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GEOMETRIC CHARACTERIZATIONS OF THE DIFFERENTIAL SHIFT PLUS ALEXANDER INTEGRAL OPERATOR

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Abstract. In this effort, we deal with a new integral operator in the open unit disk. This operator is formulated by the complex Alexander operator and its derivative. Furthermore, we introduce a new subspace of the Hardy space containing the normalized analytic functions. We shall prove that the new integral operator is closed in the subspace of normalized functions. Geometric characterizations are established in the sequel based on the maximality of Jack Lemma.

1. Introduction

The study of operators concerns with the many intersecting classes of functions on function spaces, imposed by functional operators (integral and differential). These operators can be formulated by the kinds of the operators directly or the usage of some process. The importance in this direction is to study the boundedness and the compactness of these operators. These formulations brought two sets of operators: linear operators and nonlinear operators. The information itself delivers the topological or geometrical characterizations of the spaces of functions. The data for this class of operators shows a dynamic role in mathematics, computer science and physics. To establish an operator utilizing the functional theory, and then study its characterizations, is one of the important goals of present studies in the geometric function theory and its connected areas. The aim of the current work is to impose a new operator in the open unit disk based on the complex Alexander operator.

In [1] (for recent work [2]), Alexander introduced a first order integral operator

$$A[f](z) = \int_0^z \frac{f(\xi)}{\xi} d\xi, \quad f \in A(U),$$

where $A(U)$ is the set of all normalized analytic functions in the open unit disk $U := \{z \in \mathbb{C} : |z| < 1.\}$ Note that the Alexander integral operator is the inverse of the Alexander differential operator given by the formula

$$DA[f](z) = zf'(z), \quad z \in U.$$

Based on this operator, we shall propose a modified integral operator in the open unit disk. This modification leads to define some new classes of analytic functions, specialized by the normalized class of analytic functions in the open unit disk. Our study is realized by the geometric characterizations and boundedness of the new operator.

2. Processing

Let $U := \{z : |z| < 1\}$ be the open unit disk of the complex plane and $H(U)$ be the space of holomorphic functions on the open unit disk. A holomorphic function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in U$$

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on the open unit disk belongs to the Hardy space $H^2(U)$, if its sequence of power series coefficients is square-summable:

$$H^2(U) = \{f \in H(U) : \sum_{n=0}^{\infty} |a_n|^2 < \infty\}.$$

Consequently, it can be defined a norm on $H^2(U)$ as follows [3]

$$\|f\|^2_{H^2(U)} = \sum_{n=0}^{\infty} |a_n|^2.$$

Since $L^2(U)$ is Banach space, then $H^2(U)$ is also a Banach space on U. In the sequel, we consider a subset of analytic function, which are normalized as follows: $f(0) = 0$ and $f'(0) = 1$. Thus, f is defined as follows:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U.$$

We denote this class by $A(U)$. It is clear that $A(U) \subset H(U)$ satisfying the above norm. The space H^∞ is known as the vector space of bounded holomorphic functions on U, satisfying the norm

$$\|f\|_{H^\infty} = \sup_{|z|<1} |f(z)|, \quad f \in H(U).$$

It is clear that

$$H^2(U) \subset H^\infty, \quad f \in H(U).$$

We proceed to introduce a new operator. Define the following operator $DA : A(U) \to A(U)$ as follows:

$$DA[f](z) := zf'(z), \quad z \in U, \, f \in A(U).$$

This operator is called the differentiation shift operator. Now for $f \in A(U)$, we define the differentiation shift plus complex Alexander operator as follows:

$$\begin{aligned}(\mathcal{A})\,[f](z) :&= \frac{1}{2}\Big(zf'(z) + \int_0^z \frac{f(\xi)}{\xi}\, d\xi\Big), \quad z, \xi \in U \\ &= z + \sum_{n=2}^{\infty} \alpha_n z^n, \quad f \in A(U).\end{aligned} \tag{2.1}$$

Obviously, $(\mathcal{A})\,[f] \in A(U)$. Moreover, since $DA[f]$ is a linear isometry operator and the complex Alexander operator is contraction, then the operator (2.1) is bounded in the Hardy space $H^2(U)$. Let $S^2(U)$ be the space defined by

$$S^2(U) := \Big\{f \in H(U) : f' \in H^2(U)\Big\}$$

end with the norm

$$\|f\|^2_{S^2(U)} = \|f\|^2_{H^2(U)} + \|f'\|^2_{H^2(U)}.$$

This space is subspace from H^∞, Banach algebra, and every polynomial is dense in it (see [3], Proposition 1). A direct application, we have the following proposition

Proposition 2.1. *Let $f \in S^2(U)$, then*

$$(\mathcal{A}) : S^2(U) \to S^2(U).$$

Moreover, let $S^2_0(U) := \Big\{f \in S^2(U) : f(0) = 0\Big\}$, then

$$S^2_0(U) \subset S^2(U).$$

It has been shown that the range of the Alexander operator is equal to $S^2_0(U)$. In this effort, we define a subspace $S^2_1(U)$ as follows:

$$S^2_1(U) := \Big\{ f \in S^2(U) : f(0) = 0,\ f'(0) = 1 \Big\}.$$

Then we have the following relation:

$$S^2_1(U) \subset S^2_0(U) \subset S^2(U).$$

Proposition 2.2. *Let $(\mathcal{A}) \ \in H^2(U)$. Then $rang(\mathcal{A}) \ \subset S^2_1(U)$.*

Proof. Let $g(z) \in rang(\mathcal{A})$, then there exists a normalized function $f(z) \in H^2(U)$ such that

$$g(z) = (\mathcal{A})\,[f](z) = \frac{1}{2}\Big(zf'(z) + \int_0^z \frac{f(\xi)}{\xi}\, d\,\xi \Big), \quad z, \xi \in U.$$

Then we obtain $g' \in H^2(U)$ with the properties

$$g(0) = 0, \quad g'(0) = 1.$$

Hence $g \in S^2_1(U)$.

Proposition 2.3. *Let $f, g \in A(U)$. Then*

$$\|f * g\|^2_{S^2_1(U)} \le \|f\|^2_{S^2_1(U)} \|g\|^2_{S^2_1(U)}, \quad z \in U,$$

where $*$ is represented the convolution product

$$(f * g)(z) = (z + \sum_{n=2}^{\infty} a_n z^n) * (z + \sum_{n=2}^{\infty} b_n z^n) = z + \sum_{n=2}^{\infty} a_n\, b_n z^n.$$

Proof. It is clear that $(f * g)(0) = 0$ and $(f * g)'(0) = 1$; thus $(f * g) \in S^2_1(U)$. Moreover, in view of the Young's inequality for convolutions, we have

$$\begin{aligned}
\|f * g\|^2_{S^2_1(U)} &= \|f * g\|^2_{H^2(U)} + \|(f * g)'\|^2_{H^2(U)} \\
&\le 2 \sum_{n=0}^{\infty} |a_n b_n|^2 \\
&\le 2 \sum_{n=0}^{\infty} |a_n|^2 |b_n|^2 \\
&\le \sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=0}^{\infty} |b_n|^2 \\
&= \|f\|^2_{S^2_1(U)} \|g\|^2_{S^2_1(U)}.
\end{aligned}$$

Proposition 2.4. *Let $(\mathcal{A})\,[f] \ \in H^2(U)$, $f \in A(U)$. Then*

$$rang L(f)(z) := rang\Big(2\,(\mathcal{A})\,[f]) * \ \frac{f(z)}{2} \Big) \ \subset S^2_1(U).$$

Proof. Let

$$L(f)(z) := (2\,(\mathcal{A})\,[f]) * (\frac{f(z)}{2})$$

then there exists a normalized function $f(z) \in H^2(U)$ such that

$$L(f)(z) = z + \sum_{n=2}^{\infty} \ell_n z^n \in A(U), \quad f \in A(U)$$

Since $f \in H^2(U)$ then $L(f)' \in H^2(U)$ with the properties

$$L(f)(0) = 0, \quad L(f)'(0) = 1.$$

The function $f \in A(U)$ is called starlike of order $\alpha \in [0,1)$ if and only if

$$\Re\big(\frac{zf'(z)}{f(z)}\big) > \alpha, \quad z \in U;$$

this class is denoted by $S^*(\alpha)$. And $f \in A(U)$ is called convex of order $\alpha \in [0,1)$ if and only if

$$\Re\big(1 + \frac{zf''(z)}{f'(z)}\big) > \alpha, \quad z \in U;$$

this class is denoted by $K(\alpha)$. Finally, The function $f \in A(U)$ is called bounded turning of order α if and only if $\Re\big(f'(z)\big) > \alpha$; this class is symbolled by $B(\alpha)$. Note that

$$f \in S^* \Leftrightarrow A[f] \in K.$$

We need the following result in the sequel (see [4])

Lemma 2.1. *Let $h(z)$ be analytic in U with $h(0) = 0$. Then if $|h(z)|$ approaches its maximality at a point $z_0 \in U$ when $|z| = r$, then $z_0 h'(z_0) = \epsilon\, h(z_0)$, where $\epsilon \geq 1$ is a real number.*

In addition, we request the subordination idea, which is formulated as follows: Suppose that $f(\zeta)$ and $g(\zeta)$ are analytic in the open unit disk U. Then $f(\zeta)$ is called subordinate to $g(\zeta)$ if for analytic function $\phi(\zeta)$ in U achieving $\phi(0) = 0, |\phi(\zeta)| < 1$, $(\zeta \in U)$ and $f(\zeta) = g(\phi(\zeta))$. This subordination is symbolled by

$$f(\zeta) \prec g(\zeta), \quad \zeta \in U.$$

3. Findings

In this section, we introduce sufficient conditions to study the geometric properties of the operator (2.1).

Theorem 3.1. *Let $f \in H^2(U)$. Then $(\mathcal{A})\,[f]$ is bounded on $S_1^2(U)$.*

Proof. Consider the integral operator (1) as follows:

$$(\mathcal{A})\,[f] = \frac{1}{2}\Big(zf'(z) + \int_0^z \frac{f(\xi)}{\xi}\, d\,\xi\Big), \quad z, \xi \in U,$$

we have

$$\begin{aligned}\|(\mathcal{A})\,[f]\|_{S_1^2(U)} &= \|(\mathcal{A})\,[f]\|_{H^2(U)} + \|(\mathcal{A})\,[f]'\|_{H^2(U)} \\ &\leq \|f\|_{H^2(U)} + \|f\|_{H^2(U)} + 2\max_{|z|<1} |f(z)| \\ &\leq 4\|f\|_{H^2(U)}.\end{aligned}$$

Thus the operator (1) acts from $H^2(U)$ onto $S_1^2(U)$; which is bounded.

Remark 3.1 In 1960, Biernacki showed that $f \in \mathcal{S} \Rightarrow A[f] \in \mathcal{S}$, but this brings out to be wrong (see [5], Theorem 8.11). This leads that the Alexander integral operator $A[f]$ does not cover the class $\mathcal{S}$.

Theorem 3.2. *Consider the operator (2.1). If*

$$\Re\Big(\frac{z(\mathcal{A})\,[f]''(z)}{(\mathcal{A})\,[f]'(z)}\Big) < 0, \quad z \in U,\ f \in A(U),$$

then

$$(\mathcal{A})\,[f] \in S^*.$$

Proof. Let μ be a real positive constant satisfying

$$\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)}=\frac{1+\mu\omega(z)}{1-\mu\omega(z)},\quad \omega(z)\neq\frac{1}{\mu},\ \mu>0,$$

where $\omega(z)$, $z\in U$ is a function in the open unit disk. Obviously $\omega(z)$ is analytic in U such that $\omega(0)=0$. We aim to show that $|\omega(z)|<1$ in U. Differentiating both side logarithmically, we obtain

$$1+\frac{z(\mathcal{A})\,[f]''(z)}{(\mathcal{A})\,[f]'(z)}=\frac{2\mu z\omega'(z)}{1-\mu^2\omega^2(z)}+\frac{1+\mu\omega(z)}{1-\mu\omega(z)}.$$

Thus, by the assumption we have

$$\begin{aligned}\Re\Big(1+\frac{z(\mathcal{A})\,[f]''(z)}{(\mathcal{A})\,[f]'(z)}\Big)&=\Re\Big(\frac{2\mu z\omega'(z)}{1-\mu^2\omega^2(z)}+\frac{1+\mu\omega(z)}{1-\mu\omega(z)}\Big)\\&<1,\quad z\in U,\ f\in A(U).\end{aligned}$$

If there exists a point $z_0\in U$ such that

$$\max_{|z|\leq|z_0|}|\omega(z)|=|\omega(z_0)|=1,$$

then Lemma 2.1 implies that $\omega(z_0)=e^{i\theta}$ and

$$z_0\omega'(z_0)=\epsilon\omega(z_0),\ \epsilon\geq 1.$$

Thus, we obtain

$$\begin{aligned}1+\frac{z_0(\mathcal{A})\,[f]''(z_0)}{(\mathcal{A})\,[f]'(z_0)}&=\frac{2\mu z_0\omega'(z_0)}{1-\mu^2\omega^2(z_0)}+\frac{1+\mu\omega(z_0)}{1-\mu\omega(z_0)}\\&=\frac{2\mu\epsilon e^{i\theta}}{1-\mu^2e^{2i\theta}}+\frac{1+\mu e^{i\theta}}{1-\mu e^{i\theta}}\end{aligned}$$

Since

$$\Re\Big(\frac{1}{1-\mu\,e^{i\theta}}\Big)=\frac{1}{1+\mu},$$

therefore, we conclude that

$$\begin{aligned}\Re\Big(1+\frac{z_0(\mathcal{A})\,[f]''(z_0)}{(\mathcal{A})\,[f]'(z_0)}\Big)&=\Re\Big(\frac{2\mu z_0\omega'(z_0)}{1-\mu^2\omega^2(z_0)}+\frac{1+\mu\omega(z_0)}{1-\mu\omega(z_0)}\Big)\\&=\Re\Big(\frac{2\mu\epsilon e^{i\theta}}{1-\mu^2e^{2i\theta}}+\frac{1+\mu e^{i\theta}}{1-\mu e^{i\theta}}\Big)\\&=\frac{2\mu\epsilon}{1+\mu^2}+1\\&\geq\frac{(1+\mu)^2}{1+\mu^2}\\&>1.\end{aligned}$$

Hence,

$$\Re\Big(\frac{z_0(\mathcal{A})\,[f]''(z_0)}{(\mathcal{A})\,[f]'(z_0)}\Big)>0,$$

which contradicts the assumption of the theorem. This leads that there is no $z_0\in U$ such that $|\omega(z_0)|=1$ for all $z\in U$ *i.e*

$$\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)}\prec\frac{1+\mu z}{1-\mu z},\quad z\in U,\ f\in A(U).$$

This completes the proof.

Theorem 3.3. *Consider the integral operator (2.1). If for $1<\wp<2$, such that*

$$\Re\{\frac{z(\mathcal{A})\,[f]''(z)}{(\mathcal{A})\,[f]'(z)}\}>\frac{\wp}{2},,\quad z\in U,\ f\in A(U),$$

then $(\mathcal{A})\,[f](z)\in B$.

Proof. Define a function $\psi(z)$, $z \in U$ as follows:

$$(\mathcal{A})\,[f]'(z) = (1-\psi(z))^{\wp}, \quad z \in U,$$

where, $\psi(z)$ is analytic with $\psi(0) = 0$. We need only to show that $|\psi(z)| < 1$. From the definition of ψ, we have

$$\frac{z(\mathcal{A})\,[f]''(z)}{(\mathcal{A})\,[f]'(z)} = \wp \frac{-z\psi'(z)}{1-\psi(z)}.$$

Hence, we obtain

$$\begin{aligned}\Re\{\frac{z(\mathcal{A})\,[f]''(z)}{(\mathcal{A})\,[f]'(z)}\} &= \wp\Re\{\frac{-z\psi'(z)}{1-\psi(z)}\}\\ &> \frac{\wp}{2}, \quad \wp \in (1,2).\end{aligned}$$

In view of Lemma 2.1, there exists a complex number $z_0 \in U$ such that $\psi(z_0) = e^{i\theta}$ and

$$z_0\psi'(z_0) = \epsilon\psi(z_0) = \epsilon e^{i\theta},\ \epsilon \geq 1.$$

Since

$$\Re\Big(\frac{1}{1-\psi(z_0)}\Big) = \Re\Big(\frac{1}{1-e^{i\theta}}\Big) = \frac{1}{2}$$

then, we attain

$$\begin{aligned}\Re\{\frac{z(\mathcal{A})\,[f]''(z_0)}{(\mathcal{A})\,[f]'(z_0)}\} &= \wp\Re\{\frac{-\epsilon\psi(z_0)}{1-\psi(z_0)}\}\\ &= \wp\Re\{\frac{-\epsilon e^{i\theta}}{1-e^{i\theta}}\}\\ &\leq \frac{\wp}{2}, \quad \epsilon = 1,\end{aligned}$$

which contradicts the assumption of the theorem. Hence, there is no $z_0 \in U$ with $|\psi(z_0)| = 1$, which yields that $|\psi(z)| < 1$. Moreover, we have

$$(\mathcal{A})\,[f]'(z) \prec (1-z)^{\wp},$$

which means that $\Re[(\mathcal{A})\,[f]'(z)] > 0$, equivalently, $(\mathcal{A})\,[f]'(z) \in B$. This completes the proof.

Theorem 3.4. *Consider the integral operator (2.1). If for $\wp > 1/2$, such that*

$$\Re\{\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)}\} > \frac{2\wp-1}{2\wp},$$

then

$$\frac{(\mathcal{A})\,[f](z)}{z} \prec (1-z)^{1/\wp}.$$

Proof. Define a function $w(z)$, $z \in U$ as follows:

$$\frac{(\mathcal{A})\,[f](z)}{z} = (1-w(z))^{1/\wp}, \quad z \in U,$$

where, $w(z)$ is analytic with $w(0) = 0$. We need only to show that $|w(z)| < 1$. From the definition of w, we have

$$\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)} = 1 - \frac{zw'(z)}{\wp(1-w(z))}.$$

Hence, we obtain

$$\begin{aligned}\Re\Big\{\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)}\Big\} &= \Re\Big\{1 - \frac{zw'(z)}{\wp(1-w(z))}\Big\}\\ &> \frac{2\wp-1}{2\wp}, \quad \wp > 1/2.\end{aligned}$$

In view of Lemma 2.1, there exists a complex number $z_0 \in U$ such that $w(z_0) = e^{i\theta}$ and

$$z_0 w'(z_0) = \epsilon w(z_0) = \epsilon e^{i\theta},\ \epsilon \geq 1.$$

Therefore, we arrive at

$$\begin{aligned}\Re\Big\{\frac{z_0(\mathcal{A})\,[f]'(z_0)}{(\mathcal{A})\,[f](z)}\Big\} &= \Re\Big\{1 - \frac{z_0 w'(z_0)}{\wp(1-w(z_0))}\Big\}\\ &= \Re\Big\{1 - \frac{\epsilon w(z_0)}{\wp(1-w(z_0))}\Big\}\\ &= 1 - \Re\Big\{\frac{\epsilon e^{i\theta}}{\wp(1+e^{i\theta})}\Big\}\\ &= \frac{2\wp - 1}{2\wp},\end{aligned}$$

and this is a contradiction with the assumption of the theorem. Hence, there is no $z_0 \in U$ with $|w(z_0)| = 1$, which yields that $|w(z)| < 1$. This completes the proof.

Theorem 3.5. *Let $f \in A(U)$ satisfied*

$$\Big|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big| < 1.$$

Then $(\mathcal{A})\,[f] \in B$.

Proof. Let $f \in A(U)$. Dividing (2.1) by $f(z)$, $z \in U \setminus \{0\}$ and differentiating logarithmic, we have

$$\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zDA\,[f](z)}{A\,[f](z)}.$$

Now let $\vartheta(z) = A\,[f](z) \Rightarrow \vartheta'(z) = DA\,[f](z)$. Thus, we obtain

$$\frac{zDA\,[f](z)}{A\,[f](z)} = \frac{z\vartheta'(z)}{\vartheta(z)}.$$

In view of Lemma 2.1, there exists a complex number $z_0 \in U$ such that $\vartheta(z_0) = e^{i\theta}$ and

$$z_0\vartheta'(z_0) = \epsilon\vartheta(z_0) = \epsilon e^{i\theta},\ \epsilon \geq 1.$$

Therefore, we conclude that

$$\begin{aligned}\Big|\frac{z(\mathcal{A})\,[f]'(z)}{(\mathcal{A})\,[f](z)} - 1\Big| &= \Big|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zDA\,[f](z)}{A\,[f](z)}\Big|\\ &\leq \Big|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big| + \Big|\frac{z\vartheta'(z)}{\vartheta(z)}\Big|\\ &\leq \Big|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big| + \Big|\frac{\epsilon e^{i\theta}}{e^{i\theta}}\Big|\\ &= \Big|\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\Big| + \epsilon\\ &< 1 + \epsilon =: \rho.\end{aligned}$$

Then in virtue of Theorem 5.5g P299 in [2], we obtain

$$\Big|(\mathcal{A})\,[f]'(z) - 1\Big| < 1 \Rightarrow (\mathcal{A})\,[f] \in B.$$

4. Conclusion and discussion

Here, we provided a complex integral in the open unit disk based on the Alexander operator $((\mathcal{A})\,[f])$. The new operator is achieved the differential, shift plus operator $(DA[f](z) = zf'(z),\ f \in A(U)$. Boundedness of the new integral operator is suggested in new extended space $(S_1^2(U))$. In addition, some geometric characterizations; such as univalent, starlike and bounded turning are studied. Our main tool is based on the Jack Lemma. It has proven that if $(\mathcal{A})\ : A(U) \to A(U)$. For future work, one can use the new operator to define new classes of analytic functions. Furthermore, for further investigations, one can study the subordination and superordination idea by employing the above integral. Additionally, it can be studied the connection between closed ideals of a Banach algebra together with closed invariant subspaces of the operator $DA[f]$.

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