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SOME GENERALIZED NOTIONS OF AMENABILITY MODULO AN IDEAL

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Abstract. In this paper some generalized notions of amenability modulo an ideal of Banach algebras such as uniformly (boundedly) approximately amenable (contractible) modulo an ideal of Banach algebras are investigated. Using the obtained results, uniformly (boundedly) approximately amenability (contractibility) modulo an ideal of weighted semigroup algebras are characterized.

1. Introduction

Let A be a Banach algebra and X be a Banach A-bimodule, by a derivation D we mean a bounded linear map $D : A \to X$ such that $D(ab) = a.D(b) + D(a).a$, $(a, b \in A)$. An inner derivation is a derivation D which there exists $x \in X$ such that $D(a) = ad_x(a) = a \cdot x - x \cdot a$, $(a \in A)$. A Derivation $D : A \to X$ is called approximately inner if there exists a net (ξ_α) in X such that $D(a) = \lim_{\alpha} ad_{\xi_\alpha}(a)$ $(a \in A)$ where the limit is taken in norm of X. If the above limit exists in the w^*-topology (say, X is a dual module) then D is called w^*-approximately inner. A Banach algebra A is called boundedly approximately amenable (contractible) if, for each Banach A-bimodule X and each continuous derivation $D : A \to X^*$ $(D : A \to X)$ there exist $K > 0$ and a net (ξ_α) in X^* (in X) such that for each $a \in A$ and α, $\|a.\xi_\alpha - \xi_\alpha.a\| \leq M.\|a\|$ and $D(a) = \lim_\alpha ad_{\xi_\alpha}(a)$, A is called uniformly approximately amenable (contractible) if for each Banach A-bimodule X, each continuous derivation D from A to X^* (to X) is the limit of a sequence of inner derivations in the norm topology of the set of all bounded operators from A into X^*, i.e. $\mathcal{B}(A, X^*)$ (into X, i.e. $\mathcal{B}(A, X)$). Some characterizations of these concepts of amenability are investigated in [5–7].

The concept of amenability modulo an ideal for a class of Banach algebras which could be considered as a generalization of amenability of Banach algebra was introduced by the first author and Amini in 2014 [1]. Using this idea, it is shown that a semigroup S is amenable if and only if the semigroup algebra $l^1(S)$ is amenable modulo an ideal induced by appropriate congruence σ on S, for a large class of semigroups. In further researches, it was shown that amenability modulo an ideal can be characterized by the existence of virtual diagonal modulo an ideal and approximate diagonal modulo an ideal. To see the details of these results and more on this topic, we refer to [1, 10, 11].

In this paper we shall continue the investigation of amenability modulo an ideal, in particular that of boundedly approximate amenability modulo an ideal and uniformly approximate amenability modulo an ideal of Banach algebras. Afterward, for a large class of semigroups, we introduce some characterization of amenability modulo an ideal of weighted semigroup algebras.

This paper is organized as follow; in section two, we give some basic notions of generalized amenability and amenability modulo an ideal of Banach algebras and we show that the concepts approximately contractible modulo an ideal, approximately amenable modulo an ideal and w^*−approximately amenable modulo an ideal of Banach algebras are equivalent. In section three, we investigate to the generalized notions of amenability modulo an ideal of Banach algebras such as, uniformly approximately amenable (contractible) modulo an ideal and boundedly approximately amenable (contractible) modulo an ideal of Banach algebras. In section four, we consider the generalized notions of amenability

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modulo an ideal for the weighted semigroup algebra $l^1(S)$ and we finish this section with give some examples.

2. PRELIMINARIES

In this section we recall some basic notions which we need in this paper. To see more details, reader can refer to [1, 10–12].

Definition 2.1. *Let I be a closed ideal of A. A Banach algebra A is amenable (contractible) modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, and every derivation D from A into X^* (into X) there is $\phi \in X^*$ such that $D = ad_\phi$ on the set theoretic difference $A\backslash I := \{a \in A : a \notin I\}$.*

All over this paper we fix A and I as above, unless they are otherwise specified.

Theorem 2.1. *([1, Theorem 1]) The following assertions hold.*
i) If A/I is amenable and $I^2 = I$ then A is amenable modulo I.
ii) If A is amenable modulo I then A/I is amenable.
iii) If A is amenable modulo I and I is amenable, then A is amenable.

Let A be a Banach algebra and I be a closed ideal of A. With the module actions $a.\bar{b} := \overline{ab}$ and $\bar{b}.a := \overline{ba}$, $\frac{A}{I}$ is a Banach A-bimodule where $\bar{a}$ is the image of a in $\frac{A}{I}$. Also $\frac{A}{I}\hat{\otimes}A$ can be consider as a Banach $A-$bimodule where the module actions are the linear extension of $a.(\bar{b}\otimes c) := \overline{ab}\otimes c$ and $(\bar{b}\otimes c).a := (\bar{b}\otimes ca)$, $(a, b, c \in A)$. By the diagonal operator we mean the bounded linear operator defined by the linear extension of $\pi : (\frac{A}{I}\hat{\otimes}A) \to \frac{A}{I}$ by $\pi(\bar{b}\otimes c) = \overline{bc}$. Clearly, π is a $A-$bimodule homomorphism.

Definition 2.2. (i) *By a virtual diagonal modulo I, we mean an element $M \in (\frac{A}{I}\hat{\otimes}A)^{**}$ such that;*

$$a \cdot \pi^{**}(M) - \bar{a} = 0 \ (a \in A) \ \text{ and } \ a \cdot M - M \cdot a = 0 \ (a \in A \setminus I),$$

(ii) *an approximate diagonal modulo I, we mean a bounded net $(m_\alpha)_\alpha \subseteq (\frac{A}{I}\hat{\otimes}A)$ such that;*

$$a.\pi(m_\alpha) - \bar{a} \to 0 \ (a \in A) \ \text{ and } \ a.m_\alpha - m_\alpha.a \to 0 \ (a \in A \setminus I).$$

(iii) *a diagonal modulo I, we mean an element $m \in (\frac{A}{I}\hat{\otimes}A)$ such that;*

$$a.\pi(m) - \bar{a} = 0\,(a \in A), \text{ and } a.m - m.a = 0, \ (a \in A \setminus I).$$

We recall that a bounded net $(u_\alpha)_\alpha \subseteq A$ is called approximate identity modulo I if $\lim_\alpha u_\alpha \cdot a = \lim_\alpha a \cdot u_\alpha = a \ (a \in A \setminus I)$. If A is amenable modulo I then A has an approximate identity modulo I. It is shown that a Banach algebra A is amenable modulo I if and only if A has an approximate diagonal modulo I,if and only if A has a virtual diagonal modulo I [10]. By appropriate modifications, the following Theorem may be proved in much the same way as [4, Theorem 1.9.21].

Theorem 2.2. *A is contractible modulo I if and only if A has a diagonal modulo I.*

Definition 2.3. *A Banach algebra A is called approximately amenable (contractible) modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$, every bounded derivation $D : A \to X^*$ $(D : A \to X)$ is approximately inner on the set theoretical difference $A\backslash I := \{a \in A : a \notin I\}$.*

Theorem 2.3. *The following statements are equivalent;*
a) A is approximately contractible modulo I;
b) A is approximately amenable modulo I;
c) A is w^*-approximately amenable modulo I.

Proof. It is easily seen that $(a \to b)$ and $(b \to c)$, so we only need to show that $(c \to a)$. Since A is w^*-approximately amenable modulo I, $A^\sharp$ is w^*-approximately amenable modulo I (by [11, Theorem 3.2]). Now [11, Theorem 3.3], provide us to consider a net $(M_i) \subseteq (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**}$ such that $a \cdot M_i - M_i \cdot a \to 0\,(\forall a \in A^\sharp\backslash I)$ and $\pi^{**}(M_i) \to \bar{e}$ in the w^*-topology of $(\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^{**}$ and A^{**}, respectively. Let $\epsilon > 0$ and consider finite sets $\mathcal{F} \subseteq A^\sharp\backslash I$, $\Phi \subseteq (A^\sharp\backslash I)^*$ and $\mathcal{N} \subseteq (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)^*$, so there exists j such that $(a \in \mathcal{F}, \phi \in \Phi, f \in \mathcal{N})$,

$$|\langle a.f - f.a, M_j\rangle| = |\langle f, a.M_j - M_j.a\rangle| < \epsilon \quad \text{and} \quad |\ \langle \phi, \pi^{**}(M_j) - \bar{e}\rangle\ | < \epsilon$$

Using the weak*-continuity of π^{**} and Goldstine's theorem, we can choose $m \in (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)$ such that

$$|\langle f, a.m - m.a\rangle| = |\langle a.f - f.a, m\rangle| < \epsilon, \text{ and } |\langle \phi, \pi(m) - \bar{e}\rangle| < \epsilon,$$

for each $a \in \mathcal{F}, \phi \in \Phi$ and $f \in \mathcal{N}$. Hence there exists $(m_i) \subseteq (\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)$ such that $a.m_i - m_i.a \to 0$ $(a \in A\backslash I)$ and $\pi(m_i) \to \bar{e}$ in the w-topology of $(\frac{A^\sharp}{I}\hat{\otimes}A^\sharp)$ and $A^\sharp$, respectively. Now for every finite set $F = \{a_1, a_2, , ..., a_n\} \subseteq A^\sharp\backslash I$,

$$(a_1.m_i - m_i.a_1, ..., a_n.m_i - m_i.a_n, \pi(m_i)) \to (0, ..., 0, \bar{e})$$

weakly in $(\frac{A^\sharp}{I}\hat{\otimes}A^\sharp) \oplus (A^\sharp\backslash I)$. Therefore

$$(0, ..., 0, \bar{e}) \in \bar{co}^w\{(a_1.m_i - m_i.a_1, ..., a_n.m_i - m_i.a_n, \pi(m_i))\}.$$

Set $P = \{(a_1.m_i - m_i.a_1, ..., a_n.m_i - m_i.a_n, \pi(m_i))\}$, so

$$co(P) = \{(a_1.M - M.a_1, ..., a_n.M - M.a_n, \pi(M)) \in co\{m_i\}\}.$$

We have

$$(0, ..., 0, \bar{e}) \in \bar{co}^w(P) = \bar{co}^{\|\ \|}(P).$$

The Hahn-Banach theorem implies that for each $\epsilon > 0$ there exists $u_{\epsilon,F} \in co\{m_i\}$ such that

$$\|a.u_{\epsilon,F} - u_{\epsilon,F}.a\| < \epsilon \text{ and } \|\pi(u_{\epsilon,F}) - \bar{e}\| < \epsilon, \ (a \in F).$$

Now by [11, Theorem 3.8] proof is complete. □

3. Uniformly and boundedly approximate amenability (contractibility) modulo an ideal of Banach algebras

Definition 3.1. *A Banach algebra A is uniformly approximately amenable (contractible) modulo I if for every Banach A-bimodule X such that $I \cdot X = X \cdot I = 0$ and every continuous derivation $D : A \to X^*$ ($D : A \to X$) there is a net $(x_\alpha) \subseteq X^*$ ($(x_\alpha) \subseteq X$) such that $D(a) = \lim_\alpha ad_{x_\alpha}(a)$ where the convergence is uniform for each $a \in A\backslash I$ such that $\|a\| \leq 1$,*

Lemma 3.1. *A Banach algebra A is uniformly approximately contractible modulo I if and only if $A^\sharp$ is uniformly approximately contractible modulo I.*

Proof. Let A be uniformly approximately contractible modulo I, X be a Banach $A^\sharp-$bimodule and $D : A^\sharp \to X$ be a bounded derivation. Then there are $\xi \in eXe$ and $D_1 : A^\sharp \to eXe$ such that $D = D_1 + ad_\xi$. We have $D_1(e) = 0$ and $D_1|_A \in \mathcal{Z}^1(A, eXe)$. Since A is uniformly approximately contractible modulo I, there exits $(\zeta_n) \in eXe$ such that $D_1(a, \alpha) = \lim_n ad_{\zeta_n}(a), (a \in A\backslash I, \alpha \in \mathcal{C}, \|a\| + |\alpha| \leq 1)$. Now if $(a, \alpha) \in (A\backslash I) \oplus \mathcal{C} = (A\backslash I)^\sharp$ such that $\|a\| + |\alpha| \leq 1$, then $D_1(a, \alpha) = D_1(a, 0) + \alpha D_1(e) = D_1(a) = ad_{\zeta_a}(a)$. Hence $D(a) = D_1(a) + ad_\xi(a) = \lim_n ad_{\zeta_n}(a) + ad_\xi(a) = \lim_n ad_{\zeta_n+\xi}(a)$.

Conversely, let X be a Banach $A-$bimodule and $D : A \to X^*$ be a bounded derivation. Defining $(a, \alpha).x = a.x + \alpha.x$ and $x.(a, \alpha) = x.a + \alpha.x \, (x \in X, (a, \alpha) \in A^\sharp)$ makes $A^\sharp$ into an $A^\sharp-$bimodule. Define $\widetilde{D} : A^\sharp \to X$ by $\widetilde{D}(a, \alpha) = D(a) \, ((a, \alpha) \in A^\sharp)$. Clearly $\widetilde{D}$ is a bounded derivation. Supposing $A^\sharp$ is uniformly approximately contractible modulo I, there is $(\xi_n) \subseteq X$ such that $\widetilde{D} = \lim_n ad_{\xi_n}$ on the unit ball of $(A\backslash I)^\sharp$. Now $\widetilde{D}|_A$, as required. □

Lemma 3.2. *Let X be an A-module and $(e_n) \subseteq X$ be a sequence such that for each $a \in A\backslash I$ with $\|a\| \leq 1$, $a = \lim_n a.e_n$. Then A has a right identity modulo I, i.e. there exists $u \in A$ such that $a.u = a \, (a \in A\backslash I)$.*

Proof. Let R_f denote the right multiplication by $f \in X$. Then there is $(e_n) \subseteq X$ with $\|R_f - id\| \leq 1$, so R_f is invertible. This implies that there is a $g \in \mathcal{B}(X, A)$ such that $R_f \circ g = id$. Set $u = g(f)$, so $u.f = R_f \circ g(f) = f$. Then $auf = af$ and for each $a \in A\backslash I$, $(au - a).f = 0$. This means that u is a right identity modulo I. □

Lemma 3.3. *Suppose that A is uniformly approximately contractible modulo I. Then A has an identity e on $A\backslash I$, i.e. $e.a = a.e = a \, (a \in A\backslash I)$.*

Proof. Consider A as a $A-$bimodule where the module actions are defined by $a.x = ax$ and $x.a = 0\,(a \in A, x \in X)$. Let $D : A \to A^{**}$ defined by $D(a) = \hat{a}$ be the canonical embedding. It is clear that D is a bounded derivation. Since A is uniformly approximately contractible modulo I, there is $(e_\alpha) \subseteq A^{**}$ such that $D(a) = \lim ad_{e_n}(a)\,(a \in A\backslash I, \|a\| \leq 1)$, so $a = \lim\limits_n a.e_n$. Using Lemma 3.2, A has a right identity modulo I. The same argument is true for A^{op}, and hence A has an identity e on $A\backslash I$. □

Theorem 3.1. *Let A be uniformly approximately contractible modulo I. Then A is contractible modulo I.*

Proof. By Lemma 3.3, we may suppose that A has an identity "e" on $A\backslash I$. Define $D : A \to ker\pi \subseteq (\frac{A}{I}\hat{\otimes}A)$ by $D(a) = \bar{a} \otimes e - \bar{e} \otimes a$. Then D is a bounded derivation and $\|D\| \leq 2$. Since A is uniformly approximately contractible modulo I, there is $(t_n) \in ker\pi$ such that $ad_{t_n} \to D$ uniformly for $a \in A\backslash I$, with $\|a\| \leq 1$. Suppose that $t_n = \sum\limits_i \bar{x}_i^n \otimes y_i^n$ and $s = \sum\limits_i \bar{a}_j \otimes b_j \in ker\pi$. Since $\pi(s) = \pi(t_n) = 0$, $\sum\limits_i \bar{a}_i b_i = \sum\limits_i \overline{a_i b_i} = 0$ and $\sum\limits_i \bar{x}_j^n y_j^n = \sum\limits_i \overline{x_j^n y_j^n} = 0$. Hence,

$$
\begin{aligned}
\|st_n - s\| &= \|\sum_{i,j} \bar{a}_j \bar{x}_i^n \otimes y_i^n b_j - \sum_j \bar{a}_j \otimes b_j\| \\
&= \|\sum_{i,j} \bar{a}_j \bar{x}_i^n \otimes y_i^n b_j - \sum_{i,j} \bar{a}_j b_j \bar{x}_i^n \otimes y_i^n - \sum_j \bar{a}_j \otimes b_j + \sum_j \bar{a}_j b_j \otimes e\| \\
&= \|\sum_j \bar{a}_j (\sum_i \bar{x}_i^n \otimes y_i^n b_j - \sum_i b_j \bar{x}_i^n \otimes y_i^n - \bar{e} \otimes b_j + b_j \otimes e)\| \\
&\leq \sum_j \|\sum_i \bar{x}_i^n \otimes y_i^n b_j - \sum_i b_j \bar{x}_i^n \otimes y_i^n - e \otimes b_j + b_j \otimes e\| \|\bar{a}_j\| \\
&= \sum_j \|\sum_i \bar{x}_i^n \otimes y_i^n \frac{b_j}{\|b_j\|} - \sum_i \frac{b_j}{\|b_j\|} \bar{x}_i^n \otimes y_i^n \\
&\quad - e \otimes \tfrac{b_j}{\|b_j\|} + \tfrac{b_j}{\|b_j\|} \otimes e\| \|\bar{a}_j\| \|b_j\| \\
&\leq \sum_j \sup_{\|c\| \leq 1} \|t_n.c - c.t_n - e \otimes c + c \otimes e\| \|\bar{a}_j\| \|b_j\|.
\end{aligned}
$$

It implies that $\|st_n - s\| \leq \sup\limits_{\|c\| \leq 1} \|ad_{t_n}(c) - D(c)\|$ on the unit ball of $ker\pi$, hence $st_n \to s$ uniformly on the unit ball of $ker\pi$ and by Lemma 3.2, $ker\pi$ has a right identity modulo I, u. Set $v = \bar{e} \otimes e - u$, then $\pi(v) = \bar{e} - \pi(u)$ and for each $a \in A\backslash I$, $a.v - v.a = 0$. Thus v is a diagonal modulo I and hence A is contractible modulo I(by Theorem 2.2). □

Definition 3.2. *A Banach algebra A is boundedly approximate amenable (contractible) modulo I if for each Banach A-bimodule X with $X \cdot I = I \cdot X = 0$ and each continuous derivation $D : A \to X^*$ ($D : A \to X$) there exist $K > 0$ and a net (ξ_α) in X^* (X) such that for each $a \in A\backslash I$ and α, $\|a.\xi_\alpha - \xi_\alpha.a\| \leq M.\|a\|$, and $D(a) = \lim_\alpha ad_{\xi_\alpha}(a)$.*

Theorem 3.2. *Then the following assertions hold;*

(i) if A is boundedly approximate amenable modulo I, then $\frac{A}{I}$ is boundedly approximate amenable.

(ii) if $\frac{A}{I}$ is boundedly approximate amenable and $I^2 = I$ then A is boundedly approximate amenable modulo I

Analogous assertions satisfy for uniformly approximately amenable modulo an ideal Banach algebras.

Proof. (i) Suppose that X is a Banach $\frac{A}{I}$-bimodule and $D : \frac{A}{I} \to X^*$ is a bounded derivation. Now X is a clearly Banach A- module with the module actions defined by $a.x = \pi(a).x$, $x.a = x.\pi(a)$, $(a \in A, x \in X)$ where $\pi : A \to \frac{A}{I}$ is the canonical quotient map. Since $I \cdot X = X \cdot I = 0$ and $D \circ \pi : A \to X^*$ is a bounded derivation, there is a $(\xi_\alpha) \subset X^*$ such that $\|a.\xi_\alpha - \xi_\alpha.a\| \leq M.\|a\|$ (for some $M > 0$) and $D \circ \pi = \lim\limits_\alpha ad_{\xi_\alpha}$ on $A\backslash I$. We have $\|\pi(a).\xi_\alpha - \xi_\alpha.\pi(a)\| = \|a.\xi_\alpha - \xi_\alpha.a\| \leq M.\|a\|$ and $D(\pi(a)) = D \circ \pi(a) = \lim\limits_\alpha ad_{\xi_\alpha}(a), (\pi(a) \in \dfrac{A}{I})$. Hence $\frac{A}{I}$ is boundedly approximate amenable modulo I.

(ii) Suppose that X is a Banach A-bimodule such that $X \cdot I = I \cdot X = 0$ and $D : A \to X^*$ is a bounded derivation. We can consider X as an $\frac{A}{I}$-bimodule with the module actions $a.x = \pi(a).x$, $x.a = x.\pi(a)$, $(a \in A, x \in X)$. The equality $I^2 = I$ provide us to define the well-defined bounded derivation $\overline{D} : \frac{A}{I} \to X^*$ by $\overline{D}(\pi(a)) = D(a)$ $(a \in A)$. Since $\frac{A}{I}$ is boundedly approximate amenable modulo I, there is a $(\xi_\alpha) \subset X^*$ such that $\|\pi(a).\xi_\alpha - \xi_\alpha.\pi(a)\| \leq M.\|a\|$ (for some $M > 0$) and $\overline{D} = \lim_\alpha ad_{\xi_\alpha}$. It is not far to see that the net (ad_{ξ_α}) is norm bounded in $\mathcal{B}(A, X^*)$ and $D(a) = \overline{D}(\pi(a)) = \lim_\alpha ad_{\xi_\alpha}(a)$. □

The proof of the following result is the same way as Theorem 3.2.

Corollary 3.1. *The following conditions are hold;*

(i) if A is boundedly approximate contractible modulo I, then $\frac{A}{I}$ is boundedly approximate contractible.

(ii) if $\frac{A}{I}$ is boundedly approximate contractible and $I^2 = I$ then A is boundedly approximate contractible modulo I

Analogous assertions satisfy for uniformly approximately contractible modulo an ideal.

For a Banach algebra A, it is shown that A is uniformly approximately amenable if and only if it is amenable [6, Theorem 3.1]. Using Theorem 3.2, we have the following result.

Corollary 3.2. *Suppose A is a Banach algebra and I is a closed ideal of A such that $I^2 = I$. Then A is uniformly approximate amenable modulo I if and only if it is amenable modulo I.*

Theorem 3.3. *A Banach algebra A is boundedly approximate amenable modulo I if and only if there exists a constant $M > 0$ such that for any Banach A-bimodule X with $X \cdot I = I \cdot X = 0$ and any continuous derivation $D : A \to X^*$ there is a net $(\eta_i) \subseteq X^*$ such that*

a) $\sup_i \|ad_{\eta_i}\| \leq M\|D\|$,

b) $D(a) = \lim_i ad_{\eta_i}(a)$, $(\forall a \in A\backslash I)$.

Proof. Let assumptions (a) and (b) hold, then $\|ad_{\eta_i}\| \leq M\|D\| = \frac{M\|D\|}{\|a\|}$ $(a \in A/I)$. Therefore A is boundedly approximately amenable modulo I. Conversely, let A be a boundedly approximately amenable modulo I. Consider there is no such M. Suppose that for every integer $n \in \mathbb{N}$, M_n is Banach module such that $M_n \cdot I = I \cdot M_n = 0$ and $D_n : A \to M_n^*$ is a derivation with $\|D_n\| > n$. Now $X = l^1(M_n)$ is a Banach $A-$module with dual $l^\infty(M_n^*)$. Put $D = (D_n)$, $D : A \to l^\infty(M_n^*)$ is a continuous derivation and $D(a) = (D_n(a)) = \lim_i (ad_{\eta_i^n}(a))$. Since $\|D_n\| > n$, $\|D\| \to \infty$ which is contradiction. □

The same argument of [12, Theorem 3.2 and 3.3] and minor changes, we have the following theorems;

Theorem 3.4. *A Banach algebra A is boundedly approximately amenable modulo I if and only if $A^\#$ is boundedly approximately amenable modulo I.*

Theorem 3.5. *Let A be a Banach algebra and I be a closed ideal of A. If A is boundedly approximately amenable modulo I then;*

(a) there is a net $(M_i)_i \subseteq (\frac{A^\sharp}{I} \hat{\otimes} A^\#)^{**}$ and a constant $L > 0$ such that $\bar{a}.\, M_i - M_i.\, \bar{a} \to 0$, $\pi^{**}(M_i) \to \bar{e}$, and $\|\bar{a}.\, M_i - M_i.\, \bar{a}\| \leq L\,\|\bar{a}\|$, for each $\bar{a} \in (\frac{A^\#}{I})$.

Conversely, if (a) holds and the net $(\pi^{**}(M_i))$ is bounded then A is boundedly approximately amenable modulo I.

4. Algebras related to discrete semigroups

We generally follow [3,9] for definitions and basic concepts of semigroups. For a semigroup S, the set (possibly empty) of idempotents of S is denoted by $E = E(S)$. A semigroup S is called an *E-semigroup* if $E(S)$ is a sub-semigroup of S, *E-inversive* if for each $x \in S$, there exists $y \in S$ such that $xy \in E(S)$, *regular* if the set of inverses of $a \in S$, $V(a) = \{x \in S : \ a = axa, x = xax\} \neq \phi$, *inverse semigroup* if moreover, the inverse of each element is unique, *E-unitary* if for each $x \in S$ and $e \in E(S)$, $ex \in E(S)$ implies $x \in E(S)$, *semilattice* if S is a commutative and idempotent semigroup and finally S is called *eventually inverse* if every element of S has some power that is regular and $E(S)$ is a semilattice.

By a group congruence ρ on semigroup S we mean a congruence ρ such that S/ρ is a group. The kernel of a congruence ρ on a semigroup S "$Ker\rho$" is the set $\{a \in S : a\rho \in E(S/\rho)\} = \{a \in S : (a, a^2) \in \rho\}$. We denote the least group congruence on S (if exist) by σ. The least group congruence on semigroups have also been considered by various authors [8, 13]. It is shown that if S is an E-inversive E-semigroup such that $E(S)$ is commutative (S is an *eventually semigroup*) then the relation $\sigma = \{(a,b) \in S \times S \,|\, ea = fb \text{ for some } e, f \in E_S\}$ $(\sigma' = \{(s,t) : es = et, \text{ for some } e \in E(S)\})$ is the least group congruence on S [8, 13]. We recall that a function $\omega : G \to (0, \infty)$ such that $\omega(g_1 g_2) \leq \omega(g_1)\omega(g_2)\,(g_1, g_2 \in G)$ is called a weight on group G. The weight ω on group G is called symmetric if $\omega(g) = \omega(g^{-1})(g \in G)$ and for any weight ω, by symmetrization of ω, we mean the weight defined by $\Omega_\omega(g) = \omega(g)\omega(g^{-1})$. The weighted semigroup algebra (or Beurling algebra on semigroup S) $l^1(S, \omega) = \{f \mid f : S \to \mathbb{C}, \sum_{s\in S} |f(s)|\omega(s) < \infty\}$ with $\|f\|_{1,\omega} = \sum_{s\in S} |f(s)|\omega(s)$ and convolution product is a Banach algebra. In the case $\omega = 1$, the weighted semigroup algebra $l^1(S, \omega)$ is called semigroup algebra and is denoted by $l^1(S)$. We recall the following Lemma, which is detailed in [12].

Lemma 4.1. *The following statements hold:*
(i) if S is a semigroup, ρ is a congruence on S and ω is a weight on S, then $\frac{l^1(S,\omega)}{I_\rho} \simeq l^1(S/\rho, \omega_\rho)$ where $\omega_\rho([s]_\rho) = \inf\{\omega(s) : s \in [s]_\rho\}$ is the induced weight on S/ρ and I_ρ is an ideal in $l^1(S, \omega)$ generated by the set

$$\{\delta_s - \delta_t : \, s, t \in S \;\; with\, (s, t) \in \rho\};$$

(ii) if S is an E-inversive semigroup with commuting idempotents or S is an eventually inverse semigroup, σ is the least group congruence on S and ω is a weight on S, then $l^1(S/\sigma, \omega_\sigma)) \simeq \frac{l^1(S,\omega)}{I_\sigma}$ where I_σ is a closed ideal of $l^1(S, \omega)$ and $I_\sigma^2 = I_\sigma$.

It is shown that for a locally compact group G and a weight ω on G, the Beurling algebra $L^1(G, \omega)$ is boundedly approximately contractible if and only if the Beurling algebra $L^1(G, \omega)$ is amenable, if and only if G is amenable and Ω is bounded on G [7, Corollary 2.2]. The same conclusion can be drawn for Beurling algebra of a weighted semigroup as follow;

Theorem 4.1. *Suppose that ω is a weight on semigroup S. If S is an E-inversive semigroup with commuting idempotents or S is an eventually inverse semigroup, then the followings assertions are equivalent.*
(i) The semigroup S is amenable and Ω_{ω_σ} is bounded where ω_σ is the induced weight on S/σ.
(ii) The weighted semigroup algebra $l^1(S, \omega)$ is boundedly approximately contractible modulo I_σ.

Proof. The semigroup S is amenable if and only if S/σ is amenable [1, Theorem 2], if and only if $l^1(S/\sigma, \omega_\sigma)$ is amenable (because S/σ is a group), if and only if $l^1(S/\sigma, \omega_\sigma)$ is boundedly approximately contractible (because Ω_{ω_σ} is bounded on S/σ and by [7, Corollary 2.2]), if and only if $l^1(S, \omega)$ is boundedly approximately contractible modulo I_σ (by Corollary 3.1). □

For a loccaly compact group G and a symmetric weight on ω on G, if $\lim\limits_{x\to\infty} \omega(x) = \infty$, then $L^1(G, \omega)$ is not boundedly approximately amenable [7, Corollary 2.8]. Thus we have the following corollary for the weighted semigroup algebras;

Corollary 4.1. *If S is a semigroup, ρ is a group congruence on S with $Ker\rho$ is central and ω is a weight on semigroup S such that $\lim\limits_{x\to\infty} \omega(x) = \infty\,(x \in S/\rho)$. Then $l^1(S, \omega)$ is not boundedly approximately amenable modulo I_ρ.*

Proof. Since $Ker\rho$ is central, the semigroup S is amenable if and only if S/ρ is amenable. On the other hand, S/ρ is a group and $\lim\limits_{x\to\infty} \omega(x) = \infty\,(x \in S/\rho)$, so $l^1(S/\rho, \omega_\rho)$ is not boundedly approximately amenable and consequensly $l^1(S, \omega)$ is not boundedly approximately amenable modulo I_ρ. □

We end this paper to give some illustrative examples.

Example 4.1. *(i) Let $S = \{p^m q^n : m, n \geq 0\}$ be the bicyclic semigroup generated by p, q, then $S/\sigma \simeq \mathbb{Z}$ where $\sigma = \{(s, t) \in S \times S : se = te, \text{ for some } e \in E(S)\}$ is the least group congruence on S [1]. Using Theorem 4.1, amenability of S implies that $l^1(S)$ is boundedly approximately amenable modulo*

I_σ. We note that $l^1(S)$ is not boundedly approximately amenable because $l^1(S)$ is a not approximate amenable.

(ii) Let $S = (\mathbb{N}, \vee)$ be the commutative semigroup of positive integers with maximum operation, then $E(S) = S$. Set $m\sigma n$ if and only if $km = kn$, for some $k \in E(S)\,(n, m \in \mathbb{N})$. Obviously σ is the least group congruence on S and $S/\sigma \simeq G_S$ is the maximum group image of S. Since G_S is finite, $l^1(S/\sigma)$ is contractible and consequently $l^1(S/\sigma)$ is boundedly approximately contractible and boundedly approximately amenable [2, 6]. Thus $l^1(S)$ is boundedly approximately contractible modulo I_σ and boundedly approximately amenable modulo I_σ. We note that $l^1(S)$ is not contractible because $l^1(\mathbb{N})$ has not diagonal.

(iii) Let $G = \mathbb{F}_2$ be a free group with two generators a, b, $T = (\mathbb{N}_0, +)\times(\mathbb{N}, max)$, where $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$ and $S = G\times T$. Then $E(S) = \{(1_G, e) : e \in E(T)\}$ is infinite. Under the homomorphism $\phi : (g, t) \mapsto g$, G is the maximum group homomorphism image of S. Suppose that $S/\sigma \simeq G$ where σ is a group congruence on S. Then $l^1(S)$ is not boundedly approximately amenable (contractible) modulo I_σ, since otherwise $\frac{l^1(S)}{I_\sigma} \simeq l^1(G)$ should be boundedly approximately amenable (contractible) which is contradiction.

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