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EXISTENCE OF SOLUTIONS FOR A CERTAIN BOUNDARY VALUE PROBLEM ASSOCIATED TO A FOURTH ORDER DIFFERENTIAL INCLUSION

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Abstract. Existence of solutions for a fourth order differential inclusion with cantilever boundary conditions is investigated. New results are obtained when the right hand side has convex or non convex values.

1. Introduction

Fourth order differential equations are often used in engineering and physical problems. Boundary value problems associated to fourth order differential equations appear in elasticity theory describing stationary states of the deflection of an elastic beam. The same equation can describe the "effect of the shear" when investigating transverse vibrations. As a consequence there was an intensive development of the study of such problems. In the single valued case several results concerning existence, localization and multiplicity of solutions may be found in [3], [4], [5], [8], [10], [12] etc..

This paper is devoted to the following boundary value problem

$$x^{(4)} \in F(t,x), \quad a.e. \ ([0,1]), \quad x(0) = x'(0) = x''(1) = x'''(1) = 0, \tag{1.1}$$

where $F(.,.) : [0,1] \times \mathbf{R}^n \to \mathcal{P}(\mathbf{R}^n)$ is a set-valued map.

The aim of our paper is to consider the more general framework of set-valued problems and to present three existence results for problem 1.1. Our results are obtained under several hypotheses concerning the regularity of the set-valued map F and are based on a nonlinear alternative of Leray-Schauder type, on Bressan-Colombo selection theorem for lower semicontinuous set-valued maps with decomposable values and on Kuratowski and Ryll-Nardzewski selection theorem. We mention that the methods used are rather known in the theory of differential inclusions, however their exposition in the framework of problem 1.1 is new.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel, in Section 3 we prove our results using fixed point techniques and in Section 4 we provide a Filippov type existence result.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X,d) be a metric space with the corresponding norm $|.|$ and let $I \subset \mathbf{R}$ be a compact interval. Denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(.) : I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by $\overline{A}$ the closure of A.

Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}$, $d^*(A,B) = \sup\{d(a,B); a \in A\}$, where $d(x,B) = \inf_{y \in B} d(x,y)$.

As usual, we denote by $C(I,X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I,X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)| \mathrm{d}t$.

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A subset $D \subset L^1(I,X)$ is said to be *decomposable* if for any $u(\cdot), v(\cdot) \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

Consider $T : X \to \mathcal{P}(X)$ a set-valued map. A point $x \in X$ is called a fixed point for $T(.)$ if $x \in T(x)$. $T(.)$ is said to be bounded on bounded sets if $T(B) := \cup_{x\in B} T(x)$ is a bounded subset of X for all bounded sets B in X. $T(.)$ is said to be compact if $T(B)$ is relatively compact for any bounded sets B in X. $T(.)$ is said to be totally compact if $\overline{T(X)}$ is a compact subset of X. $T(.)$ is said to be upper semicontinuous if for any $x_0 \in X$, $T(x_0)$ is a nonempty closed subset of X and if for each open set D of X containing $T(x_0)$ there exists an open neighborhood V_0 of x_0 such that $T(V_0) \subset D$. Let E a Banach space, $Y \subset E$ a nonempty closed subset and $T(.) : Y \to \mathcal{P}(E)$ a multifunction with nonempty closed values. $T(.)$ is said to be lower semicontinuous if for any open subset $D \subset E$, the set $\{y \in Y; T(y) \cap D \neq \emptyset\}$ is open. $T(.)$ is called completely continuous if it is upper semicontinuous and totally compact on X.

It is well known that a compact set-valued map $T(.)$ with nonempty compact values is upper semicontinuous if and only if $T(.)$ has a closed graph.

We recall the following nonlinear alternative of Leray-Schauder type proved in [11] and its consequences.

Theorem 2.1. *Let D and $\overline{D}$ be the open and closed subsets in a normed linear space X such that $0 \in D$ and let $T : \overline{D} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in \partial D$ (the boundary of D) such that $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.1. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to \mathcal{P}(X)$ be a completely continuous set-valued map with compact convex values. Then either*

i) the inclusion $x \in T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$.

Corollary 2.2. *Let $B_r(0)$ and $\overline{B_r(0)}$ be the open and closed balls in a normed linear space X centered at the origin and of radius r and let $T : \overline{B_r(0)} \to X$ be a completely continuous single valued map with compact convex values. Then either*

i) the equation $x = T(x)$ has a solution, or

ii) there exists $x \in X$ with $|x| = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

If $F(.,.) : I \times X \to \mathcal{P}(X)$ is a set-valued map with compact values we define $S_F : C(I,X) \to \mathcal{P}(L^1(I,X))$ by

$$S_F(x) := \{f \in L^1(I,X); \quad f(t) \in F(t, x(t)) \quad a.e.\ (I)\}.$$

We say that $F(.,.)$ is of *lower semicontinuous type* if $S_F(.)$ is lower semicontinuous with nonempty closed and decomposable values. The next result is proved in [2].

Theorem 2.2. *Let S be a separable metric space and $G(.) : S \to \mathcal{P}(L^1(I,X))$ be a lower semicontinuous set-valued map with closed decomposable values.*

Then $G(.)$ has a continuous selection (i.e., there exists a continuous mapping $g(.) : S \to L^1(I,X)$ such that $g(s) \in G(s) \quad \forall s \in S$).

A set-valued map $G : I \to \mathcal{P}(X)$ with nonempty compact convex values is said to be *measurable* if for any $x \in X$ the function $t \to d(x, G(t))$ is measurable. A set-valued map $F(.,.) : I \times X \to \mathcal{P}(X)$ is said to be *Carathéodory* if $t \to F(t,x)$ is measurable for any $x \in X$ and $x \to F(t,x)$ is upper semicontinuous for almost all $t \in I$. Moreover, $F(.,.)$ is said to be L^1-*Carathéodory* if for any $l > 0$ there exists $h_l(.) \in L^1(I, \mathbf{R})$ such that $\sup\{|v|; v \in F(t,x)\} \leq h_l(t)$ a.e. (I), $\forall x \in \overline{B_l(0)}$.

Theorem 2.3. *Let X be a Banach space, let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a L^1-Carathéodory set-valued map with $S_F(x) \neq \emptyset$ for all $x(.) \in C(I,X)$ and let $\Gamma : L^1(I,X) \to C(I,X)$ be a linear continuous mapping.*

Then the set-valued map $\Gamma \circ S_F : C(I,X) \to \mathcal{P}(C(I,X))$ defined by

$$(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$$

has compact convex values and has a closed graph in $C(I,X)\times C(I,X)$.

The proof of theorem above may be found in [9], Note that if $dimX<\infty$, and $F(.,.)$ is as in Theorem 2.5, then $S_F(x)\neq\emptyset$ for any $x(.)\in C(I,X)$ (e.g., [9]).

We recall also a selection result ([1]) which is a version of the celebrated Kuratowski and Ryll-Nardzewski selection theorem.

Lemma 2.1. *Consider X a separable Banach space, B is the closed unit ball in X, $H:I\to\mathcal{P}(X)$ is a set-valued map with nonempty closed values and $g:I\to X, L:I\to\mathbf{R}_+$ are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset \quad a.e.(I),$$

then the set-valued map $t\to H(t)\cap(g(t)+L(t)B)$ has a measurable selection.

In what follows $I=[0,1]$, and $V=\{x\in W^{4,1}(I,\mathbf{R}); x(0)=x'(0)=x''(1)=x'''(1)=0\}$ with the norm $||x||_V=||x^{(4)}||_1$. By a solution of problem (1.1) we mean a function $x(.)\in V$ for which there exists a function $f(.)\in L^1(I,\mathbf{R})$ with $f(t)\in F(t,x(t))$, *a.e.* (I) such that $x^{(4)}(t)=f(t)$ *a.e.* (I).

The next technical result is proved in [3].

Lemma 2.2. *If $f(.):[0,1]\to\mathbf{R}$ is an integrable function, then the solution of the boundary value problem*

$$x^{(4)}=f(t),\quad a.e.\ ([0,1]),\quad x(0)=x'(0)=x''(1)=x'''(1)=0$$

is given by

$$x(t)=\int_0^1 G(t,s)f(s)ds,$$

where

$$G(t,s):=\begin{cases}\frac{s^2}{6}(3t-s), & if\quad 0\le s<t\le 1,\\ \frac{t^2}{6}(3s-t), & if\quad 0\le t<s\le 1\end{cases}$$

Obviously, $|G(t,s)|\le\frac{1}{2}\ \forall t.s\in I$.

3. Existence via fixed points

We are able now to present the two existence results for problem (1.1) using fixed point techniques. We consider first the case when $F(.,.)$ is convex valued.

Hypothesis H1. i) $F(.,.):I\times\mathbf{R}\to\mathcal{P}(\mathbf{R})$ has nonempty compact convex values and is Carathéodory.

ii) There exist $\varphi(.)\in L^1(I,\mathbf{R})$ with $\varphi(t)>0$ a.e. (I) and there exists a nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ such that

$$\sup\{|v|;\quad v\in F(t,x)\}\le\varphi(t)\psi(|x|)\quad a.e.\ (I),\quad \forall x\in\mathbf{R}.$$

Theorem 3.1. *Assume that Hypothesis H1 is satisfied and there exists $r>0$ such that*

$$r>\frac{1}{2}|\varphi|_1\psi(r). \tag{3.1}$$

Then problem 1.1 has at least one solution $x(.)$ such that $|x(.)|_C<r$.

Proof. Let $X=W^{4,1}(I,\mathbf{R})$ and consider $r>0$ as in 3.1. It is obvious that the existence of solutions to problem 1.1 reduces to the existence of the solutions of the integral inclusion

$$x(t)\in\int_0^1 G(t,s)F(s,x(s))ds,\quad t\in I. \tag{3.2}$$

Consider the set-valued map $T:\overline{B_r(0)}\to\mathcal{P}(W^{4,1}(I,\mathbf{R}))$ defined by

$$T(x):=\{v(.)\in W^{4,1}(I,\mathbf{R});\ v(t):=\int_0^1 G(t,s)f(s)ds,\ f\in\overline{S_F(x)}\}. \tag{3.3}$$

We show that $T(.)$ satisfies the hypotheses of Corollary 2.1. First, we show that $T(x)\subset W^{4,1}(I,\mathbf{R})$ is convex for any $x\in W^{4,1}(I,\mathbf{R})$.

If $v_i\in T(x)$ then there exist $f_i\in S_F(x)$ such that for any $t\in I$ one has $v_i(t)=\int_0^1 G(t,s)f_i(s)ds$, $i=1,2$.

Let $0 \le \alpha \le 1$. Then for any $t \in I$ we have

$$(\alpha v_1 + (1-\alpha)v_2)(t) = \int_0^1 G(t,s)[\alpha f_1(s) + (1-\alpha)f_2(s)]ds.$$

The values of $F(.,.)$ are convex, thus $S_F(x)$ is a convex set and hence $\alpha f_1 + (1-\alpha)f_2 \in T(x)$.

Secondly, we show that $T(.)$ is bounded on bounded sets of $W^{4,1}(I,\mathbf{R})$.

Let $B \subset W^{4,1}(I,\mathbf{R})$ be a bounded set. Then there exist $m > 0$ such that $|x|_C \le m\ \forall x \in B$. If $v \in T(x)$ there exists $f \in S_F(x)$ such that $v(t) = \int_0^1 G(t,s)f(s)ds$. One may write for any $t \in I$

$$|v(t)| \le \int_0^1 |G(t,s)|.|f(s)|ds \le \int_0^1 |G(t,s)|\varphi(s)\psi(|x(t)|)ds$$

and therefore $|v|_C \le \frac{1}{2}|\varphi|_1\psi(m) \quad \forall v \in T(x)$, i.e., $T(B)$ is bounded.

We show next that $T(.)$ maps bounded sets into equi-continuous sets.

Let $B \subset W^{4,1}(I,\mathbf{R})$ be a bounded set as before and $v \in T(x)$ for some $x \in B$. There exists $f \in S_F(x)$ such that $v(t) = \int_0^1 G(t,s)f(s)ds$. Then for any $t, \tau \in I$ we have

$$|v(t) - v(\tau)| \le |\int_0^1 G(t,s)f(s)ds - \int_0^1 G(\tau,s)f(s)ds| \le$$

$$\int_0^1 |G(t,s) - G(\tau,s)|.|f(s)|ds \le \int_0^1 |G(t,s) - G(\tau,s)|\varphi(s)\psi(m)ds.$$

It follows that $|v(t) - v(\tau)| \to 0$ as $t \to \tau$. Therefore, $T(B)$ is an equi-continuous set in $W^{4,1}(I,\mathbf{R})$.

We apply now Arzela-Ascoli's theorem we deduce that $T(.)$ is completely continuous on $W^{4,1}(I,\mathbf{R})$.

In the next step of the proof we prove that $T(.)$ has a closed graph.

Let $x_n \in W^{4,1}(I,\mathbf{R})$ be a sequence such that $x_n \to x^*$ and $v_n \in T(x_n)\ \forall n \in \mathbf{N}$ such that $v_n \to v^*$. We prove that $v^* \in T(x^*)$. Since $v_n \in T(x_n)$, there exists $f_n \in S_F(x_n)$ such that $v_n(t) = \int_0^1 G(t,s)f_n(s)ds$. Define $\Gamma : L^1(I,\mathbf{R}) \to W^{4,1}(I,\mathbf{R})$ by $(\Gamma(f))(t) := \int_0^1 G(t,s)f(s)ds$. One has $\max_{t\in I}|v_n(t) - v^*(t)| = |v_n(.) - v^*(.)|_C \to 0$ as $n \to \infty$

We apply Theorem 2.3 to find that $\Gamma \circ S_F$ has closed graph and from the definition of Γ we get $v_n \in \Gamma \circ S_F(x_n)$. Since $x_n \to x^*$, $v_n \to v^*$ it follows the existence of $f^* \in S_F(x^*)$ such that $v^*(t) = \int_0^1 G(t,s)f^*(s)ds$.

Therefore, $T(.)$ is upper semicontinuous and compact on $\overline{B_r(0)}$. We apply Corollary 2.1 to deduce that either i) the inclusion $x \in T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $\lambda x \in T(x)$ for some $\lambda > 1$. Assume that ii) is true. With the same arguments as in the second step of our proof we get $r = |x(.)|_C \le \frac{1}{2}|\varphi|_1\psi(r)$ which contradicts 3.1. Hence only i) is valid and theorem is proved. □

We consider now the case when $F(.,.)$ is not necessarily convex valued. Our existence result in this case is based on the Leray-Schauder alternative for single valued maps and on Bressan Colombo selection theorem.

Hypothesis H2. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has compact values, $F(.,.)$ is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable and $x \to F(t,x)$ is lower semicontinuous for almost all $t \in I$.

ii) There exist $\varphi(.) \in L^1(I,\mathbf{R})$ with $\varphi(t) > 0$ a.e. (I) and there exists a nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ such that

$$\sup\{|v|; \quad v \in F(t,x)\} \le \varphi(t)\psi(|x|) \quad a.e.\ (I), \quad \forall x \in \mathbf{R}.$$

Theorem 3.2. *Assume that Hypothesis H2 is satisfied and there exists $r > 0$ such that condition 3.1 is satisfied.*

Then problem 1.1 has at least one solution on I.

Proof. We note first that if Hypothesis H2 is satisfied then $F(.,.)$ is of lower semicontinuous type (e.g., [7]). Therefore, we apply Theorem 2.2 with $S = W^{4,1}(I,\mathbf{R})$ and $G(.) = S_F(.)$ to deduce that there exists a continuous mapping $f(.): W^{4,1}(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that $f(x) \in S_F(x)$ $\forall x \in W^{4,1}(I,\mathbf{R})$. We consider the corresponding problem

$$x(t) = \int_0^1 G(t,s)f(x(s))ds, \quad t \in I \tag{3.4}$$

in the space $X = W^{4,1}(I,\mathbf{R})$. It is clear that if $x(.) \in W^{4,1}(I,\mathbf{R})$ is a solution of the problem (3.4) then $x(.)$ is a solution to problem 1.1.

Let $r > 0$ that satisfies condition 3.1 and define the set-valued map $T : \overline{B_r(0)} \to \mathcal{P}(W^{4,1}(I,\mathbf{R}))$ by $(T(x))(t) := \int_0^1 G(t,s)f(x(s))ds$.

Obviously, the integral equation 3.4 is equivalent with the operator equation

$$x(t) = (T(x))(t), \quad t \in I.$$

It remains to show that $T(.)$ satisfies the hypotheses of Corollary 2.2.

We show that $T(.)$ is continuous on $\overline{B_r(0)}$. From Hypotheses H2 ii) we have

$$|f(x(t))| \le \varphi(t)\psi(|x(t)|) \quad a.e.\ (I)$$

for all $x(.) \in W^{4,1}(I,\mathbf{R})$. Let $x_n, x \in \overline{B_r(0)}$ such that $x_n \to x$. Then

$$|f(x_n(t))| \le \varphi(t)\psi(r) \quad a.e.\ (I).$$

From Lebesgue's dominated convergence theorem and the continuity of $f(.)$ we obtain, for all $t \in I$

$$\lim_{n\to\infty}(T(x_n))(t) = \int_0^1 G(t,s)f(x_n(s))ds = \int_0^1 G(t,s)f(x(s))ds = (T(x))(t)$$

i.e., $T(.)$ is continuous on $\overline{B_r(0)}$.

Repeating the arguments in the proof of Theorem 3.1 with corresponding modifications it follows that $T(.)$ is compact on $\overline{B_r(0)}$. We apply Corollary 2.2 and we find that either i) the equation $x = T(x)$ has a solution in $\overline{B_r(0)}$, or ii) there exists $x \in X$ with $|x|_C = r$ and $x = \lambda T(x)$ for some $\lambda < 1$.

As in the proof of Theorem 3.1 if the statement ii) holds true, then we obtain a contradiction to 3.1. Thus only the statement i) is true and problem 1.1 has a solution $x(.) \in W^{4,1}(I,\mathbf{R})$ with $|x(.)|_C < r$ □

4. A Filippov type existence result

In this section we consider the, even, more general problem

$$x^{(4)} \in F(t,x,V(x)(t)), \quad a.e.\ ([0,1]), \quad x(0) = x'(0) = x''(1) = x'''(1) = 0, \tag{4.1}$$

where $F : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map, $V : C([0,1],\mathbf{R}) \to C([0,1],\mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t,s,x(s))ds$ with $k(.,.,.) : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function. We show that Filippov's ideas ([6]) can be suitably adapted in order to obtain the existence of solutions for problem 4.1.

In order to prove our results we need the following hypotheses.

Hypothesis H3. i) *$F(.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I,(0,\infty))$ such that, for almost all $t \in I, F(t,.,.)$ is $L(t)$-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t,x_1,y_1), F(t,x_2,y_2)) \le L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall\ x_1,x_2,y_1,y_2 \in \mathbf{R}.$$

iii) *$k(.,.,.) : I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a function such that $\forall x \in \mathbf{R}$, $(t,s) \to k(t,s,x)$ is measurable.*

iv) *$|k(t,s,x) - k(t,s,y)| \le L(t)|x-y| \quad a.e.\ (t,s) \in I \times I, \quad \forall\, x,y \in \mathbf{R}$.*

We use next the following notations

$$M(t) := L(t)(1 + \int_0^t L(u)du), \quad t \in I, \quad M_0 = \int_0^1 M(t)dt.$$

Theorem 4.1. *Assume that Hypothesis H3 is satisfied and $M_0 < 2$. Let $y(.) \in C(I, \mathbf{R})$ be such that $y(0) = y'(0) = y''(1) = y'''(1) = 0$ and there exists $p(.) \in L^1(I, \mathbf{R}_+)$ with $d(y^{(4)}(t), F(t, y(t), V(y)(t))) \le p(t)$ a.e. (I).*

Then there exists $x(.)$ a solution of problem 4.1 satisfying for all $t \in I$

$$|x(t) - y(t)| \le \frac{1}{2 - M_0} \int_0^1 p(t)dt.$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and $F(t, y(t), V(y)(t)) \cap \{y^{(4)}(t) + p(t)[-1, 1]\} \neq \emptyset$ a.e. (I).

It follows from Lemma 2.1 that there exists a measurable selection $f_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|f_1(t) - y^{(4)}(t)| \le p(t) \quad a.e.\ (I) \tag{4.2}$$

Define $x_1(t) = \int_0^1 G(t, s)f_1(s)ds$ and one has $|x_1(t) - y(t)| \le \frac{1}{2}\int_0^1 p(t)dt$.

We claim that it is enough to construct the sequences $x_n(.) \in C(I, \mathbf{R})$, $f_n(.) \in L^1(I, \mathbf{R})$, $n \ge 1$ with the following properties

$$x_n(t) = \int_0^1 G(t, s)f_n(s)ds, \quad t \in I, \tag{4.3}$$

$$f_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\ (I), \tag{4.4}$$

$$|f_{n+1}(t) - f_n(t)| \le L(t)(|x_n(t) - x_{n-1}(t)| + \int_0^t L(s)|x_n(s) - x_{n-1}(s)|ds) \quad a.e.\ (I) \tag{4.5}$$

If this construction is realized then from 4.2-4.5 we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \le \frac{1}{2}(\frac{M_0}{2})^n \int_0^1 p(t)dt \quad \forall n \in \mathbf{N}.$$

Indeed, assume that the last inequality is true for $n - 1$ and we prove it for n. One has

$$|x_{n+1}(t) - x_n(t)| \le \int_0^1 |G(t, t_1)|.|f_{n+1}(t_1) - f_n(t_1)|dt_1 \le$$

$$\frac{1}{2}\int_0^1 L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \int_0^{t_1} L(s)|x_n(s) - x_{n-1}(s)|ds]dt_1 \le \frac{1}{2}$$

$$\int_0^1 L(t_1)(1 + \int_0^{t_1} L(s)ds)dt_1.(\frac{1}{2})^n M_0^{n-1} \int_0^1 p(t)dt = \frac{1}{2}(\frac{M_0}{2})^n \int_0^1 p(t)dt$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence converging uniformly to some $x(.) \in C(I, \mathbf{R})$. Therefore, by 4.5, for almost all $t \in I$, the sequence $\{f_n(t)\}$ is Cauchy in $\mathbf{R}$. Let $f(.)$ be the pointwise limit of $f_n(.)$. Moreover, one has

$$\begin{aligned} &|x_n(t) - y(t)| \le |x_1(t) - y(t)| + \textstyle\sum_{i=1}^{n-1} |x_{i+1}(t) - x_i(t)| \le \\ &\tfrac{1}{2}\textstyle\int_0^1 p(t)dt + \sum_{i=1}^{n-1}(\frac{1}{2}\int_0^1 p(t)dt)(\frac{M_0}{2})^i = \frac{\frac{1}{2}\int_0^1 p(t)dt}{1 - \frac{M_0}{2}}. \end{aligned} \tag{4.6}$$

On the other hand, from 4.2, 4.5 and 4.6 we obtain for almost all $t \in I$

$$|f_n(t) - y^{(4)}(t)| \le \sum_{i=1}^{n-1} |f_{i+1}(t) - f_i(t)| + |f_1(t) - D_C^q y(t)| \le L(t)\frac{\int_0^1 p(t)dt}{2 - M_0} + p(t).$$

Hence the sequence $f_n(.)$ is integrably bounded and therefore $f(.) \in L^1(I, \mathbf{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in 4.2, 4.4 we deduce that $x(.)$ is a solution of 1.1. Finally, passing to the limit in 4.6 we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), f_n(.)$ with the properties in 4.2-4.5. The construction will be done by induction.

Since the first step is already realized, assume that for some $N \ge 1$ we already constructed $x_n(.) \in C(I, \mathbf{R})$ and $f_n(.) \in L^1(I, \mathbf{R})$, $n = 1, 2, ...N$ satisfying 4.2, 4.5 for $n = 1, 2, ...N$ and 4.4 for $n = 1, 2, ...N - 1$. The set-valued map $t \to F(t, x_N(t), V(x_N)(t))$ is measurable. Moreover, the map

$t \to L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t \in I$

$$F(t, x_N(t)) \cap \{f_N(t) + L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds)[-1,1]\} \neq \emptyset.$$

Lemma 2.1 yields that there exist a measurable selection $f_{N+1}(.)$ of $F(., x_N(.), V(x_N)(.))$ such that for almost all $t \in I$

$$|f_{N+1}(t) - f_N(t)| \leq L(t)(|x_N(t) - x_{N-1}(t)| + \int_0^t L(s)|x_N(s) - x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in 4.2 with $n = N + 1$. Thus $f_{N+1}(.)$ satisfies 4.4 and 4.5 and the proof is complete. □

The assumptions in Theorem 4.1 are satisfied, in particular, for $y(.) = 0$ and therefore with $p(.) = L(.)$. We obtain the following consequence of Theorem 4.1.

Corollary 4.1. *Assume that Hypothesis H3 is satisfied,* $M_0 < 2$ *and* $d(0, F(t, 0, V(0)(t)) \leq L(t)$ *a.e.* (I)*. Then there exists* $x(.)$ *a solution of problem 4.1 satisfying for all* $t \in I$*,* $|x(t)| \leq \frac{1}{2-M_0}\int_0^1 L(t)dt$*.*

If F does not depend on the last variable, Hypothesis H3 becames

Hypothesis H4. i) $F(.,.) : I \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ has nonempty closed values and is $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R})$ measurable.

ii) There exists $L(.) \in L^1(I, (0, \infty))$ such that, for almost all $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t, x_1), F(t, x_2)) \leq L(t)|x_1 - x_2| \quad \forall\, x_1, x_2 \in \mathbf{R}.$$

Denote $L_0 = \int_0^1 L(t)dt$.

Corollary 4.2. *Assume that Hypothesis H4 is satisfied,* $M_0 < 2$ *and* $d(0, F(t, 0) \leq L(t)$ *a.e.* (I)*. Then there exists* $x(.)$ *a solution of problem 1.1 satisfying for all* $t \in I$

$$|x(t)| \leq \frac{L_0}{2 - L_0}.$$

References

[1] J.P. Aubin and H. Frankowska, Set-valued Analysis, Birkhäuser, Basel, 1990.

[2] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, Studia Math. 90 (1988), 69-86.

[3] A. Cabada, R. Precup, L. Saavedra and S.A. Tersian, Multiple positive solutions to a fourth-order boundary value problem, Electronic J. Differ. Equ. 2016 (2016), Art. 254.

[4] J.A. Cid, D. Franco and F. Minhos, Positive fixed points and fourth-order equations, Bull. Lond. Math. Soc. 41 (2009), 72-78.

[5] R. Enguica and L. Sanchez, Existence and localization of solutions for fourth-order boundary value problems, Electron. J. Differ. Equ. 2007 (2007), Art. 127.

[6] A.F. Filippov, Classical solutions of differential equations with multivalued right hand side, SIAM J. Control 5 (1967), 609-621.

[7] M. Frignon and A. Granas, Théorèmes d'existence pour les inclusions différentielles sans convexité, C. R. Acad. Sci. Paris, Ser. I 310 (1990), 819-822.

[8] J.R. Graef, L. Kong, Q. Kong and B. Yang, Positive solutions to a fourth order boundary value problem, Results Math. 59 (2011), 141-155.

[9] A. Lasota and Z. Opial, An application of the Kakutani-Ky-Fan theorem in the theory of ordinary differential equations, Bull. Acad. Polon. Sci. Math., Astronom. Physiques 13 (1965), 781-786.

[10] F. Li, Q. Zhang and Z. Liang, Existence and multiplicity of solutions of a kind of fourth-order boundary value problem, Nonlinear Anal. 62 (2005), 803-816.

[11] D. O' Regan, Fixed point theory for closed multifunctions, Arch. Math. (Brno) 34 (1998), 191-197.

[12] L. Yang, H. Chen and X. Yang, The multiplicity of solutions for fourth-order equations generated from a boundary condition, Appl. Math. Lett. 24 (2011), 1599-1603.

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