 = \{b \in B : \ d(a, b) = \operatorname{dist}(A, B) \text{ for some } a \in A\}$, while $CB(B)$ is the set of all nonempty closed and bounded subsets of B.

A point $x^* \in X$ is said to be a best proximity point of $T\colon A \to CB(B)$ if $d(x^*, Tx^*) = \operatorname{dist}(A, B)$.

The set B is said to be approximatively compact with respect to the set A, if each $\{v_n\}$ in B with $d(x, v_n) \to d(x, B)$ for some x in A, has a convergent subsequence [20].

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. F-contraction; F-contraction of Hardy Rogers and Ciric type; best proximity point.

A class of all functions $F\colon (0,\infty) \to \mathbb{R}$ satisfying the conditions:

(F_1) F is strictly increasing, that is, for each $a_1, a_2 \in (0,\infty)$ with $a_1 < a_2$, we have $F(a_1) < F(a_2)$,

(F_2) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers we have $\lim_{n\to\infty} \mathfrak{d}_n = 0$ if and only if $\lim_{n\to\infty} F(\mathfrak{d}_n) = -\infty$,

(F_3) For each sequence $\{\mathfrak{d}_n\}$ of positive real numbers with $\lim_{n\to\infty} \mathfrak{d}_n = 0$, there exists $k \in (0,1)$ such that $\lim_{n\to\infty} {\mathfrak{d}_\mathfrak{n}}^k F(\mathfrak{d}_\mathfrak{n}) = 0$,

is called class $\mathfrak{F}$.

A contraction involving a function $F \in \mathfrak{F}$ is called an F-contraction. This class was introduced by Wardowski in [21]. In time, the functions from this class were used by various authors to generalize their contractive conditions: Cosentino and Vetro [22]; Minak *et al.* [23]; Sgroi and Vetro [24]; Paesano and Vetro [25]; Piri and Kumam [26]; Acar *et al.* [27]; Batra and Vashistha [28].

Recently, Basha and Shahzad [20] proved the following best proximity point theorem:

Theorem 2.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty and $T\colon A \to B$ is a mapping such that for each $x_1, x_2, u_1, u_2 \in A$ with $d(u_1, Tx_1) = dist(A,B) = d(u_2, Tx_2)$, we have*

$$d(u_1,u_2) \le a_1 d(x_1,x_2) + a_2 d(x_1,u_1) + a_3 d(x_2,u_2) + a_4[d(x_1,u_2) + d(x_2,u_1)] \tag{2.1}$$

where $a_1, a_2, a_3, a_4 \ge 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 < 1$. Further assume that the following conditions hold:

(i) $T(A_0)$ is contained in B_0;

(ii) B is approximatively compact with respect to A.

Then T has a best proximity point.

In this paper we introduce some new F type proximal contractions and prove some best proximity point theorems for such contractions. Our results generalize some existing best proximity point results. In particular Theorem 2.1 becomes a special case of one of our results (Theorem 3.1).

3. Main results

We begin this section with the following definition.

Definition 3.1. Let A and B be two nonempty subsets of a metric space (X,d). A mapping $T\colon A \to CB(B)$ is called α_F-proximal contraction of Hardy Rogers type if there exist two functions $\alpha\colon A\times A \to [0,\infty)$, $F \in \mathfrak{F}$ and a constant $\tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1$, $v_2 \in Tx_2$ with $\alpha(x_1,x_2) \ge 1$ and $d(u_1,v_1) = dist(A,B) = d(u_2,v_2)$, we have

$$\alpha(u_1,u_2) \ge 1 \text{ and } \tau + F(d(u_1,u_2)) \le F(N(x_1,x_2)) \tag{3.1}$$

whenever $\min\{d(u_1,u_2), N(x_1,x_2)\} > 0$, where

$$N(x_1,x_2) = a_1 d(x_1,x_2) + a_2 d(x_1,u_1) + a_3 d(x_2,u_2) + a_4[d(x_1,u_2) + d(x_2,u_1)]$$

with $a_1, a_2, a_3, a_4 \ge 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \ne 1$.

Remark 3.1. By taking $F(x) = \ln x$ for each $x \in (0,\infty)$, one can see that (3.1) reduces to (2.1). Therefore, (3.1) is a proper generalization/extension of (2.1).

Theorem 3.1. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty and $T\colon A \to CB(B)$ is an α_F-proximal contraction of Hardy Rogers type and satisfying the following conditions:*

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1,x_2) \ge 1$ and $d(x_2,v_1) = dist(A,B)$;

(iii) T is continuous, or,

for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \ge 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n,x) \ge 1$ for each $n \in \mathbb{N}$;

(iv) B is approximatively compact with respect to A.

Then T has a best proximity point.

Proof. By hypothesis (ii), we have $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ for which

$$\alpha(x_1, x_2) \geq 1 \text{ and } d(x_2, v_1) = \text{dist}(A, B).$$

As $v_2 \in Tx_2 \subseteq B_0$, there is $x_3 \in A_0$ satisfying

$$d(x_3, v_2) = \text{dist}(A, B).$$

From (3.1), we get $\alpha(x_2, x_3) \geq 1$ and

$$\begin{aligned} \tau + F(d(x_2, x_3)) &\leq F(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + a_4[d(x_1, x_3) + d(x_2, x_2)]) \\ &\leq F(a_1 d(x_1, x_2) + a_2 d(x_1, x_2) + a_3 d(x_2, x_3) + a_4[d(x_1, x_2) + d(x_2, x_3)]) \\ &= F((a_1 + a_2 + a_4) d(x_1, x_2) + (a_3 + a_4) d(x_2, x_3)). \end{aligned} \tag{3.2}$$

As F is strictly increasing, from (3.2), we get

$$d(x_2, x_3) < (a_1 + a_2 + a_4) d(x_1, x_2) + (a_3 + a_4) d(x_2, x_3).$$

That is,

$$(1 - a_3 - a_4) d(x_2, x_3) < (a_1 + a_2 + a_4) d(x_1, x_2).$$

As $a_1 + a_2 + a_3 + 2a_4 = 1$, the above inequality implies that

$$d(x_2, x_3) < d(x_1, x_2).$$

Thus by (3.2), we have

$$\tau + F(d(x_2, x_3)) \leq F(d(x_1, x_2)). \tag{3.3}$$

From above we have $x_2, x_3 \in A_0$ and $v_2 \in Tx_2$ satisfying

$$\alpha(x_2, x_3) \geq 1 \text{ and } d(x_3, v_2) = dist(A, B).$$

As $v_3 \in Tx_3 \subseteq B_0$, there is $x_4 \in A_0$ such that

$$d(x_4, v_3) = dist(A, B).$$

From (3.1), we get $\alpha(x_3, x_4) \geq 1$ and

$$\begin{aligned} \tau + F(d(x_3, x_4)) &\leq F(a_1 d(x_2, x_3) + a_2 d(x_2, x_3) + a_3 d(x_3, x_4) + a_4[d(x_2, x_4) + d(x_3, x_3)]) \\ &\leq F(a_1 d(x_2, x_3) + a_2 d(x_2, x_3) + a_3 d(x_3, x_4) + a_4[d(x_2, x_3) + d(x_3, x_4)]) \\ &= F((a_1 + a_2 + a_4) d(x_2, x_3) + (a_3 + a_4) d(x_3, x_4)). \end{aligned}$$

After simplification we get

$$\tau + F(d(x_3, x_4)) \leq F(d(x_2, x_3)). \tag{3.4}$$

From (3.4) and (3.3), we obtain

$$F(d(x_3, x_4)) \leq F(d(x_1, x_2)) - 2\tau.$$

Continuing the same process we get sequences $\{x_n\}$ in A_0 and $\{v_n\}$ in B_0 such that $v_n \in Tx_n$, $\alpha(x_n, x_{n+1}) \geq 1$, $d(x_{n+1}, v_n) = dist(A, B)$ and

$$F(d(x_n, x_{n+1})) \leq F(d(x_1, x_2)) - n\tau \text{ for each } n \in \mathbb{N} \setminus \{1\}. \tag{3.5}$$

Letting $n \to \infty$ in (3.5), we get $\lim_{n\to\infty} F(d(x_n, x_{n+1})) = -\infty$. Thus, by property (F_2), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Let $d_n = d(x_n, x_{n+1})$ for each $n \in \mathbb{N}$. From (F_3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} d_n^k F(d_n) = 0.$$

From (3.5) we have

$$d_n^k F(d_n) - d_n^k F(d_1) \leq -d_n^k n\tau \leq 0 \text{ for each } n \in \mathbb{N}. \tag{3.6}$$

Letting $n \to \infty$ in (3.6), we get

$$\lim_{n\to\infty} n d_n^k = 0.$$

This implies that there exists $n_1 \in \mathbb{N}$ such that $n d_n^k \leq 1$ for each $n \geq n_1$. Thus, we have

$$d_n \leq \frac{1}{n^{1/k}}, \quad \text{for each } n \geq n_1. \tag{3.7}$$

To prove that $\{x_n\}$ is a Cauchy sequence in A, consider $m, n \in \mathbb{N}$ with $m > n > n_1$. By using the triangular inequality and (3.7), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= \sum_{i=n}^{m-1} d_i \leq \sum_{i=n}^{\infty} d_i \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ is convergent series, we get $\lim_{n\to\infty} d(x_n, x_m) = 0$, which implies that $\{x_n\}$ is a Cauchy sequence in A. Since A is closed subset of a complete metric space, there exists x^* in A such that $x_n \to x^*$ as $n \to \infty$. As $d(x_{n+1}, v_n) = \operatorname{dist}(A, B)$, we have $\lim_{n\to\infty} d(x^*, v_n) = \operatorname{dist}(A, B)$. As B is approximatively compact with respect to A, we get a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in Tx_{n_k}$ that converges to v^*. Thus,

$$d(x^*, v^*) = \lim_{k\to\infty} d(x_{n_k}, v_{n_k}) = dist(A, B).$$

By hypothesis (iii), when T is continuous, we get $v^* \in Tx^*$. Hence $\operatorname{dist}(A, B) \leq d(x^*, Tx^*) \leq d(x^*, v^*) = \operatorname{dist}(A, B)$. This implies that $\operatorname{dist}(A, B) = d(x^*, Tx^*)$. Now we prove the theorem for second assumption of hypothesis (iii), that is, $\alpha(x_n, x^*) \geq 1$ for each $n \in \mathbb{N}$. Since $x^* \in A_0$, then $Tx^* \subseteq B_0$. This implies that for $z^* \in Tx^*$, we have $w^* \in A_0$ such that $d(w^*, z^*) = \operatorname{dist}(A, B)$. Further note that $d(x_{n+1}, v_n) = \operatorname{dist}(A, B)$.

We claim that $d(x^*, w^*) = 0$.

Suppose on contrary that $d(x^*, w^*) \neq 0$ Now from (3.1), we get

$$d(x_{n+1}, w^*) < a_1 d(x_n, x^*) + a_2 d(x_n, x_{n+1}) + a_3 d(x^*, w^*) + a_4[d(x_n, w^*) + d(x^*, x_{n+1})].$$

Letting $n \to \infty$, we get

$$d(x^*, w^*) \leq (a_3 + a_4) d(x^*, w^*),$$

which is only possible when $d(x^*, w^*) = 0$. Thus we get

$$\operatorname{dist}(A, B) \leq d(x^*, Tx^*) \leq d(x^*, z^*) = \operatorname{dist}(A, B),$$

and this completes the proof. □

Example 3.1. Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in X$. Take $A = \{(0, x) : -1 \leq x \leq 1\}$ and $B = \{(1, x) : -1 \leq x \leq 1\}$. Define

$$T\colon A \to CB(B), \quad T(0, x) = \begin{cases} \left\{\left(1, \dfrac{x+1}{2}\right)\right\} \text{ if } x \geq 0 \\ \{(1, x), (1, x^2)\} \text{ otherwise,} \end{cases}$$

and

$$\alpha\colon A \times A \to [0, \infty), \quad \alpha((0, x), (0, y)) = \begin{cases} 1 \text{ if } x, y \in [0, 1] \\ 0 \text{ otherwise.} \end{cases}$$

Take $F(x) = \ln x$ for each $x \in (0, \infty)$ and $\tau = \frac{1}{2}$. It is easy to see that T is α_F-proximal contraction of Hardy Rogers type with $a_0 = 1$ and $a_2 = a_3 = a_4 = 0$. For each $x \in A_0$, we have $Tx \subseteq B_0$. Also for $x_1 = (0, \frac{1}{2}) \in A_0$ and $v_1 = (1, \frac{3}{4}) \in Tx_1$, we have $x_2 = (0, \frac{3}{4})$ such that $\alpha(x_1, x_2) = 1$ and $d(x_2, v_1) = \operatorname{dist}(A, B)$. Moreover, for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) = 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) = 1$ for each $n \in \mathbb{N}$. Further note that B is approximatively compact with respect to A, therefore, by Theorem 3.1, T has a best proximity point.

Remark 3.2. Note that Theorem 2.1 is not applicable in the above example. Therefore, our theorem properly generalizes/extends Theorem 2.1.

Definition 3.2. Let A and B be two nonempty subsets of a metric space (X, d). A mapping $T\colon A \to CB(B)$ is called α_F-proximal contraction of Ciric type if there exist two functions $\alpha\colon A \times A \to [0, \infty)$, continuous F in $\mathfrak{F}$ and a constant $\tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1$, $v_2 \in Tx_2$ with $\alpha(x_1, x_2) \geq 1$ and $d(u_1, v_1) = \operatorname{dist}(A, B) = d(u_2, v_2)$, we have

$$\alpha(u_1, u_2) \geq 1 \text{ and } \tau + F(d(u_1, u_2)) \leq F(M(x_1, x_2)) \tag{3.8}$$

whenever $\min\{d(u_1,u_2), M(x_1,x_2)\} > 0$, where

$$M(x_1,x_2) = \max\Big\{d(x_1,x_2), d(x_1,u_1), d(x_2,u_2), \frac{d(x_1,u_2)+d(x_2,u_1)}{2}\Big\}.$$

Theorem 3.2. *Let A and B be nonempty closed subsets of a complete metric space (X,d). Assume that A_0 is nonempty and $T\colon A \to CB(B)$ is an α_F-proximal contraction of Ciric type satisfying the following conditions:*

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) there exist $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ such that $\alpha(x_1,x_2) \geq 1$ and $d(x_2,v_1) = \operatorname{dist}(A,B)$;

(iii) T is continuous, or,

for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n,x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n,x) \geq 1$ for each $n \in \mathbb{N}$;

(iv) B is approximatively compact with respect to A.

Then T has a best proximity point.

Proof. By hypothesis (ii), we have $x_1, x_2 \in A_0$ and $v_1 \in Tx_1$ for which

$$\alpha(x_1,x_2) \geq 1 \text{ and } d(x_2,v_1) = \operatorname{dist}(A,B).$$

As $v_2 \in Tx_2 \subseteq B_0$, there is $x_3 \in A_0$ satisfying

$$d(x_3,v_2) = \operatorname{dist}(A,B).$$

From (3.8), we get $\alpha(x_2,x_3) \geq 1$ and

$$\begin{aligned}\tau + F(d(x_2,x_3)) &\leq F\Big(\max\Big\{d(x_1,x_2), d(x_1,x_2), d(x_2,x_3), \frac{d(x_1,x_3)+d(x_2,x_2)}{2}\Big\}\Big)\\ &= F\Big(\max\{d(x_1,x_2), d(x_2,x_3)\}\Big)\\ &= F(d(x_1,x_2)),\end{aligned} \tag{3.9}$$

otherwise we have a contradiction. From above we have $x_2, x_3 \in A_0$ and $v_2 \in Tx_2$ satisfying

$$\alpha(x_2,x_3) \geq 1 \text{ and } d(x_3,v_2) = \operatorname{dist}(A,B).$$

As $v_3 \in Tx_3 \subseteq B_0$, there is $x_4 \in A_0$ such that

$$d(x_4,v_3) = \operatorname{dist}(A,B).$$

From (3.8), we get $\alpha(x_3,x_4) \geq 1$ and

$$\begin{aligned}\tau + F(d(x_3,x_4)) &\leq F\Big(\max\Big\{d(x_2,x_3), d(x_2,x_3), d(x_3,x_4), \frac{d(x_2,x_4)+d(x_3,x_3)}{2}\Big\}\Big)\\ &= F\Big(\max\{d(x_2,x_3), d(x_3,x_4)\}\Big)\\ &= F(d(x_2,x_3)),\end{aligned} \tag{3.10}$$

otherwise we have a contradiction. From (3.9) and (3.10), we have

$$F(d(x_3,x_4)) \leq F(d(x_1,x_2)) - 2\tau.$$

Continuing the same process we get sequences $\{x_n\}$ in A_0 and $\{v_n\}$ in B_0 such that $v_n \in Tx_n$, $\alpha(x_n,x_{n+1}) \geq 1$, $d(x_{n+1},v_n) = \operatorname{dist}(A,B)$ and

$$F(d(x_n,x_{n+1})) \leq F(d(x_1,x_2)) - n\tau \text{ for each } n \in \mathbb{N} - \{1\}.$$

Working on the same lines as the proof of Theorem 3.1 is done.

We prove that $\{x_n\}$ is a Cauchy sequence in A.

Since A is closed subset of a complete metric space, there exists x^* in A such that $x_n \to x^*$ as $n \to \infty$. As $d(x_{n+1},v_n) = \operatorname{dist}(A,B)$. Thus, we have $\lim_{n\to\infty} d(x^*,v_n) = \operatorname{dist}(A,B)$. Since B is approximatively compact with respect to A, we get a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ with $v_{n_k} \in Tx_{n_k}$ that converges to v^*. Thus,

$$d(x^*,v^*) = \lim_{k\to\infty} d(x_{n_k+1},v_{n_k}) = dist(A,B).$$

By hypothesis (iii), when T is continuous, we get $v^* \in Tx^*$. Hence $\operatorname{dist}(A,B) \leq d(x^*,Tx^*) \leq d(x^*,v^*) = \operatorname{dist}(A,B)$. Now assume that we have $\alpha(x_n,x^*) \geq 1$ for each $n \in \mathbb{N}$. Since $x^* \in A_0$, then

$Tx^* \subseteq B_0$. This implies that for $z^* \in Tx^*$, we have $w^* \in A_0$ such that $d(w^*, z^*) = \text{dist}(A, B)$. Further note that $d(x_{n+1}, v_n) = \text{dist}(A, B)$.

We claim that $d(x^*, w^*) = 0$.

On contrary assume that $d(x^*, w^*) \neq 0$. Now, from (3.8), we get

$$\tau + F(d(x_{n+1}, w^*)) < F\Big(\max\Big\{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, w^*), \frac{d(x_n, w^*) + d(x_{n+1}, x^*)}{2}\Big).$$

Letting $n \to \infty$, we obtain

$$\tau + F(d(x^*, w^*)) \leq F(d(x^*, w^*)),$$

which is not possible. Hence, we have $d(x^*, w^*) = 0$. Thus we get

$$\text{dist}(A, B) \leq d(x^*, Tx^*) \leq d(x^*, z^*) = \text{dist}(A, B),$$

and this completes the proof. □

Example 3.2. Let $X = \mathbb{R} \times \mathbb{R}$ be endowed with a metric $d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$ for each $x, y \in X$. Take $A = \{(0, x) : -1 \leq x \leq 1\}$ and $B = \{(1, x) : -1 \leq x \leq 1\}$. Define

$$T\colon A \to CB(B), \quad T(0, x) = \begin{cases} \{(1, \frac{x}{3}), (1, \frac{x}{2})\} \text{ if } x \geq 0 \\ \{(1, x), (1, x^2)\} \text{ otherwise,} \end{cases}$$

and

$$\alpha\colon A \times A \to [0, \infty) \quad \alpha((0, x), (0, y)) = \begin{cases} 1 \text{ if } x, y \in [0, 1] \\ 0 \text{ otherwise.} \end{cases}$$

Take $F(x) = \ln x$ for each $x \in (0, \infty)$ and $\tau = \frac{1}{2}$. It is easy to see that T is α_F-proximal contraction of Ciric type. For each $x \in A_0$, we have $Tx \subseteq B_0$. Also for $x_1 = (0, \frac{1}{3}) \in A_0$ and $v_1 = (1, \frac{1}{6}) \in Tx_1$, we have $x_2 = (0, \frac{1}{6})$ such that $\alpha(x_1, x_2) = 1$ and $d(x_2, v_1) = \text{dist}(A, B)$. Moreover, for any sequence $\{x_n\} \subseteq A$ such that $x_n \to x$ as $n \to \infty$ and $\alpha(x_n, x_{n+1}) = 1$ for each $n \in \mathbb{N}$, we have $\alpha(x_n, x) = 1$ for each $n \in \mathbb{N}$. Further, note that B is approximatively compact with respect to A. Therefore, by Theorem 3.2, T has a best proximity point.

4. Consequences

By taking $\alpha(x, y) = 1$ for each $x, y \in A$, the following two theorems immediately follow from our results.

Theorem 4.1. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T\colon A \to CB(B)$ is a mapping for which there exist a function $F \in \mathfrak{F}$ and a constant $\tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1$, $v_2 \in Tx_2$ with $d(u_1, v_1) = \text{dist}(A, B) = d(u_2, v_2)$, we have*

$$\tau + F(d(u_1, u_2)) \leq F(N(x_1, x_2))$$

whenever $\min\{d(u_1, u_2), N(x_1, x_2)\} > 0$, where

$$N(x_1, x_2) = a_1 d(x_1, x_2) + a_2 d(x_1, u_1) + a_3 d(x_2, u_2) + a_4[d(x_1, u_2) + d(x_2, u_1)]$$

with $a_1, a_2, a_3, a_4 \geq 0$ satisfying $a_1 + a_2 + a_3 + 2a_4 = 1$ and $a_3 \neq 1$. Further assume that the following conditions hold:

(i) for each $x \in A_0$, we have $Tx \subseteq B_0$;

(ii) B is approximatively compact with respect to A.

Then T has a best proximity point.

Theorem 4.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Assume that A_0 is nonempty and $T\colon A \to CB(B)$ is a mapping for which there exist a continuous function $F \in \mathfrak{F}$ and a constant $\tau > 0$ such that for each $x_1, x_2, u_1, u_2 \in A$ and $v_1 \in Tx_1$, $v_2 \in Tx_2$ with $d(u_1, v_1) = \text{dist}(A, B) = d(u_2, v_2)$, we have*

$$\tau + F(d(u_1, u_2)) \leq F(M(x_1, x_2))$$

whenever $\min\{d(u_1,u_2), M(x_1,x_2)\} > 0$, *where*

$$M(x_1,x_2) = \max\Big\{d(x_1,x_2), d(x_1,u_1), d(x_2,u_2), \frac{d(x_1,u_2)+d(x_2,u_1)}{2}\Big\}.$$

Further assume that the following conditions hold:

(i) for each $x \in A_0$, *we have* $Tx \subseteq B_0$;

(ii) B *is approximatively compact with respect to* A.

Then T *has a best proximity point.*

When we take $X = A = B$, we get the following fixed point theorems from our results:

Theorem 4.3. *Let* (X,d) *be a complete metric space. Assume* $T\colon X \to CB(X)$ *is a mapping for which there are two functions* $\alpha\colon A \times A \to [0,\infty)$, $F \in \mathfrak{F}$ *and a constant* $\tau > 0$ *such that for each* $x_1, x_2 \in X$ *and* $u_1 \in Tx_1$, $u_2 \in Tx_2$ *with* $\alpha(x_1,x_2) \geq 1$, *we have*

$$\alpha(u_1,u_2) \geq 1 \text{ and } \tau + F(d(u_1,u_2)) \leq F(N(x_1,x_2))$$

whenever $\min\{d(u_1,u_2), N(x_1,x_2)\} > 0$, *where*

$$N(x_1,x_2) = a_1 d(x_1,x_2) + a_2 d(x_1,u_1) + a_3 d(x_2,u_2) + a_4[d(x_1,u_2) + d(x_2,u_1)]$$

with $a_1, a_2, a_3, a_4 \geq 0$ *satisfying* $a_1+a_2+a_3+2a_4 = 1$ *and* $a_3 \neq 1$. *Further assume that* T *is continuous, or, for any sequence* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for each* $n \in \mathbb{N}$, *we have* $\alpha(x_n,x) \geq 1$ *for each* $n \in \mathbb{N}$. *Then* T *has a fixed point.*

Theorem 4.4. *Let* (X,d) *be a complete metric space. Assume* $T\colon X \to CB(X)$ *is a mapping for which there is* $\alpha\colon A \times A \to [0,\infty)$, *continuous function,* F *in* $\mathfrak{F}$ *and* $\tau > 0$ *such that for each* $x_1, x_2 \in X$ *and* $u_1 \in Tx_1$, $u_2 \in Tx_2$ *with* $\alpha(x_1,x_2) \geq 1$, *we have*

$$\alpha(u_1,u_2) \geq 1 \text{ and } \tau + F(d(u_1,u_2)) \leq F(M(x_1,x_2))$$

whenever $\min\{d(u_1,u_2), M(x_1,x_2)\} > 0$, *where*

$$M(x_1,x_2) = \max\Big\{d(x_1,x_2), d(x_1,u_1), d(x_2,u_2), \frac{d(x_1,u_2)+d(x_2,u_1)}{2}\Big\}.$$

Further assume that T *is continuous, or, for any sequence* $\{x_n\} \subseteq X$ *such that* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n,x_{n+1}) \geq 1$ *for each* $n \in \mathbb{N}$, *we have* $\alpha(x_n,x) \geq 1$ *for each* $n \in \mathbb{N}$. *Then* T *has a fixed point.*

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