International Journal of Analysis and Applications
ISSN 2291-8639
Volume 14, Number 1 (2017), 20-26
http://www.etamaths.com

ALGEBRAIC HYPER-STRUCTURES ASSOCIATED TO NASH EQUILIBRIUM POINT AND APPLICATIONS

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Abstract. In this paper, we generalize some concepts of the game theory such as Nash equilibrium point, saddle point and existence theorems on hyper-structures. Based on new definitions and theorems, we obtain some important results in the game theory. A few suitable examples have been given for better understanding.

1. Introduction and preliminaries

Algebraic hyperstructures are suitable generalizations of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. More exactly, if H is a non-empty set and $\mathcal{P}^*(H)$ is the set of all non-empty subsets of H, then we consider maps of the following type: $f_i : H \times H \longrightarrow \mathcal{P}^*(H)$, where $i \in \{1, 2, \ldots, n\}$ and n is a positive integer. The maps f_i are called (*binary*) *hyperoperations*. For all x, y of H, $f_i(x, y)$ is called the (*binary*) *hyperproduct* of x and y. An algebraic system $(H, f_1, \ldots, f_n)$ is called a (*binary*) *hyperstructure*. Usually, $n = 1$ or $n = 2$. Under certain conditions, imposed to the maps f_i, we obtain the so-called semihypergroups, hypergroups, hyperrings or hyperfields. Sometimes, external hyperoperations are considered, which are maps of the following type: $h : R \times H \longrightarrow \mathcal{P}^*(H)$, where $R \neq H$. Usually, R is endowed with a ring or a hyperring structure. Several books have been written on this topic, see [1, 2, 6, 13]. Hyperstructure theory both extends some well-known group results and introduce new topics leading us to a wide variety of applications, as well as to a broadening of the investigation fields, for example see [4, 5, 8, 10–12]. A recent book on hyperstructures [2] points out on their applications in rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Another book [6] is devoted especially to the study of hyperring theory. Several kinds of hyperrings are introduced and analyzed. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. The theory of suitable modified hyperstructures can serve as a mathematical background in the field of quantum communication systems.

Optimization theory is the study of the extremal values of a function: its minima and maxima. In mathematics, optimization refers to choosing the best element from some set of available alternatives. Nonlinear programming deals with the problem of optimizing an objective function in the presence of some constraints. In [8], we generalized the optimization theory on algebraic hyperstructures. The game theory is another framework which has been generalized of optimization theory. The famous mathematician Von Numan has been proposed his important game theory in 1928. Game theory is an important branch of applied mathematics in which decision maker chooses his strategy with regards to strategies of other players. In this theory any player tries to choose his best strategy for obtaining maximum pay off function. The methods and applications in the game theory are very different. In [9], bileaner two person nonzero sum game has been considered.

In the mathematical domain the extensions of the previous works are very popular. Our contribution in this paper is the extensions of some previous concepts in game theory on the hyperstructures and have considered several new examples that we can't solve them via usual game theory. It means that we have obtained a new widespread in the game theory.

2010 *Mathematics Subject Classification.* 20N20, 91A99.
Key words and phrases. algebraic hyperstructure; optimization; game theory; Nash equilibrium point.

2. Game Theory

In this paper, we address a hyper-structures as follows:

$$\star : H \times H \to H \otimes H \subseteq \mathcal{P}^*(H), \tag{2.1}$$
$$\cdot : F \times H \to H,$$
$$+ : H \times H \to H,$$

where $H \neq \emptyset$, $\star$ is a commutative hyperoperation such that $\star(H \times H) = H \otimes H$, $\cdot$ and $+$ are commutative binary operations and F is a filed. Henceforth, let $F = \mathbb{R}$. Convex and concave functions play an important role in almost all branches of mathematics as well as other areas of science and engineering. Convex and concave functions have many special and important properties. In this paper we use some these properties in game theory . Let define $P^*(X) = \{x \star y \in H \otimes H : x,\ y \in X\}$ for all non-empty subset X in H. Now, we formulate the n-person game theory problem on hyper-structures as follows: Define the function

$$f_i : P^*(X_1) \times \cdots \times P^*(X_n) \to \mathbb{R},\ i = 1, \cdots, n,$$

where X_i is any non-empty subset of H and f_i and X_i are a pay off function and a strategy set of i-th player, respectively. Let $W = X_1 \times \cdots \times X_n$. The next definition plies an important role in the following discussions.

Definition 2.1. *The n-tuple $(\overline{x_1} \star \overline{y_1}, \cdots, \overline{x_n} \star \overline{y_n}) \in P^*(X_1) \times \cdots \times P^*(X_n)$ is called Nash equilibrium point, if the following inequalities hold.*

$$f_1(\overline{x_1} \star \overline{y_1}, \cdots, \overline{x_n} \star \overline{y_n}) \leqslant f_1(x_1 \star y_1, \cdots, \overline{x_n} \star \overline{y_n}), \tag{2.2}$$

for all $x_1 \star y_1 \in P^*(X_1)$,

$$\vdots$$

$$f_n(\overline{x_1} \star \overline{y_1}, \cdots, \overline{x_n} \star \overline{y_n}) \leqslant f_n(\overline{x_1} \star \overline{y_1}, \cdots, x_n \star y_n),$$

for all $x_n \star y_n \in P^*(X_n)$.

Henceforth, for simplicity let consider $n = 2$. As a special case we consider the following situation. If we have

$$f_1(x_1 \star y_1, x_2 \star y_2) + f_2(x_1 \star y_1, x_2 \star y_2) = 0,$$

for all $(x_1 \star y_1, x_1 \star y_1) \in P^*(X_1) \times P^*(X_2)$. Thus, the Nash equilibrium point satisfies in the following inequalities,

$$f_1(\overline{x_1} \star \overline{y_1}, x_2 \star y_2) \leqslant f_1(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) \leqslant f_1(x_1 \star y_1, \overline{x_2} \star \overline{y_2}), \tag{2.3}$$

for all $(x_1 \star y_1, x_2 \star y_2) \in P^*(X_1) \times P^*(X_2)$, or equivalently

$$f_2(x_1 \star y_1, \overline{x_2} \star \overline{y_2}) \leqslant f_2(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) \leqslant f_2(\overline{x_1} \star \overline{y_1}, x_2 \star y_2), \tag{2.4}$$

for all $(x_1 \star y_1, x_2 \star y_2) \in P^*(X_1) \times P^*(X_2)$. The next definition plays an important role in the game theory.

Definition 2.2. *The pair $(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) \in P^*(X_1) \times P^*(X_2)$ is called the saddle point, if satisfies the first inequalities (3) or second inequalities (4).*

Suppose that

$$\overline{v} = \inf_{x_2, y_2 \in X_2} \sup_{x_1, y_1 \in X_1} f(x_1 \star y_1, x_2 \star y_2) \text{ and } \underline{v} = \sup_{x_1, y_1 \in X_1} \inf_{x_2, y_2 \in X_2} f(x_1 \star y_1, x_2 \star y_2).$$

Clearly, we have $\underline{v} \leq \overline{v}$.

Definition 2.3. *The strategy $\overline{x_1},\ \overline{y_1} \in X_1$ is called max-min if*

$$\underline{v} = \inf_{x_2, y_2 \in X_2} f(\overline{x_1} \star \overline{y_1}, x_2 \star y_2),$$

and similarly, the strategy $\overline{x_2} \star \overline{y_2} \in X_2$ is called min-max, if

$$\overline{v} = \sup_{x_1, y_1 \in X_1} f(x_1 \star y_1, \overline{x_2} \star \overline{y_2}).$$

The following theorem gives us a necessary and sufficient condition that guaranties the existence of saddle point.

Theorem 2.1. *Suppose that the pay-off function $f(x_1 \star y_1, x_2 \star y_2)$ on $X_1 \times X_2$ is given. There is a saddle point $(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2})$ if and only if*

$$\sup_{x_1,y_1 \in X_1} \inf_{x_2,y_2 \in X_2} f(x_1 \star y_1, x_2 \star y_2) = \inf_{x_2,y_2 \in X_2} \sup_{x_1,y_1 \in X_1} f(x_1 \star y_1, x_2 \star y_2). \tag{2.5}$$

In addition $\overline{x_1} \star \overline{y_1} \in P^*(X_1)$ and $\overline{x_2} \star \overline{y_2} \in P^*(X_2)$ are max-min and min-max, respectively.

Proof. Suppose that $(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2})$ is a saddle point. We have

$$\overline{v} \leq \sup_{x_1,y_1 \in X_1} f(x_1 \star y_1, \overline{x_2} \star \overline{y_2}) = f(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) = \inf_{x_2,y_2 \in X_2} f(\overline{x_1} \star \overline{y_1}, x_2 \star y_2) \leq \underline{v},$$

so the equation (5) is held. Now, suppose that we have the equation (5) and $\overline{x_1} \star \overline{y_1}$ and $\overline{x_2} \star \overline{y_2}$ are max-min and min-max strategies, respectively. Then,

$$\begin{aligned} f(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) \quad & \leq \sup_{x_1,y_1 \in X_1} f(x_1 \star y_1, \overline{x_2} \star \overline{y_2}) = \overline{v} \\ & = \underline{v} = \inf_{x_2,y_2 \in X_2} f(\overline{x_1} \star \overline{y_1}, x_2 \star y_2) \\ & \leq f(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}). \end{aligned}$$

This completes the proof. □

The saddle point does not always exist. The following example denotes such a situation.

Example 2.1. *Suppose that $H = \mathbb{R}$ and $X_1 = X_2 = [0,1]$. We define $f : P^*([0,1]) \longrightarrow \mathbb{R}$ and $\star$ respectively as follows:*

$$x \star y = x, \ \ f(x_1 \star y_1, x_2 \star y_2) = 3x_1^2 - 5x_1x_2 + 3x_2^2.$$

Clearly, $M(x_1 \star y_1) = \min_{x_2,y_2 \in X_2} f(x_1 \star y_1, x_2 \star y_2) = \frac{11x_1^2}{12}$. Then, we have: $\underline{v} = \max_{x_1,y_1 \in X_1} \frac{11x_1^2}{12} = \frac{11}{12}$, $\overline{x_1} = 1$. Similarly, $N(x_2 \star y_2) = \max_{x_1,y_1 \in X_1} f(x_1 \star y_1, x_2 \star y_2) = \max\{3x_2^2, 3x_2^2 - 5x_2 + 3\}$, $\overline{x_2} = \frac{3}{5}$. Therefore, we have $\overline{v} = \min_{x_2,y_2 \in X_2} N(x_2 \star y_2) = \frac{27}{25}$, $\underline{v} < \overline{v}$. Now, we obtain max-min and min-max strategies. As

$$\min_{x_2,y_2 \in X_2} f(1 \star y_1, x_2 \star y_2) = \min_{x_2 \in X_2} 3 - 5x_2 + 3x_2^2 = \frac{11}{12} = \underline{v},$$
$$\max_{x_1,y_1 \in X_1} f(x_1 \star y_1, \frac{3}{5} \star y_2) = \max_{x_1 \in X_1} 3x_1^2 - 3x_1 + \frac{27}{25} = \frac{27}{25} = \overline{v}.$$

Therefore, $\{1 \star y_1 : \ y_1 \in [0,1]\}$ and $\{\frac{3}{5} \star y_2 : \ y_2 \in [0,1]\}$ are max-min and min-max strategy set, respectively. According to Theorem (2.5), $(1 \star y_1, \frac{3}{5} \star y_2)$ is not a saddle point.

The above example shows that we must generalize the previous saddle point definition.

Definition 2.4. *Let $\epsilon > 0$. The pair $(x_1^\epsilon \star y_1^\epsilon, x_2^\epsilon \star y_2^\epsilon) \in P^*(X_1) \times P^*(X_2)$ is called ϵ-saddle point of $f(x_1 \star y_1, x_2 \star y_2)$ on $X_1 \times X_2$, if*

$$f(x_1 \star y_1, x_2^\epsilon \star y_2^\epsilon) - \epsilon \leq f(x_1^\epsilon \star y_1^\epsilon, x_2^\epsilon \star y_2^\epsilon) \leq f(x_1^\epsilon \star y_1^\epsilon, x_2 \star y_2) + \epsilon \tag{2.6}$$

for all $x_1, \ y_1 \in X_1, \ x_2, \ y_2 \in X_2$.

Lemma 2.1. *Let $\overline{x_1} \star \overline{y_1}, \ \overline{x_2} \star \overline{y_2}$ are max-min and min-max strategies, respectively and $\epsilon = \overline{v} - \underline{v} \geq 0$. Then, $(\overline{x_1} \star \overline{y_1}, \ \overline{x_2} \star \overline{y_2})$ is an $\epsilon-$saddle point of $f(x_1 \star y_1, x_2 \star y_2)$ on $X_1 \times X_2$.*

Proof. If $\overline{v} = \underline{v}$, so $\overline{x_1} \star \overline{y_1}, \ \overline{x_2} \star \overline{y_2}$ is a saddle point. Let $\epsilon = \overline{v} - \underline{v} > 0$. Clearly, $\overline{x_1} \star \overline{y_1}, \ \overline{x_2} \star \overline{y_2}$ is an $\epsilon-$saddle point. □

Using the above definition, we consider the following example.

Example 2.2. *In Example 2, let $\epsilon = \overline{v} - \underline{v}$. As*

$$\begin{aligned} f(x_1 \star y_1, \tfrac{3}{5} \star y_2) - \epsilon \quad & = 3x_1^2 - 3x_1 + \tfrac{27}{25} - (\tfrac{27}{25} - \tfrac{11}{12}) \\ & \leq f(1 \star y_1, \tfrac{3}{5} \star y_2) = \tfrac{27}{25} \\ & \leq f(1 \star y_1, x_2 \star y_2) + \epsilon \\ & = 3 - 5x_2 + 3x_2^2 + (\tfrac{27}{25} - \tfrac{11}{12}), \end{aligned}$$

for all $x_1,\ y_1 \in X_1,\ x_2,\ y_2 \in X_2$, *so* $(x_1^\epsilon \star y_1^\epsilon, x_2^\epsilon \star y_2^\epsilon) = (1 \star y_1, \frac{3}{5} \star y_2)$, *is a* $\epsilon-$*saddle point of* $f(x_1 \star y_1, x_2 \star y_2)$.

Similar to saddle point, we can generalize the concepts of min-max and max-min as follows:

Definition 2.5. *Let* $\epsilon > 0$. *The strategies* $x_1^\epsilon \star y_1^\epsilon$ *and* $x_2^\epsilon \star y_2^\epsilon$ *are* $\epsilon-$ *max-min and* $\epsilon-$*min-max, if* $\inf\limits_{x_2, y_2 \in X_2} f(x_1^\epsilon \star y_1^\epsilon, x_2 \star y_2) \geq \underline{v} - \epsilon$ *and* $\sup\limits_{x_1, y_1 \in X_1} f(x_1 \star y_1, x_2^\epsilon \star y_2^\epsilon) \leq \overline{v} + \epsilon$.

In the remaining part, we consider the some new topology characters of hyper-structures [7] and their applications in game theory. Let H be a metric space, $X,\ Y$ be compact subsets of H and

$$\begin{aligned} Y(x_1 \star y_1) &= \text{Arg} \min_{x_2,\ y_2 \in X_2} f(x_1 \star y_1, x_2 \star y_2) \\ &= \{\hat{x_2} \star \hat{y_2} | \hat{x_2}, \hat{y_2} \in X_2,\ f(x_1 \star y_1, \hat{x_2} \star \hat{y_2}) \\ &= \min_{x_2,\ y_2 \in X_2} f(x_1 \star y_1, x_2 \star y_2)\}. \end{aligned}$$

Theorem 2.2. *Suppose that the pay off function* $f(x_1 \star y_1, x_2 \star y_2)$ *is a continuous function on* $X_1 \times X_2$ *and* $P^*(X_1),\ P^*(X_2)$ *are compact sets in* $H \otimes H$. *Then, the function* $g(x_1 \star y_1) = \min\limits_{x_2,\ y_2 \in X_2} f(x_1 \star y_1, x_2 \star y_2)$ *on* X_1 *is a continuous function.*

Proof. Let $\{x_1^k\}$ and $\{y_1^k\}$ be two sequences in X_1 such that convergence to $\overline{x_1}$ and $\overline{y_1}$, respectively. By considering $g(x_1^k \star y_1^k)$, we can prove that it converges to $g(\overline{x_1} \star \overline{y_1})$. In contradiction, there are subsequences $\{x_1^{k_l}\},\ \{y_1^{k_l}\}$ in X_1 such that $\lim\limits_{l \to \infty} g(x_1^{k_l} \star y_1^{k_l}) \neq g(\overline{x_1} \star \overline{y_1})$. Choosing sequence $\{x_2^{k_l} \star y_2^{k_l} \in Y(x_1^{k_l} \star y_1^{k_l})\}$, based on compactness of X_2, we have $\lim\limits_{l \to \infty} x_2^{k_l} \star y_2^{k_l} = \overline{x_2} \star \overline{y_2}$. We must show that $\overline{x_2} \star \overline{y_2} \in Y(\overline{x_1} \star \overline{y_1})$. By definition $x_2^{k_l} \star y_2^{k_l}$, we have

$$g(x_1^{k_l} \star y_1^{k_l}) = f(x_1^{k_l} \star y_1^{k_l}, x_2^{k_l} \star y_2^{k_l}) \leq f(x_1^{k_l} \star y_1^{k_l}, x_2 \star y_2),$$

for all $x_2,\ y_2 \in X_2$. In the above inequality, when $l \to \infty$, we conclude that

$$f(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) \leq f(\overline{x_1} \star \overline{y_1}, x_2 \star y_2),$$

for all $x_2,\ y_2 \in X_2$. Therefore, $\overline{x_2} \star \overline{y_2} \in Y(\overline{x_1} \star \overline{y_1})$, that is

$$\lim_{l \to \infty} g(x_1^{k_l} \star y_1^{k_l}) = \lim_{l \to \infty} f(x_1^{k_l} \star y_1^{k_l}, x_2^{k_l} \star y_2^{k_l}) = f(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) = g(\overline{x_1} \star \overline{y_1}).$$

□

Under what conditions is the function $Y(x_1 \star y_1)$ continuous? We assert the sufficient condition, that guaranties the continuous function $Y(x_1 \star y_1)$.

Theorem 2.3. *Suppose that the conditions of previous theorem have been satisfied and for any* $x_1,\ y_1 \in X_1$, $Y(x_1 \star y_1) = \{x_2 \star y_2\}$ *be a singleton. Then,* $Y(x_1 \star y_1)$ *is a continuous function on* X_1.

Proof. Suppose that the function $Y(x_1 \star y_1)$ isn't continuous in $\overline{x_1} \star \overline{y_1} \in P^*(X_1)$, so there is a sequence $\{\overline{x_1^k} \star \overline{y_1^k}\}$ in $P^*(X_1)$, such that it converges $\{\overline{x_1} \star \overline{y_1}\}$, but $\{\overline{x_1^k} \star \overline{y_1^k}\} = \{x_2^k \star y_2^k\}$ does not convergent to $Y(\overline{x_1} \star \overline{y_1}) = \{x_2 \star y_2\}$. Therefore, there is an $\epsilon-$Neighborhood $N_\epsilon^\star(Y(\overline{x_1} \star \overline{y_1})) \subset P^*(X_2)$ that does not contain infinite number of elements $\{x_2^k \star y_2^k\}$. Rely on compactness of $P^*(X_2) - N_\epsilon^\star(Y(\overline{x_1} \star \overline{y_1}))$, we have subsequence $\{Y(x_1^{k_l} \star y_1^{k_l})\} = \{x_2^{k_l} \star y_2^{k_l}\} \subset P^*(X_2) - N_\epsilon^\star(Y(\overline{x_1} \star \overline{y_1}))$, such that it convergent to $x_2' \star y_2' \neq \overline{x_2} \star \overline{y_2}$. According to the previous theorem, we conclude that $x_2' \star y_2' = Y(\overline{x_1} \star \overline{y_1})$, which it contradicts to singleton. □

Definition 2.6. *Let* $f : P^*(X) \to \mathbb{R}$, *where* X *is non-empty convex subset in* H. *The function* f *is called a convex function on* $P^*(X)$ *if*

$$f([\lambda x_1 + (1-\lambda)x_2] \star [\lambda y_1 + (1-\lambda)y_2]) \leq \lambda f(x_1 \star y_1) + (1-\lambda) f(x_2 \star y_2)$$

for each $x_1, x_2, y_1, y_2 \in X,\ x_1 \star y_1,\ x_2 \star y_2 \in P^*(X)$ *and for all* $0 \leq \lambda \leq 1$. *The function is called strictly convex on* $P^*(X)$ *if the inequality is satisfied as a strict inequality for each distinct* $x_1 \star y_1,\ x_2 \star y_2 \in P^*(X)$ *and* $0 < \lambda < 1$. *The function* f *is called concave (strictly concave) on* X *if* $-f$ *is convex (strictly convex) on* X.

The following function is an example of convex function [8].

Example 2.3. *Let $H = \mathbb{R}^+ \times \mathbb{R}^+$. Suppose that*

$$\begin{array}{l} z_{\min} = \min\{x_1, x_2, y_1, y_2\}, \\ z_{\max} = \max\{x_1, x_2, y_1, y_2\}, \\ (x_1, y_1) \star (x_2, y_2) = [z_{\min},\ z_{\max}] \times [z_{\min},\ z_{\max}] \subseteq \mathbb{R}^+ \times \mathbb{R}^+ \end{array}$$

and $f : H \otimes H \to \mathbb{R}$ is defined by $f((x_1, y_1) \star (x_2, y_2)) = z_{\max} - z_{\min}$, for all $(x_1, y_1),\ (x_2, y_2) \in X$, where X is any non-empty convex subset in H. Suppose that $((\bar{x}_1, \bar{y}_1), (\bar{x}_2, \bar{y}_2), \bar{z})$ and $((\hat{x}_1, \hat{y}_1), (\hat{x}_2, \hat{y}_2), \hat{z})$ in $epi^\star f$ and $\bar{x}_\lambda = \lambda\bar{x}_1 + (1-\lambda)\bar{x}_2$, $\bar{y}_\lambda = \lambda\bar{y}_1 + (1-\lambda)\bar{y}_2$, $\hat{x}_\lambda = \lambda\hat{x}_1 + (1-\lambda)\hat{x}_2$, $\hat{y}_\lambda = \lambda\hat{y}_1 + (1-\lambda)\hat{y}_2$. One can show that

$$\max\{\bar{x}_\lambda, \bar{y}_\lambda, \hat{x}_\lambda, \hat{y}_\lambda\} - \min\{\bar{x}_\lambda, \bar{y}_\lambda, \hat{x}_\lambda, \hat{y}_\lambda\} \leq \lambda\bar{z} + (1-\lambda)\hat{z}.$$

That is, $((\bar{x}_\lambda, \bar{y}_\lambda),\ (\hat{x}_\lambda, \hat{y}_\lambda),\ \lambda\bar{z} + (1-\lambda)\hat{z}) \in epi^\star f$. Therefore, $epi^\star f$ is a convex set, so we conclude that the function f is also a convex function.

Theorem 2.4. *Let $H = \mathbb{R}^n$, X_1, X_2 be convex and compact subsets of H and $f(x_1 \star y_1, x_2 \star y_2)$ is a continuous function on $X_1 \times X_2$. Suppose that for any $x_2 \star y_2 \in P^*(X_2)$, $f(x_1 \star y_1, x_2 \star y_2)$ be a concave function with respect to $x_1 \star y_1 \in P^*(X_1)$ and for any $x_1 \star y_1 \in P^*(X_1)$ it is a convex function with respect to $x_2 \star y_2 \in P^*(X_2)$. Then, the function $f(x_1 \star y_1, x_2 \star y_2)$ has a saddle point.*

Proof. : At first , we consider the special case that the function $f(x_1 \star y_1, x_2 \star y_2)$ be a strictly convex w.r.t $x_2 \star y_2 \in P^*(X_2)$. Then, for any $x_1 \star y_1 \in P^*(X_1)$, the function $f(x_1 \star y_1, x_2 \star y_2)$ obtains its unique minimum on X_2 in $Y(x_1 \star y_1)$. According to previous theorems, we conclude $g(x_1 \star y_1)$ and $Y(x_1 \star y_1)$ are continuous function on X_1. Suppose that the function $g(x_1 \star y_1)$ obtains its minimum on X_1 in $\overline{x_1} \star \overline{y_1}$. We can show that $(\overline{x_1} \star \overline{y_1}, Y(\overline{x_1} \star \overline{y_1}))$ is a saddle point of $f(x_1 \star y_1, x_2 \star y_2)$ on $X_1 \times X_2$. Let $x_1 \star y_1 \in P^*(X_1)$ be arbitrary, $0 < \lambda < 1$ and $Y = Y([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1])$. According to concavity of the function $f(x_1 \star y_1, x_2 \star y_2)$ with respect to $x_1 \star y_1 \in P^*(X_1)$, we have

$$\begin{array}{l} (1-\lambda)g(\overline{x_1} \star \overline{y_1}) + \lambda f(x_1 \star y_1, Y([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1])) \\ \leq (1-\lambda)f(\overline{x_1} \star \overline{y_1}, Y([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1])) \\ \quad + \lambda f(x_1 \star y_1, Y([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1])) \\ \leq f([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1], Y([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1])) \\ = g([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1]) \\ \leq g(\overline{x_1} \star \overline{y_1}). \end{array}$$

Therefore, $\lambda f(x_1 \star y_1, Y([(1-\lambda)\overline{x_1} + \lambda x_1] \star [(1-\lambda)\overline{y_1} + \lambda y_1])) \leq \lambda g(\overline{x_1} \star \overline{y_1})$. Divided by λ and it tends to zero, we have the following inequalities:

$$\begin{array}{ll} f(x_1 \star y_1, Y(\overline{x_1} \star \overline{y_1})) & \leq g(\overline{x_1} \star \overline{y_1}) \\ & = f(\overline{x_1} \star \overline{y_1}, Y(\overline{x_1} \star \overline{y_1})) \\ & \leq f(\overline{x_1} \star \overline{y_1}, x_2 \star y_2), \end{array}$$

for all $x_1,\ y_1 \in X_1$ and $x_2,\ y_2 \in X_2$. In general we consider the following perturbed function $f_\epsilon(x_1 \star y_1, x_2 \star y_2) = f(x_1 \star y_1, x_2 \star y_2) + \epsilon h(x_2 \star y_2)$ where $h(x_2 \star y_2)$ is a continuous and strictly convex function on X_2. Clearly $f_\epsilon(x_1 \star y_1, x_2 \star y_2)$ is a continuous, concave with respect to $x_1 \star y_1$ and strictly convex with respect to $x_2 \star y_2$ function. Using previous discussion, we have

$$f_\epsilon(x_1 \star y_1, x_2^\epsilon \star y_2^\epsilon) \leq f_\epsilon(x_1^\epsilon \star y_1^\epsilon, x_2^\epsilon \star y_2^\epsilon) \leq f_\epsilon(x_1^\epsilon \star y_1^\epsilon, x_2 \star y_2), \tag{2.7}$$

for all $x_1,\ y_1 \in X_1$ and $x_2,\ y_2 \in X_2$. Let $\epsilon = \epsilon_k$ in the inequalities (7) and $\epsilon_k \to 0^+$. Because of compactness of X_1 and X_2, we conclude that $x_1^\epsilon \star y_1^\epsilon \to \overline{x_1} \star \overline{y_1}$ and $x_2^\epsilon \star y_2^\epsilon \to \overline{x_2} \star \overline{y_2}$. Therefore, $(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2})$ is a saddle point of the function $f(x_1 \star y_1, x_2 \star y_2)$. □

3. Applications

In this section, we consider some examples in game theory and explain our theory in the previous section.

3.1. **Examples.** The following examples show that, how we can use Nash equilibrium point in practice. First of all, we show that the Nash equilibrium point gives a generalization of usual optimization problem.

Example 3.1. *Since the optimization problem is a special case of the game theory with one player, so we obtain the optimization model as follows:*

$$f_1(\overline{x_1} \star \overline{y_1}) \leqslant f_1(x_1 \star y_1),$$

for all $x_1 \star y_1 \in P^*(X_1)$*. This means that* $\overline{x_1} \star \overline{y_1} \in \mathrm{Argmin}\{f(x_1 \star y_1) : \ x_1 \star y_1 \in P^*(X_1)\}$.

Example 3.2. *Let* $H = \{f \in C | f : \mathbb{R} \to \mathbb{R}\}$ *and* $X_1, \ X_2 \subseteq H$*, where* X_1 *and* X_2 *are such continuous functions that have left and right inverse. We define the following function.*

$$f_1(g, h) = \int_{D_g \bigcap D_h} [(g - h)(x)]^2 dx,$$

where g *and* h *are the left and right inverse of an arbitrary function and* D_g *and* D_h *are the domains of* g *and* h*, respectively. Let* $f_2(g, h) = -f_1(g, h)$*. Clearly* $(g, g) = (f, f)$ *is a saddle point, i.e., the left and right inverse are the same.*

3.2. **Numerical Examples.**

Example 3.3. *In Example 3, we define* $X_1 = \{(x, 0) : 0 \leq x \leq M\}$*,* $X_2 = \{(0, y) : 0 \leq y \leq N\}$*. Clearly,* $X_1, \ X_2$ *are compact and convex subsets of* H*. According to Example 3, the function* $h(x_1 \star y_1) = z_{max} - z_{min} = max\{x_1^1, x_1^2\}$ *and* $e(x_2 \star y_2) = z_{max} - z_{min} = max\{y_2^1, y_2^2\}$*, where* $x_1 = (x_1^1, 0)$*,* $y_1 = (x_1^2, 0)$*,* $x_2 = (0, y_2^1)$*,* $y_2 = (0, y_2^2)$*,* $x_1, \ y_1 \in X_1$*,* $x_2, \ y_2 \in X_2$ *are convex functions. Let*

$$f(x_1 \star y_1, \ x_2 \star y_2) = e(x_2 \star y_2) - h(x_1 \star y_1).$$

We can show that the function $f(x_1 \star y_1, x_2 \star y_2)$ *is a continuous, concave function with respect to* $x_1 \star y_1) \in P^*(X_1)$ *and convex function with respect to* $x_2 \star y_2) \in P^*(X_2)$*. Therefore, the all assumptions of Theorem (2.11) has been held. Clearly,* $\{(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) : \ max\{x_1^1, x_1^2\} = max\{y_2^1, y_2^2\}\}$ *is the set of saddle points of the function* $f(x_1 \star y_1, x_2 \star y_2)$ *on* $X_1 \times X_2$.

Example 3.4. *Let define* $H = \mathbb{R}$ *and* $X_1 = X_2 = H$*,* $x \star y = [min\{x, y\}, max\{x, y\}]$*,*

$$f(x_1 \star y_1, x_2 \star y_2) = (x_2^2 + y_2^2) - (x_1^2 + y_1^2),$$

for all $x_1, \ y_1 \in X_1$ *and* $x_2, \ y_2 \in X_2$*. In this example,* $X_1, \ X_2$ *are not compact subset of* H*. We choose* $\overline{x_1} = \overline{y_1} = \overline{x_2} = \overline{y_2} = 0$*. Then,* $\overline{x_1} \star \overline{y_1} = \{0\}$*,* $\overline{x_2} \star \overline{y_2} = \{0\}$ *and* $f(\{0\}, \{0\}) = 0$*. Therefore,* $(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) = (\{0\}, \{0\})$ *is a saddle point, that is*

$$f(x_1 \star y_1, \overline{x_2} \star \overline{y_2}) = -(x_1^2 + y_1^2) \leq f(\overline{x_1} \star \overline{y_1}, \overline{x_2} \star \overline{y_2}) = 0 \leq f(\overline{x_1} \star \overline{y_1}, x_2 \star y_2) = (x_2^2 + y_2^2),$$

for all $x_1, \ y_1 \in X_1$ *and* $x_2, \ y_2 \in X_2$*. Thus, Theorem (2.11) is a sufficient condition.*

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