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SOME INTEGRAL INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

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Abstract. In this paper, firstly we extend some generalization of the Hermite-Hadamard inequality and Bullen inequality to generalized convex functions. Then, we give some important integral inequalities related to these inequalities.

1. Introduction

Definition 1.1 (Convex function). *The function $f : [a,b] \subset R \to R$, is said to be convex if the following inequality holds*

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

for all $x, y \in [a,b]$ and $t \in [0,1]$. We say that f is concave if $(-f)$ is convex.

The classical Hermite-Hadamard inequality which was first published in [8] gives us an estimate of the mean value of a convex function $f : I \to \mathbb{R}$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

In [1], Bullen proved the following inequality which is known as Bullen's inequality for convex function.

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right].$$

An account the history of this inequality can be found in [3]. Surveys on various generalizations and developments can be found in [12] and [2]. Recently in [5], the author established this inequality for twice differentiable functions. In the case where f is convex then there exists an estimation better than (1.1).

In [6], Farissi gave the refinement of the inequality (1.1) as follows:

Theorem 1.1. *Assume that $f : I \to \mathbb{R}$ is a convex function on I. Then for all $\lambda \in [0,1]$, we have*

$$f\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq L(\lambda) \leq \frac{f(a)+f(b)}{2},$$

where

$$l(\lambda) := \lambda f\left(\frac{\lambda b+(2-\lambda)a}{2}\right)+(1-\lambda)f\left(\frac{(1+\lambda)b+(1-\lambda)a}{2}\right)$$

and

$$L(\lambda) := \frac{1}{2}\left(f(\lambda b+(1-\lambda)a)+\lambda f(a)+(1-\lambda)f(b)\right).$$

For more information recent developments to above inequalities, please refer to *[2]- [7], [9]- [11], [14]* and so on.

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2. Preliminaries

Recall the set R^{α} of real line numbers and use the Gao-Yang-Kang's idea to describe the definition of the local fractional derivative and local fractional integral, see [15,16] and so on.

Recently, the theory of Yang's fractional sets [15] was introduced as follows.

For $0<\alpha\leq 1$, we have the following α-type set of element sets:

Z^{α} : The α-type set of integer is defined as the set $\{0^{\alpha},\pm 1^{\alpha},\pm 2^{\alpha},...,\pm n^{\alpha},...\}$.

Q^{α} : The α-type set of the rational numbers is defined as the set $\{m^{\alpha}=\left(\frac{p}{q}\right)^{\alpha}: p,q\in Z,\ q\neq 0\}$.

J^{α} : The α-type set of the irrational numbers is defined as the set $\{m^{\alpha}\neq\left(\frac{p}{q}\right)^{\alpha}: p,q\in Z,\ q\neq 0\}$.

R^{α} : The α-type set of the real line numbers is defined as the set $R^{\alpha}=Q^{\alpha}\cup J^{\alpha}$.

If a^{α},b^{α} and c^{α} belongs the set R^{α} of real line numbers, then

(1) $a^{\alpha}+b^{\alpha}$ and $a^{\alpha}b^{\alpha}$ belongs the set R^{α};

(2) $a^{\alpha}+b^{\alpha}=b^{\alpha}+a^{\alpha}=(a+b)^{\alpha}=(b+a)^{\alpha}$;

(3) $a^{\alpha}+(b^{\alpha}+c^{\alpha})=(a+b)^{\alpha}+c^{\alpha}$;

(4) $a^{\alpha}b^{\alpha}=b^{\alpha}a^{\alpha}=(ab)^{\alpha}=(ba)^{\alpha}$;

(5) $a^{\alpha}\left(b^{\alpha}c^{\alpha}\right)=\left(a^{\alpha}b^{\alpha}\right)c^{\alpha}$;

(6) $a^{\alpha}\left(b^{\alpha}+c^{\alpha}\right)=a^{\alpha}b^{\alpha}+a^{\alpha}c^{\alpha}$;

(7) $a^{\alpha}+0^{\alpha}=0^{\alpha}+a^{\alpha}=a^{\alpha}$ and $a^{\alpha}1^{\alpha}=1^{\alpha}a^{\alpha}=a^{\alpha}$.

The definition of the local fractional derivative and local fractional integral can be given as follows.

Definition 2.1. *[15] A non-differentiable function $f:R\rightarrow R^{\alpha}$, $x\rightarrow f(x)$ is called to be local fractional continuous at x_0, if for any $\varepsilon>0$, there exists $\delta>0$, such that*

$$|f(x)-f(x_0)|<\varepsilon^{\alpha}$$

holds for $|x-x_0|<\delta$, where $\varepsilon,\delta\in R$. If $f(x)$ is local continuous on the interval (a,b), we denote $f(x)\in C_{\alpha}(a,b)$.

Definition 2.2. *[15] The local fractional derivative of $f(x)$ of order α at $x=x_0$ is defined by*

$$f^{(\alpha)}(x_0)=\left.\frac{d^{\alpha}f(x)}{dx^{\alpha}}\right|_{x=x_0}=\lim_{x\rightarrow x_0}\frac{\Delta^{\alpha}\left(f(x)-f(x_0)\right)}{\left(x-x_0\right)^{\alpha}},$$

where $\Delta^{\alpha}\left(f(x)-f(x_0)\right)\cong\Gamma(\alpha+1)\left(f(x)-f(x_0)\right)$.

If there exists $f^{(k+1)\alpha}(x)=\overbrace{D_x^{\alpha}...D_x^{\alpha}}^{k+1\text{ times}}f(x)$ for any $x\in I\subseteq R$, then we denoted $f\in D_{(k+1)\alpha}(I)$, where $k=0,1,2,...$

Definition 2.3. *[15] Let $f(x)\in C_{\alpha}\left[a,b\right]$. Then the local fractional integral is defined by,*

$${}_aI_b^{\alpha}f(x)=\frac{1}{\Gamma(\alpha+1)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(\alpha+1)}\lim_{\Delta t\rightarrow 0}\sum_{j=0}^{N-1}f(t_j)(\Delta t_j)^{\alpha},$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\left\{\Delta t_1,\Delta t_2,...,\Delta t_{N-1}\right\}$, where $\left[t_j,t_{j+1}\right]$, $j=0,...,N-1$ and $a=t_0<t_1<...<t_{N-1}<t_N=b$ is partition of interval $\left[a,b\right]$.

Here, it follows that ${}_aI_b^{\alpha}f(x)=0$ if $a=b$ and ${}_aI_b^{\alpha}f(x)=-{}_bI_a^{\alpha}f(x)$ if $a<b$. If for any $x\in\left[a,b\right]$, there exists ${}_aI_x^{\alpha}f(x)$, then we denoted by $f(x)\in I_x^{\alpha}\left[a,b\right]$.

Definition 2.4 (Generalized convex function)**.** *[15] Let $f:I\subseteq R\rightarrow R^{\alpha}$. For any $x_1,x_2\in I$ and $\lambda\in\left[0,1\right]$, if the following inequality*

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda^{\alpha}f(x_1)+(1-\lambda)^{\alpha}f(x_2)$$

holds, then f is called a generalized convex function on I.

Here are two basic examples of generalized convex functions:

(1) $f(x)=x^{\alpha p}$, $x\geq 0$, $p>1$;

(2) $f(x)=E_{\alpha}(x^{\alpha})$, $x\in R$ where $E_{\alpha}(x^{\alpha})=\sum\limits_{k=0}^{\infty}\frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ is the Mittag-Lrffer function.

Theorem 2.1. *[13] Let $f \in D_\alpha(I)$, then the following conditions are equivalent*

a) f is a generalized convex function on I

b) $f^{(\alpha)}$ is an increasing function on I

c) for any $x_1, x_2 \in I$,

$$f(x_2) - f(x_1) \geq \frac{f^{(\alpha)}(x_1)}{\Gamma(1+\alpha)} (x_2 - x_1)^\alpha .$$

Corollary 2.1. *[13] Let $f \in D_{2\alpha}(a,b)$. Then f is a generalized convex function (or a generalized concave function) if and only if*

$$f^{(2\alpha)}(x) \geq 0 \left(or\ f^{(2\alpha)}(x) \leq 0 \right)$$

for all $x \in (a,b)$.

Lemma 2.1. *[15]*

(1) (Local fractional integration is anti-differentiation) Suppose that $f(x) = g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$${}_aI_b^\alpha f(x) = g(b) - g(a).$$

(2) (Local fractional integration by parts) Suppose that $f(x), g(x) \in D_\alpha[a,b]$ and $f^{(\alpha)}(x)$, $g^{(\alpha)}(x) \in C_\alpha[a,b]$, then we have

$${}_aI_b^\alpha f(x)g^{(\alpha)}(x) = f(x)g(x)|_a^b - {}_aI_b^\alpha f^{(\alpha)}(x)g(x).$$

Lemma 2.2. *[15] We have*

i) $\dfrac{d^\alpha x^{k\alpha}}{dx^\alpha} = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha}$;

ii) $\dfrac{1}{\Gamma(\alpha+1)} \int_a^b x^{k\alpha}(dx)^\alpha = \dfrac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right)$, $k \in R$.

Lemma 2.3 (Generalized Hölder's inequality)**.** *[15] Let $f, g \in C_\alpha[a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| (dx)^\alpha \leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q (dx)^\alpha \right)^{\frac{1}{q}}.$$

In [13], Mo et al. proved the following generalized Hermite-Hadamard inequality for generalized convex function:

Theorem 2.2 (Generalized Hermite-Hadamard inequality)**.** *Let $f(x) \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \leq \frac{f(a)+f(b)}{2^\alpha}. \tag{2.1}$$

The aim of this paper is to extend the generalized Hermite-Hadamard inequalities and generalized Bullen inequalities to generalized convex functions.

3. Main Results

Theorem 3.1 (Generalized Hermite–Hadamard-type inequality)**.** *Let $f(x) \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then*

$$f\left(\frac{a+b}{2}\right) \leq h(\lambda) \leq \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} {}_aI_b^\alpha f(x) \leq H(\lambda) \leq \frac{f(a)+f(b)}{2^\alpha}, \tag{3.1}$$

where

$$h(\lambda) := \lambda^\alpha f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda)^\alpha f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$H(\lambda) := \frac{1}{2^\alpha}\left[f(\lambda b + (1-\lambda)a) + \lambda^\alpha f(a) + (1-\lambda)^\alpha f(b)\right].$$

Proof. Let f be a generalized convex. Then, applying (2.1) on the subinterval $[a, \lambda b + (1-\lambda)a]$, with $\lambda \neq 0$, we have

$$\begin{aligned} & f\left(\frac{\lambda b + (2-\lambda)a}{2}\right) \\ \leq & \frac{1}{\lambda^{\alpha}(b-a)^{\alpha}} \int_{a}^{\lambda b+(1-\lambda)a} f(t)(dt)^{\alpha} \\ \leq & \frac{f(a) + f(\lambda b + (1-\lambda)a)}{2^{\alpha}}. \end{aligned} \tag{3.2}$$

Applying (2.1) again on $[\lambda b + (1-\lambda)a, b]$, with $\lambda \neq 1$, we get

$$\begin{aligned} & f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ \leq & \frac{1}{(1-\lambda)^{\alpha}(b-a)^{\alpha}} \int_{\lambda b+(1-\lambda)a}^{b} f(t)(dt)^{\alpha} \\ \leq & \frac{f(\lambda b + (1-\lambda)a) + f(b)}{2^{\alpha}}. \end{aligned} \tag{3.3}$$

Multiplying (3.2) by λ^{α}, (3.3) by $(1-\lambda)^{\alpha}$, and adding the resulting inequalities, we get:

$$h(\lambda) \leq \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(x) \leq H(\lambda) \tag{3.4}$$

where $h(\lambda)$ and $H(\lambda)$ are defined as in Theorem 3.1.

Using the fact that f is a generalized convex function, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}\right) \\ = & f\left(\lambda\frac{\lambda b + (2-\lambda)a}{2} + (1-\lambda)\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ \leq & \lambda^{\alpha} f\left(\frac{\lambda v + (2-\lambda)a}{2}\right) + (1-\lambda)^{\alpha} f\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right) \\ \leq & \frac{\lambda^{\alpha}}{2^{\alpha}}\left[f(\lambda b + (1-\lambda)a) + f(a)\right] + \frac{(1-\lambda)^{\alpha}}{2^{\alpha}}\left[f(b) + f(\lambda b + (1-\lambda)a)\right] \\ = & \frac{1}{2^{\alpha}}\left[f(\lambda b + (1-\lambda)a) + \lambda^{\alpha} f(a) + (1-\lambda)^{\alpha} f(b)\right] \\ \leq & \frac{f(a) + f(b)}{2^{\alpha}}. \end{aligned} \tag{3.5}$$

Then by (3.4) and (3.5), we get (3.1). □

Theorem 3.2. *Let $g(x) \in D_{2\alpha}[a,b]$ such that there exist constants $m, M \in R^{\alpha}$ so that $m \leq g^{(2\alpha)}(x) \leq M$ for $x \in [a,b]$. Then*

$$\begin{aligned} & \frac{m(b^{\alpha} + a^{\alpha}b^{\alpha} + a^{\alpha})}{\Gamma(1+3\alpha)} - \frac{m}{\Gamma(1+2\alpha)}\left(\frac{a^{2\alpha} + b^{2\alpha}}{2^{\alpha}}\right) \\ \leq & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} g(x) - g\left(\frac{a+b}{2}\right) \\ \leq & \frac{M}{\Gamma(1+2\alpha)}\left(\frac{a^{2\alpha} + b^{2\alpha}}{2^{\alpha}}\right) - \frac{M(b^{\alpha} + a^{\alpha}b^{\alpha} + a^{\alpha})}{\Gamma(1+3\alpha)}. \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & \frac{m}{\Gamma(1+2\alpha)}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right)-\frac{m\left(b^{\alpha}+a^{\alpha}b^{\alpha}+a^{\alpha}\right)}{\Gamma(1+3\alpha)} \\ \leq\ & \frac{g(a)+g(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}g(x) \\ \leq\ & \frac{M\left(b^{\alpha}+a^{\alpha}b^{\alpha}+a^{\alpha}\right)}{\Gamma(1+3\alpha)}-\frac{M}{\Gamma(1+2\alpha)}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right). \end{aligned} \tag{3.7}$$

Proof. Let $f(x)=g(x)-\frac{m}{\Gamma(1+2\alpha)}x^{2\alpha}$, then $f^{(2\alpha)}(x)=g^{(2\alpha)}(x)-m\geq 0$, which shows that f is generalized convex on (a,b). Appliying ineqaulity (2.1) for f , then we have

$$\begin{aligned} & g\left(\frac{a+b}{2}\right)-\frac{m}{\Gamma(1+2\alpha)}\left(\frac{a+b}{2}\right)^{2\alpha} \\ =\ & f\left(\frac{a+b}{2}\right) \\ \leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x) \\ =\ & \frac{1}{(b-a)^{\alpha}}\int_{a}^{b}\left[g(x)-\frac{m}{\Gamma(1+2\alpha)}x^{2\alpha}\right](dx)^{\alpha} \\ =\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}g(x)-\frac{1}{(b-a)^{\alpha}}\frac{m}{\Gamma(1+2\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(b^{3\alpha}-a^{3\alpha}\right). \end{aligned}$$

This implies that

$$\begin{aligned} & \frac{m\left(b^{\alpha}+a^{\alpha}b^{\alpha}+a^{\alpha}\right)}{\Gamma(1+3\alpha)}-\frac{m}{\Gamma(1+2\alpha)}\left(\frac{a+b}{2}\right)^{2\alpha} \\ \leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}g(x)-g\left(\frac{a+b}{2}\right) \end{aligned}$$

which proves the first inequality in (3.6). To prove the second part of (3.6), we apply the same argument for the generalized convex mapping $f(x)=\frac{M}{\Gamma(1+2\alpha)}x^{2\alpha}-g(x);\ \ x\in[a,b]$.

By applying the second part of the generalized Hermite-Hadamard inequality (2.1) for the mapping $f(x)=g(x)-\frac{m}{\Gamma(1+2\alpha)}x^{2\alpha}$ as follows

$$\begin{aligned} & \frac{g(a)+g(b)}{2^{\alpha}}-\frac{m}{\Gamma(1+2\alpha)}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right) \\ =\ & \frac{f(a)+f(b)}{2^{\alpha}} \\ \geq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x) \\ =\ & \frac{1}{(b-a)^{\alpha}}\int_{a}^{b}\left[g(x)-\frac{m}{\Gamma(1+2\alpha)}x^{2\alpha}\right](dx)^{\alpha} \\ =\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}g(x)-\frac{1}{(b-a)^{\alpha}}\frac{m}{\Gamma(1+2\alpha)}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(b^{3\alpha}-a^{3\alpha}\right). \end{aligned}$$

This is equivalent to

$$\begin{aligned} & \frac{m}{\Gamma(1+2\alpha)}\left(\frac{a^{2\alpha}+b^{2\alpha}}{2^{\alpha}}\right)-\frac{m\left(b^{\alpha}+a^{\alpha}b^{\alpha}+a^{\alpha}\right)}{\Gamma(1+3\alpha)} \\ \leq\ & \frac{g(a)+g(b)}{2^{\alpha}}-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}g(x) \end{aligned}$$

which proves the rest part of (3.7). The second part is established in a similar manner; and we omit the details which completes the proof. □

We prove the following inequality which is generalized Bullen inequality for generalized convex function.

Theorem 3.3 (Generalized Bullen inequality)**.** *Let $f(x)\in I_{x}^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a<b$. Then we have the inequality*

$$\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\leq\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]. \tag{3.8}$$

Proof. Using the Theorem 2.2, we find that

$$\begin{aligned} & \frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\frac{1}{\Gamma(1+\alpha)}\int_{a}^{b}f(x)\,(dx)^{\alpha} \\ =\ & \frac{2^{\alpha}\Gamma(1+\alpha)}{(b-a)^{\alpha}}\left[\frac{1}{\Gamma(1+\alpha)}\int_{a}^{\frac{a+b}{2}}f(x)\,(dx)^{\alpha}+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}f(x)\,(dx)^{\alpha}\right] \\ \leq\ & \frac{f\left(\frac{a+b}{2}\right)+f(a)}{2^{\alpha}}+\frac{f(b)+f\left(\frac{a+b}{2}\right)}{2^{\alpha}} \\ =\ & f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}. \end{aligned}$$

Hence, the proof is completed. □

Theorem 3.4. *Let $I\subseteq\mathbb{R}$ be an interval, $f:I^{0}\subseteq\mathbb{R}\rightarrow\mathbb{R}^{\alpha}$ (I^{0} is the interior of I) such that $f\in D_{2\alpha}(I^{0})$ and $f^{(\alpha)}\in C_{\alpha}[a,b]$ for $a,b\in I^{0}$ with $a<b$. Then, for all $x\in[a,b]$, we have the following identity*

$$\begin{aligned} & \frac{1}{2^{\alpha}(b-a)^{\alpha}(\Gamma(1+\alpha))^{2}}\int_{a}^{b}\left(x-\frac{a+b}{2}\right)^{\alpha}p(x)f^{(2\alpha)}(x)\,(dx)^{\alpha} \\ =\ & \frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x) \end{aligned} \tag{3.9}$$

where

$$p(x)=\begin{cases}(a-x)^{\alpha}, & \left[a,\frac{a+b}{2}\right) \\ (b-x)^{\alpha}, & \left[\frac{a+b}{2},b\right].\end{cases}$$

Proof. Using the local fractional integration by parts, we have

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \left(x-\frac{a+b}{2}\right)^{\alpha} p(x)f^{(2\alpha)}(x)(dx)^{\alpha}$$

$$= \frac{1}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}} \left(x-\frac{a+b}{2}\right)^{\alpha}(a-x)^{\alpha} f^{(2\alpha)}(x)(dx)^{\alpha}$$

$$+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b} \left(x-\frac{a+b}{2}\right)^{\alpha}(b-x)^{\alpha} f^{(2\alpha)}(x)(dx)^{\alpha}$$

$$= \left(x-\frac{a+b}{2}\right)^{\alpha}(a-x)^{\alpha} f^{(\alpha)}(x)\Big|_a^{\frac{a+b}{2}}$$

$$-\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}} \left(\frac{3a+b}{2}-2x\right)^{\alpha} f^{(\alpha)}(x)(dx)^{\alpha}$$

$$+\left(x-\frac{a+b}{2}\right)^{\alpha}(b-x)^{\alpha} f^{(\alpha)}(x)\Big|_{\frac{a+b}{2}}^{b}$$

$$-\frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b} \left(\frac{a+3b}{2}-2x\right)^{\alpha} f^{(\alpha)}(x)(dx)^{\alpha}.$$

Using the local fractional integration by parts again, we find that

$$\frac{1}{\Gamma(1+\alpha)}\int_a^b \left(x-\frac{a+b}{2}\right)^{\alpha} p(x)f^{(2\alpha)}(x)(dx)^{\alpha}$$

$$= \Gamma(1+\alpha)(b-a)^{\alpha} f\left(\frac{a+b}{2}\right)+\Gamma(1+\alpha)(b-a)^{\alpha}\frac{f(a)+f(b)}{2^{\alpha}}$$

$$-\frac{2^{\alpha}(\Gamma(1+\alpha))^{2}}{\Gamma(1+\alpha)}\int_a^b f(x)(dx)^{\alpha}.$$

If we devide the resulting equality with $2^{\alpha}\Gamma(1+\alpha)(b-a)^{\alpha}$, then we complete the proof. □

Theorem 3.5. *Suppose that the assumptions of Theorem 3.4 are satisfied, then we have the following inequality*

$$\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_aI_b^{\alpha}f(x)\right|$$

$$\leq \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{8^{\alpha}\Gamma(1+\alpha)}\left(B(p+1,p+1)\right)^{\frac{1}{p}}\left\|f^{(2\alpha)}(x)\right\|_{q}$$

where, $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\left\|f^{(2\alpha)}\right\|_{q}$ *is defined by*

$$\left\|f^{(2\alpha)}\right\|_{q}=\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b \left|f^{(2\alpha)}(x)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}$$

and $B(x,y)$ is defined by

$$B(x,y)=\frac{1}{\Gamma(1+\alpha)}\int_0^1 t^{(x-1)\alpha}(1-t)^{(y-1)\alpha}(dt)^\alpha .$$

Proof. Taking madulus in (3.9) and using generalized Hölder inequality, we have

$$\left|\frac{1}{2^\alpha}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^\alpha}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\,{}_aI_b^\alpha f(x)\right| \tag{3.10}$$

$$\begin{aligned}
&\leq \frac{1}{2^\alpha(b-a)^\alpha(\Gamma(1+\alpha))^2}\int_a^b\left|x-\frac{a+b}{2}\right|^\alpha |p(x)|\left|f^{(2\alpha)}(x)\right|(dx)^\alpha\\
&\leq \frac{1}{2^\alpha(b-a)^\alpha\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|f^{(2\alpha)}(x)\right|^q(dx)^\alpha\right)^{\frac{1}{q}}\\
&\quad\times\left(\frac{1}{\Gamma(1+\alpha)}\int_a^b\left|x-\frac{a+b}{2}\right|^{p\alpha}|p(x)|^p(dx)^\alpha\right)^{\frac{1}{p}}\\
&= \frac{\left\|f^{(2\alpha)}\right\|_q}{2^\alpha(b-a)^\alpha\Gamma(1+\alpha)}\left(\frac{1}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{p\alpha}(x-a)^{p\alpha}(dx)^\alpha\right.\\
&\quad\left.+\frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)^{p\alpha}(b-x)^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}\\
&= \frac{\left\|f^{(2\alpha)}\right\|_q}{2^\alpha(b-a)^\alpha\Gamma(1+\alpha)}(K_1+K_2)^{\frac{1}{p}} .
\end{aligned}$$

For calculating integral K_1, using changing variable with $x=(1-t)a+t\frac{a+b}{2}$, we obtain

$$\begin{aligned}
K_1 &= \frac{1}{\Gamma(1+\alpha)}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^{p\alpha}(x-a)^{p\alpha}(dx)^\alpha\\
&= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha}\frac{1}{\Gamma(1+\alpha)}\int_0^1(1-t)^{p\alpha}t^{p\alpha}(dt)^\alpha\\
&= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha}B(p+1,p+1).
\end{aligned} \tag{3.11}$$

Similarliy, using changing variable with $x=(1-t)\frac{a+b}{2}+tb$, we have

$$\begin{aligned}
K_2 &= \frac{1}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^b\left(x-\frac{a+b}{2}\right)^{p\alpha}(b-x)^{p\alpha}(dx)^\alpha\\
&= \left(\frac{b-a}{2}\right)^{(2p+1)\alpha}B(p+1,p+1)
\end{aligned} \tag{3.12}$$

Putting (3.11) and (3.12) in (3.10), we obtain

$$\begin{aligned}&\left|\frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\right|\\ \leq\;& \frac{\left\|f^{(2\alpha)}\right\|_{q}}{2^{\alpha}(b-a)^{\alpha}\Gamma(1+\alpha)}\left(2^{\alpha}\frac{(b-a)^{(2p+1)\alpha}}{2^{(2p+1)\alpha}}B(p+1,p+1)\right)^{\frac{1}{p}}\\ =\;& \frac{(b-a)^{\left(1+\frac{1}{p}\right)\alpha}}{8^{\alpha}\Gamma(1+\alpha)}\left(B(p+1,p+1)\right)^{\frac{1}{p}}\left\|f^{(2\alpha)}\right\|_{q}\end{aligned}$$

which completes the proof. □

Theorem 3.6. *The assumptions of Theorem 3.4 are satisfied. If the mapping*

$$\varphi(x)=\begin{cases}(a-x)^{\alpha}\left(x-\frac{a+b}{2}\right)^{\alpha}f^{(2\alpha)}(x), & \left[a,\frac{a+b}{2}\right)\\ (b-x)^{\alpha}\left(x-\frac{a+b}{2}\right)^{\alpha}f^{(2\alpha)}(x), & \left[\frac{a+b}{2},b\right].\end{cases}$$

is a generalized convex, then we have the inequality

$$\begin{aligned}&\frac{(b-a)^{2\alpha}}{64^{\alpha}\left(\Gamma(1+\alpha)\right)^{2}}\left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right)+f^{(2\alpha)}\left(\frac{a+3b}{4}\right)\right]\\ \leq\;& \frac{1}{2^{\alpha}}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right]-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\,{}_{a}I_{b}^{\alpha}f(x)\\ \leq\;& \frac{(b-a)^{2\alpha}}{128^{\alpha}\left(\Gamma(1+\alpha)\right)^{2}}\left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right)+f^{(2\alpha)}\left(\frac{a+3b}{4}\right)\right].\end{aligned}$$

Proof. Applying the first inequality (2.1) for the mapping φ, we get

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_{a}^{\frac{a+b}{2}}\varphi(x)(dx)^{\alpha}\\ \geq\;& \varphi\left(\frac{3a+b}{4}\right)=\frac{(b-a)^{2\alpha}}{16^{\alpha}}f^{(2\alpha)}\left(\frac{3a+b}{4}\right)\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\varphi(x)(dx)^{\alpha}\\ \geq\;& \varphi\left(\frac{a+3b}{4}\right)=\frac{(b-a)^{2\alpha}}{16^{\alpha}}f^{(2\alpha)}\left(\frac{a+3b}{4}\right).\end{aligned}\tag{3.14}$$

Applying the inequality (3.8) for the mapping φ, we have

$$\begin{aligned}&\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_{a}^{\frac{a+b}{2}}\varphi(x)(dx)^{\alpha}\\ \leq\;& \frac{1}{2^{\alpha}}\left[\varphi\left(\frac{3a+b}{4}\right)+\frac{\varphi(a)+\varphi\left(\frac{a+b}{2}\right)}{2^{\alpha}}\right]\\ =\;& \frac{(b-a)^{2\alpha}}{32^{\alpha}}f^{(2\alpha)}\left(\frac{3a+b}{4}\right)\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}
&\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\varphi(x)(dx)^{\alpha} \\
\leq\ & \frac{1}{2^{\alpha}}\left[\varphi\left(\frac{a+3b}{4}\right)+\frac{\varphi\left(\frac{a+b}{2}\right)+\varphi(b)}{2^{\alpha}}\right] \\
=\ & \frac{(b-a)^{2\alpha}}{32^{\alpha}}f^{(2\alpha)}\left(\frac{a+3b}{4}\right).
\end{aligned} \tag{3.16}$$

Adding the inequalities (3.13)-(3.16) and from Theorem 3.4, we write

$$\begin{aligned}
&\frac{(b-a)^{2\alpha}}{16^{\alpha}}\left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right)+f^{(2\alpha)}\left(\frac{a+3b}{4}\right)\right] \\
\leq\ & \frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\frac{2^{\alpha}}{\Gamma(1+\alpha)}\int_{\frac{a+b}{2}}^{b}\varphi(x)(dx)^{\alpha} \\
=\ & 4^{\alpha}(\Gamma(1+\alpha))^{2}\left[\frac{1}{2^{\alpha}}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2^{\alpha}}\right)-\frac{\Gamma(1+\alpha)}{(b-a)^{\alpha}}\ {}_{a}I_{b}^{\alpha}f(x)\right] \\
\leq\ & \frac{(b-a)^{2\alpha}}{32^{\alpha}}\left[f^{(2\alpha)}\left(\frac{3a+b}{4}\right)+f^{(2\alpha)}\left(\frac{a+3b}{4}\right)\right].
\end{aligned}$$

If we devide the resulting inequality with $4^{\alpha}(\Gamma(1+\alpha))^{2}$, then we complete the proof. □

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