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OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR FUNCTIONAL DYNAMIC EQUATIONS WITH DAMPING ON TIME SCALES

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Abstract. In this paper, we study oscillatory behavior of second-order dynamic equations with damping under some assumptions on time scales. New theorems extend and improve the results in the literature. Illustrative examples are given.

1. Introduction

During the past decades, the questions regarding the study of oscillatory properties of differential equations with damping or distributed deviating arguments have become an important area of research due to the fact that such equations arise in many real life problems.

In 1988, Hilger introduced the theory of time scales in his Ph.D. thesis [1] in order to unify continuous and discrete analysis; see also [4]. Preliminaries about time scale calculus can be found in [2, 3] and omitted here.

There has been much research achievement about the oscillation of dynamic equations on time scales in the last few years; see the papers [5-8, 10,11, 13-16, 18-20] and the references therein.

In [9], Chen et al. investigated the oscillation of a second-order nonlinear dynamic equation with positive and negative coefficients of the form

$$(r(t)x^{\Delta}(t))^{\Delta} + p(t)f(x(\xi(t))) - q(t)f(x(\delta(t))) = 0.$$

In [17], Şenel concerned with the oscillatory behavior of all solutions of nonlinear second order damped dynamic equation

$$(r(t)\Psi(x^{\Delta}(t)))^{\Delta} + p(t)\Psi(x^{\Delta}(t)) + q(t)f(x^{\sigma}(t)) = 0.$$

In [12], Erbe et al. studied the oscillatory behavior of the solutions of the second order nonlinear functional dynamic equation

$$(a(t)(x^{\Delta}(t))^{\gamma})^{\Delta} + \sum_{i=0}^{n} p_i(t)\Phi_{\alpha_i}(x(g_i(t))) = 0,$$

on an arbitrary time scale $\mathbb{T}$.

In this study, we are concerned with the oscillation of solutions of second order dynamic equations with damping terms of the following form

$$(r(t)g(x(t), x^{\Delta}(t)))^{\Delta} + p(t)g(x(t), x^{\Delta}(t)) + q_1(t)f_1(x(\tau_1(t))) + q_2(t)f_2(x(\tau_2(t))) = 0 \tag{1.1}$$

on a time scale $\mathbb{T}$ such that $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$.

This paper is organized as follows. In this section we give some assumptions and lemmas that we need through our work. In Section 2, we establish some new sufficient conditions for oscillation of (1.1). Finally, in Section 3, we present some examples to illustrate our results.

Now, we mention some definitions and lemmas from calculus on time scales which can be found in [2-3].

Lemma 1.1. *Assume that $g : \mathbb{T} \to \mathbb{R}$ is strictly increasing and that $\tilde{\mathbb{T}} := g(\mathbb{T}) = \{g(t) : t \in \mathbb{T}\}$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous functions, g is differentiable with rd-continuous derivative,*

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and $a, b \in \mathbb{T}$, *then*

$$\int_a^b f(t)g^{\triangle}(t)\triangle t = \int_{g(a)}^{g(b)} (f \circ g^{-1})(s)\tilde{\triangle} s,$$

where g^{-1} *is the inverse function of* g *and* $\tilde{\triangle}$ *denotes the derivative on* $\tilde{\mathbb{T}}$.

Lemma 1.2. *Every rd-continuous function has an antiderivative. In particular if* $t_0 \in \mathbb{T}$, *then* F *defined by*

$$F(t) := \int_{t_0}^t f(\tau)\triangle\tau \quad \text{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

Lemma 1.3. *Assume that* $f : \mathbb{T} \to \mathbb{R}$ *is strictly increasing and that* $\tilde{\mathbb{T}} := f(\mathbb{T}) = \{f(t) : t \in \mathbb{T}\}$ *is a time scale. Let* $g : \tilde{\mathbb{T}} \to \mathbb{R}$. *If* $f^{\triangle}(t)$ *and* $g^{\tilde{\triangle}}(f(t))$ *exist for* $t \in \mathbb{T}^{\kappa}$, *then*

$$(g \circ f)^{\triangle} = (g^{\tilde{\triangle}} \circ f)f^{\triangle},$$

where $\tilde{\triangle}$ *denotes the derivative on* $\tilde{\mathbb{T}}$.

Definition 1.1. *A function* $p : \mathbb{T} \to \mathbb{R}$ *is said to be regressive provided* $1 + \mu(t)p(t) \neq 0$ *for all* $t \in \mathbb{T}^{\kappa}$, *where* $\mu(t) = \sigma(t) - t$. *The set of all regressive rd-continuous functions* $p : \mathbb{T} \to \mathbb{R}$ *is denoted by* $\mathcal{R}$.

Let $p \in \mathcal{R}$ for all $t \in \mathbb{T}$. The exponential function on $\mathbb{T}$ is defined by

$$e_p(t, s) = \exp\left(\int_s^t \zeta_{\mu(r)}(p(r))\Delta r\right)$$

where $\zeta_{\mu(s)}$ is the cylinder transformation given by

$$\zeta_{\mu(r)}(p(r)) := \begin{cases} \frac{1}{\mu(r)} Log(1 + \mu(r)p(r)), & \text{if } \mu(r) > 0; \\ p(r), & \text{if } \mu(r) = 0. \end{cases}$$

The exponential function $y(t) = e_p(t, s)$ is the solution to the initial value problem $y^{\Delta} = p(t)y$, $y(s) = 1$. Other properties of the exponential function are given in the following lemma [3, Theorem 1.39].

Lemma 1.4. *Let* $p, q \in \mathcal{R}$. *Then*

i. $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
ii. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
iii. $\frac{1}{e_p(t,s)} = e_{\ominus}(t, s)$ *where,* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
iv. $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
v. $e_p(t, s)e_p(s, r) = e_p(t, r)$;
vi. $e_p(t, s)e_q(t, s) = e_{p\oplus q}(t, s)$;
vii. $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;
viii. $\left(\frac{1}{e_p(.,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(.,s)}$.

Throughout this paper we assume that the followings:

(C_1) $t_0 \in \mathbb{T}$ and $[t_0, \infty)_{\mathbb{T}} = \{t \in \mathbb{T} : t \geq t_0\}$,
(C_2) $r \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ and $\int_{t_0}^{\infty} \frac{1}{r(t)}\Delta t = \infty$,
(C_3) $p, q_1, q_2 \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$
(C_4) $\tau_1, \tau_2 \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\lim_{t\to\infty} \tau_1(t) = \lim_{t\to\infty} \tau_2(t) = \infty$, τ_2 has inverse function $\tau_2^{-1} \in C_{rd}(\mathbb{T}, \mathbb{T})$, $v := \tau_2^{-1} \circ \tau_1 \in C_{rd}(\mathbb{T}, \mathbb{T})$, $\tau_1^{\Delta}, v^{\Delta} \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$, $\tau_1(t), v(t) \leq t$ for $t \in [t_0, \infty)_{\mathbb{T}}$, $\tau_1([t_0, \infty)_{\mathbb{T}}) = [\tau_1(t_0), \infty)_{\mathbb{T}}$, $v([t_0, \infty)_{\mathbb{T}}) = [v(t_0), \infty)_{\mathbb{T}}$, where $\tau_1([t_0, \infty)_{\mathbb{T}}) = \{\tau_1(t) : t \in [t_0, \infty)_{\mathbb{T}}\}$ and $v([t_0, \infty)_{\mathbb{T}}) = \{v(t) : t \in [t_0, \infty)_{\mathbb{T}}\}$,
(C_5) $f_1, f_2 \in C(\mathbb{R}, \mathbb{R})$, there exist positive constants L_1, L_2, M such that $\frac{f_1(u)}{u} \geq L_1$, $0 < \frac{f_2(u)}{u} \leq L_2$ and $| f_2(u) | \leq M$ for $u \neq 0$ and $\frac{q_1(t)}{e_{-\frac{p}{r}}(\sigma(t), t_0)} L_1 - q_2(v(t))L_2 v^{\Delta}(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$,
(C_6) $g \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, there exist positive constants L_3 such that $\frac{g(u,v)}{v} \leq L_3$ and $vg(u, v) >$

0 for $v \neq 0$,

(C_7) $\displaystyle\int_t^{\infty}\left[\frac{1}{r(s)}\int_{v(s)}^{s} q_2(u)\Delta u\right]\Delta s < \infty$ for every sufficiently large $t \in \mathbb{T}$,

(C_8) $-\frac{p(t)}{r(t)}$ is positively regressive, which means $1-\mu(t)\frac{p(t)}{r(t)} > 0$.

The following lemma has an important role to prove our main results.

Lemma 1.5. *Assume that* $(C_1)-(C_7)$ *hold. Furthermore, suppose that* x *is a positive solution of* (1.1) *on* $[t_0,\infty)_{\mathbb{T}}$*, then* $\left(\frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}(t,t_0)}}\right)^{\Delta} < 0$ *and* $x^{\Delta}(t) > 0$ *on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. Easily we get

$$\begin{aligned}
\left(\frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}(t,t_0)}}\right)^{\Delta} &= \frac{(r(t)g(x(t),x^{\Delta}(t)))^{\Delta}e_{-\frac{p}{r}(t,t_0)} - (e_{-\frac{p}{r}(t,t_0)})^{\Delta}r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}(t,t_0)}e_{-\frac{p}{r}(\sigma(t),t_0)}}\\
&= \frac{(r(t)g(x(t),x^{\Delta}(t)))^{\Delta} + p(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}(\sigma(t),t_0)}}\\
&= \frac{-q_1(t)f_1(x(\tau_1(t))) - q_2(t)f_2(x(\tau_2(t)))}{e_{-\frac{p}{r}(\sigma(t),t_0)}}\\
&< 0.
\end{aligned}$$

This implies that $\frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}(t,t_0)}}$ is decreasing.

We claim that $x^{\Delta}(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$.

If not, then there is $t \geq t_1$ such that $\frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}(t,t_0)}} \leq \frac{r(t_1)g(x(t_1),x^{\Delta}(t_1))}{e_{-\frac{p}{r}(t_1,t_0)}} := a < 0$. From (C_6), we get $x^{\Delta}(t) \leq \frac{a}{L_3}\frac{e_{-\frac{p}{r}(t,t_0)}}{r(t)}$. Integrating from t_1 to t and using decreasing of $e_{-\frac{p}{r}(.,t_0)}$, we have

$$x(t) - x(t_1) \leq \frac{ae_{-\frac{p}{r}(t_1,t_0)}}{L_3}\int_{t_1}^{t}\frac{1}{r(s)}\Delta s.$$

So $x(t) \leq -\infty$. This implies that $x(t)$ is eventually negative which is a contradiction.

Hence, $x^{\Delta}(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. □

2. Main Results

In this section, we'll obtain some new oscillation criteria of second-order dynamic equation (1.1) with damping by using the generalized Riccati transformation and the inequality technique.

Theorem 2.1. *Assume that* $(C_1)-(C_8)$ *hold. Furthermore, suppose that there exists a positive function* $\alpha \in C_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ *such that for every sufficiently large* T,

$$\limsup_{t\to\infty}\int_T^t\left(\left[\frac{q_1(s)L_1}{e_{-\frac{p}{r}(\sigma(s),t_0)}} - q_2(v(s))L_2v^{\Delta}(s)\right]\alpha(s) - \frac{(\alpha_+^{\Delta}(-s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right)\Delta s = \infty, \tag{2.1}$$

where $\alpha_+^{\Delta}(s) = max\{\alpha^{\Delta}(s),0\}$*. Then every solution of* (1.1) *is oscillatory.*

Proof. Assume that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume x is an eventually positive solution of (1.1). That is, there exists $t_1 \in \mathbb{T}$ for $t \geq t_1$ and $x(t) > 0$. We defined the function z by

$$\begin{aligned}
z(t) &= \int_{t_1}^{t}\frac{g(x(s),x^{\Delta}(s))}{e_{-\frac{p}{r}(s,t_0)}}\Delta s + \int_t^{\infty}\frac{1}{r(s)}\int_{v(s)}^{s}q_2(u)f_2(x(\tau_2(u)))\Delta u\Delta s\\
&\geq \int_{t_1}^{t}\frac{g(x(s),x^{\Delta}(s))}{e_{-\frac{p}{r}(s,t_0)}}\Delta s\\
&\geq 0.
\end{aligned}$$

Thus, we get

$$\begin{aligned} z^{\Delta}(t) &= \left(\int_{t_1}^{t} \frac{g(x(s),x^{\Delta}(s))}{e_{-\frac{p}{r}}(s,t_0)}\Delta s\right)^{\Delta} + \left(\int_{t}^{\infty} \frac{1}{r(s)}\int_{v(s)}^{s} q_2(u)f_2(x(\tau_2(u)))\Delta u\Delta s\right)^{\Delta} \\ &= \frac{g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(t,t_0)} - \frac{1}{r(t)}\int_{v(t)}^{t} q_2(u)f_2(x(\tau_2(u)))\Delta u \end{aligned}$$

and

$$\begin{aligned} r(t)z^{\Delta}(t) &= \frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(t,t_0)} - \int_{v(t)}^{t} q_2(u)f_2(x(\tau_2(u)))\Delta u \\ &= \frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(t,t_0)} - \int_{v(t_1)}^{t} q_2(u)f_2(x(\tau_2(u)))\Delta u + \int_{v(t_1)}^{v(t)} q_2(u)f_2(x(\tau_2(u)))\Delta u. \end{aligned}$$

Making substitution $s = v(u)$, we have

$$\begin{aligned} \int_{t_1}^{t} q_2(v(u))f_2(x(\tau_1(u)))v^{\Delta}(u)\Delta u &= \int_{v(t_1)}^{v(t)} q_2(s)f_2(x(\tau_1(v^{-1}(s))))\tilde{\Delta} s \\ &= \int_{v(t_1)}^{v(t)} q_2(s)f_2(x(\tau_2(s)))\tilde{\Delta} s \quad \text{for} \quad t \in [t_1,\infty)_{\mathbb{T}}. \end{aligned} \tag{2.2}$$

According to condition $v([t_0,\infty)_{\mathbb{T}}) = [v(t_0),\infty)_{\mathbb{T}}$ in (C_4), we get that the derivative Δ on $\mathbb{T}$ is equal to the derivative $\tilde{\Delta}$ on $\tilde{\mathbb{T}} := v([t_0,\infty)_{\mathbb{T}})$ in (2.2). Hence, we conclude

$$\int_{t_1}^{t} q_2(v(u))f_2(x(\tau_1(u)))v^{\Delta}(u)\Delta u = \int_{v(t_1)}^{v(t)} q_2(s)f_2(x(\tau_2(s)))\Delta s \quad \text{for} \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

Thus, for $t \in [t_1,\infty)_{\mathbb{T}}$, we get

$$\begin{aligned} r(t)z^{\Delta}(t) &= \frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(t,t_0)} - \int_{v(t_1)}^{t} q_2(u)f_2(x(\tau_2(u)))\Delta u + \int_{t_1}^{t} q_2(v(u))f_2(x(\tau_1(u)))v^{\Delta}(u)\Delta u \\ (r(t)(z^{\Delta}(t))^{\Delta} &= \left(\frac{r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(t,t_0)}\right)^{\Delta} - q_2(t)f_2(x(\tau_2(t))) + q_2(v(t))f_2(x(\tau_1(t)))v^{\Delta}(t) \\ &= \frac{(r(t)g(x(t),x^{\Delta}(t)))^{\Delta}e_{-\frac{p}{r}}(t,t_0) - (e_{-\frac{p}{r}}(t,t_0))^{\Delta}r(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(t,t_0)e_{-\frac{p}{r}}(\sigma(t),t_0)} - q_2(t)f_2(x(\tau_2(t))) \\ &\quad +q_2(v(t))f_2(x(\tau_1(t)))v^{\Delta}(t) \\ &= \frac{(r(t)g(x(t),x^{\Delta}(t)))^{\Delta} + p(t)g(x(t),x^{\Delta}(t))}{e_{-\frac{p}{r}}(\sigma(t),t_0)} - q_2(t)f_2(x(\tau_2(t))) \\ &\quad +q_2(v(t))f_2(x(\tau_1(t)))v^{\Delta}(t) \\ &= \frac{-q_1(t)f_1(x(\tau_1(t))) - q_2(t)f_2(x(\tau_2(t)))}{e_{-\frac{p}{r}}(\sigma(t),t_0)} - q_2(t)f_2(x(\tau_2(t))) \\ &\quad +q_2(v(t))f_2(x(\tau_1(t)))v^{\Delta}(t) \\ &\le \frac{-q_1(t)L_1x(\tau_1(t))}{e_{-\frac{p}{r}}(\sigma(t),t_0)} + q_2(v(t))L_2x(\tau_1(t))v^{\Delta}(t) \\ &= -\left[\frac{q_1(t)L_1}{e_{-\frac{p}{r}}(\sigma(t),t_0)} - q_2(v(t))L_2v^{\Delta}(t)\right]x(\tau_1(t)) \\ &= -Q(t)x(\tau_1(t)) < 0, \end{aligned}$$

where $Q(t) = \dfrac{q_1(t)L_1}{e_{-\frac{p}{r}}(\sigma(t),t_0)} - q_2(v(t))L_2v^{\Delta}(t)$.

Thus, there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $r(t)z^{\Delta}(t)$ strictly decreasing on $[t_2, \infty)_{\mathbb{T}}$ and either eventually positive or eventually negative. Since $r(t) > 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, $z^{\Delta}(t)$ is also either eventually positive or eventually negative.

We claim

$$z^{\Delta}(t) > 0 \quad \text{for} \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{2.3}$$

Assume that (2.3) does not hold, then there exists $t_\xi \in [t_2, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t_\xi) < 0$. Since $r(t)z^{\Delta}(t)$ is strictly decreasing on $[t_2, \infty)_{\mathbb{T}}$, it is clear that $r(t)z^{\Delta}(t) \le r(t_\xi)z^{\Delta}(t_\xi) = -c < 0$ for $t \in [t_\xi, \infty)_{\mathbb{T}}$. Thus, we obtain $z^{\Delta}(t) \le -c\frac{1}{r(t)}$ for $t \in [t_\xi, \infty)_{\mathbb{T}}$. By integrating both sides of the last inequality from t_ξ to t, we get

$$z(t) - z(t_\xi) \le -c \int_{t_\xi}^{t} \frac{1}{r(s)} \Delta(s) \quad \text{for} \quad t \in [t_\xi, \infty)_{\mathbb{T}}.$$

Noticing (C_2), we have $\lim_{t\to\infty} z(t) = -\infty$. This contradicts $z(t) \ge 0$. Therefore, (2.3) holds. Thus, we have $z^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$.

Define the function w by generalized Riccati substitution

$$w(t) := \alpha(t) \frac{r(t)z^{\Delta}(t)}{x(\tau_1(t))}.$$

There exist $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $w(t) > 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$.

Easily, we get

$$\begin{aligned} w^{\Delta} &= (rz^{\Delta})^{\Delta} \frac{\alpha}{x \circ \tau_1} + (rz^{\Delta})^{\sigma} \left(\frac{\alpha}{x \circ \tau_1} \right)^{\Delta} \\ &= (rz^{\Delta})^{\Delta} \frac{\alpha}{x \circ \tau_1} + (rz^{\Delta})^{\sigma} \left(\frac{\alpha^{\Delta}}{(x \circ \tau_1)^{\sigma}} - \frac{(x \circ \tau_1)^{\Delta} \alpha}{x \circ \tau_1 (x \circ \tau_1)^{\sigma}} \right) \\ &\le (rz^{\Delta})^{\Delta} \frac{\alpha}{x \circ \tau_1} + \alpha_+^{\Delta} \frac{(rz^{\Delta})^{\sigma}}{(x \circ \tau_1)^{\sigma}} - \alpha \frac{(rz^{\Delta})^{\sigma}}{(x \circ \tau_1)^{\sigma}} \frac{(x \circ \tau_1)^{\Delta}}{x \circ \tau_1} \\ &= (rz^{\Delta})^{\Delta} \frac{\alpha}{x \circ \tau_1} + \alpha_+^{\Delta} \frac{w^{\sigma}}{\alpha^{\sigma}} - \alpha \frac{w^{\sigma}}{\alpha^{\sigma}} \frac{(x \circ \tau_1)^{\Delta}}{x \circ \tau_1}, \end{aligned}$$

where $\alpha_+^{\Delta}(s) = max\{\alpha^{\Delta}(s), 0\}$.

Thus, we have

$$w^{\Delta} \le -Q\alpha + \alpha_+^{\Delta} \frac{w^{\sigma}}{\alpha^{\sigma}} - \alpha \frac{w^{\sigma}}{\alpha^{\sigma}} \frac{(x \circ \tau_1)^{\Delta}}{x \circ \tau_1}.$$

From the chain rule, we know that

$$(x \circ \tau_1)^{\Delta} = (x^{\tilde{\Delta}} \circ \tau_1)\tau_1^{\Delta}.$$

According to condition $\tau_1([t_0, \infty)_{\mathbb{T}}) = [\tau_1(t_0), \infty)_{\mathbb{T}}$ in (C_4), we get that the derivative Δ on $\mathbb{T}$ is equal to the derivative $\tilde{\Delta}$ on $\tilde{\mathbb{T}} := \tau_1([t_0, \infty)_{\mathbb{T}})$. So, we have

$$w^{\Delta} \le -Q\alpha + \alpha_+^{\Delta} \frac{w^{\sigma}}{\alpha^{\sigma}} - \alpha \frac{w^{\sigma}}{\alpha^{\sigma}} \frac{(x^{\Delta} \circ \tau_1)\tau_1^{\Delta}}{x \circ \tau_1}.$$

Also

$$\begin{aligned} z^{\Delta}(t) &= g(x(t), x^{\Delta}(t)) - \frac{1}{r(t)} \int_{v(t)}^{t} q_2(u) f_2(x(\tau_2(u))) \Delta u \\ &\le g(x(t), x^{\Delta}(t)) \\ &\le L_3 x^{\Delta}(t), \end{aligned}$$

implies that

$$-x^{\Delta}(t) \le -\frac{z^{\Delta}(t)}{L_3}$$

and so

$$w^{\Delta} \le -Q\alpha + \alpha_+^{\Delta} \frac{w^{\sigma}}{\alpha^{\sigma}} - \alpha \frac{w^{\sigma}}{\alpha^{\sigma}} \frac{(z^{\Delta} \circ \tau_1)}{x \circ \tau_1} \frac{\tau_1^{\Delta}}{L_3}.$$

Since $\tau_1(t) \le t \le \sigma(t)$ and $r(t)z^{\Delta}(t)$ is strictly decreasing on $[t_2, \infty)_{\mathbb{T}}$, we get

$$(r \circ \tau_1)(z^{\Delta} \circ \tau_1) \ge (rz^{\Delta})^{\sigma}$$

and

$$(z^{\Delta} \circ \tau_1) \ge \frac{(rz^{\Delta})^{\sigma}}{(r \circ \tau_1)}.$$

Thus, we get

$$\begin{aligned} w^{\Delta} &\le -Q\alpha + \alpha_+^{\Delta}\frac{w^{\sigma}}{\alpha^{\sigma}} - \frac{\alpha}{L_3}\frac{w^{\sigma}}{\alpha^{\sigma}}\frac{(rz^{\Delta})^{\sigma}}{(r \circ \tau_1)}\frac{\tau_1^{\Delta}}{x \circ \tau_1} \\ &= -Q\alpha + \alpha_+^{\Delta}\frac{w^{\sigma}}{\alpha^{\sigma}} - \frac{\alpha}{L_3}\left(\frac{w^{\sigma}}{\alpha^{\sigma}}\right)^2\frac{(x \circ \tau_1)^{\sigma}}{x \circ \tau_1}\frac{\tau_1^{\Delta}}{r \circ \tau_1}. \end{aligned} \tag{2.4}$$

From (C_4) we see that $\tau_1(t)$ is strictly increasing on $[t_0, \infty)_{\mathbb{T}}$. Since $t \le \sigma(t)$, we have $\tau_1(t) \le \tau_1^{\sigma}(t)$. Since $x^{\Delta}(t) > 0$, we get $x \circ \tau_1(t) \le x \circ \tau_1^{\sigma}(t)$. Hence, from (2.4) there exist a sufficiently large $t_4 \in [t_3, \infty)_{\mathbb{T}}$ such that

$$\begin{aligned} w^{\Delta} &\le -Q\alpha + \alpha_+^{\Delta}\frac{w^{\sigma}}{\alpha^{\sigma}} - \frac{\alpha}{L_3}\left(\frac{w^{\sigma}}{\alpha^{\sigma}}\right)^2\frac{\tau_1^{\Delta}}{r \circ \tau_1} \\ &= -Q\alpha - \left[\frac{\alpha_+^{\Delta}}{2}\sqrt{\frac{(r \circ \tau_1)L_3}{\alpha\tau_1^{\Delta}}} - \frac{w^{\sigma}}{\alpha^{\sigma}}\sqrt{\frac{\alpha\tau_1^{\Delta}}{(r \circ \tau_1)L_3}}\right]^2 + \frac{(\alpha_+^{\Delta})^2(r \circ \tau_1)L_3}{4\alpha\tau_1^{\Delta}} \\ &\le -Q\alpha + \frac{(\alpha_+^{\Delta})^2(r \circ \tau_1)L_3}{4\alpha\tau_1^{\Delta}}. \end{aligned} \tag{2.5}$$

Integrating both sides of the last inequality from t_4 to t, we get

$$w(t) - w(t_4) \le -\int_{t_4}^{t}\left[(Q(s)\alpha(s)) - \frac{(\alpha_+^{\Delta}(s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right]\Delta s$$

Since $w(t) > 0$ for $t \in [t_3, \infty)_{\mathbb{T}}$ we have

$$\int_{t_4}^{t}\left[(Q(s)\alpha(s)) - \frac{(\alpha_+^{\Delta}(s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right]\Delta s \le w(t_4) - w(t) \le w(t_4)$$

and

$$\limsup_{t\to\infty}\int_{t_4}^{t}\left[(Q(s)\alpha(s)) - \frac{(\alpha_+^{\Delta}(s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right]\Delta s \le w(t_4) < \infty,$$

which is a contradiction to (2.1). The proof is completed. □

Theorem 2.2. *Assume that $(C_1) - (C_8)$ hold. Let H be an rd-continuous function defined as follows:*

$$H : D_{\mathbb{T}} \equiv \{(t, s) : t \ge s \ge t_0, t, s \in [t_0, \infty)_{\mathbb{T}}\} \to \mathbb{R},$$

such that

$$H(t,t) = 0, \quad \text{for } \ t \ge t_0,$$
$$H(t,s) > 0, \quad \text{for } \ t > s \ge t_0,$$

and H has a nonpositive rd-continuous delta partial derivative H_s^{Δ} with respect to the second variable and

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\left(Q(s)\alpha(s) - \frac{(\alpha_+^{\Delta}(-s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right)\Delta s = \infty \tag{2.6}$$

for every sufficiently large T, where $Q(t) = \frac{q_1(t)L_1}{e_{-\frac{p}{r}(\sigma(t),t_0)}} - q_2(v(t))L_2v^{\Delta}(t)$ and $\alpha_+^{\Delta}(s) = max\{\alpha^{\Delta}(s), 0\}$, then all solutions of (1.1) *are oscillatory.*

Proof. Assume that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume x is an eventually positive solution of (1.1). Proceeding as in the proof of the Theorem 2.1, we get

$$w^{\Delta}(t) \leq -Q(t)\alpha(t) + \frac{(\alpha_+^{\Delta})^2(t)(r\circ\tau_1(t))L_3}{4\alpha(t)\tau_1^{\Delta}(t)}.$$

Multiplying by $H(t,s)$ and then integrating from t_4 to t, we obtain

$$\int_{t_4}^{t} H(t,s)w^{\Delta}(s)\triangle s \leq \int_{t_4}^{t} H(t,s)\Big(-Q(s)\alpha(s) + \frac{(\alpha_+^{\Delta})^2(s)(r\circ\tau_1(s))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\Big)\Delta s.$$

Since

$$\int_{t_4}^{t} H(t,s)w^{\Delta}(s)\Delta s = H(t,s)w(s)\mid_{s=t_4}^{s=t} - \int_{t_4}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s,$$

we get

$$-H(t,t_4)w(t_4) \leq \int_{t_4}^{t} H(t,s)\Big(-Q(s)\alpha(s) + \frac{(\alpha_+^{\Delta})^2(s)(r\circ\tau_1(s))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\Big)\Delta s.$$

Thus, we have

$$\int_{t_4}^{t} H(t,s)\Big(Q(s)\alpha(s) - \frac{(\alpha_+^{\Delta})^2(s)(r\circ\tau_1(s))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\Big)\Delta s \leq H(t,t_4)w(t_4)$$

and so

$$\frac{1}{H(t,t_4)}\int_{t_4}^{t} H(t,s)\Big(Q(s)\alpha(s) - \frac{(\alpha_+^{\Delta})^2(s)(r\circ\tau_1(s))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\Big)\Delta s \leq w(t_4) < \infty,$$

which contradicts with (2.6). This completes the proof. □

Theorem 2.3. *Assume that $(C_1)-(C_8)$ hold. Let H be an rd-continuous function defined as follows:*

$$H : \mathbb{D}_{\mathbb{T}} \equiv \{(t,s) : t \geq s \geq t_0, t,s \in [t_0,\infty)_{\mathbb{T}}\} \to \mathbb{R},$$

such that

$$H(t,t) = 0, \quad \textit{for} \ \ t \geq t_0,$$
$$H(t,s) > 0, \quad \textit{for} \ \ t > s \geq t_0,$$

and H has an rd-continuous $\Delta-$partial derivative H_s^{Δ} on $\mathbb{D}_{\mathbb{T}}$ with respect to the second variable.

Let $h : \mathbb{D}_{\mathbb{T}} \to \mathbb{R}$ be an rd-continuous function satisfying

$$H_s^{\Delta}(t,s) + H(t,s)\frac{\alpha_+^{\Delta}(s)}{\alpha^{\sigma}(s)} = \frac{h(t,s)}{\alpha^{\sigma}(s)}\sqrt{H(t,s)}, \qquad (t,s) \in \mathbb{D}_{\mathbb{T}}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)}\int_{T}^{t} \Big(H(t,s)Q(s)\alpha(s) - \frac{[h(t,s)]^2(r\circ\tau_1)(s)L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\Big)\Delta s = \infty \tag{2.7}$$

for every sufficiently large T, where $Q(t) = \frac{q_1(t)L_1}{e_{-\frac{p}{r}(\sigma(t),t_0)}} - q_2(v(t))L_2v^{\Delta}(t)$ and $\alpha_+^{\Delta}(s) = max\{\alpha^{\Delta}(s), 0\}$, then all the solutions of (1.1) *are oscillatory.*

Proof. Assume that x is a nonoscillatory solution of (1.1). Without loss of generality, we may assume x is an eventually positive solution of (1.1). Proceeding as in the proof of the Theorem 2.1, we have (2.5). Multiplying (2.5) by $H(t,s)$ and then integrating from t_4 to t, we obtain

$$\begin{aligned}\int_{t_4}^{t} H(t,s)Q(s)\alpha(s)\Delta s \quad &\leq \quad -\int_{t_4}^{t} H(t,s)w^{\Delta}(s)\Delta s + \int_{t_4}^{t} H(t,s)\frac{\alpha_+^{\Delta}(s)}{\alpha^{\sigma}(s)}w^{\sigma}(s)\Delta s \\ &\quad -\int_{t_4}^{t} H(t,s)\frac{\alpha(s)\tau_1^{\Delta}(s)}{(\alpha^{\sigma}(s))^2(r\circ\tau_1)(s)L_3}[w^{\sigma}(s)]^2\Delta s.\end{aligned}$$

Thus, using

$$\int_{t_4}^{t} H(t,s)w^{\Delta}(s)\Delta s \quad = \quad [H(t,s)w(s)]_{s=t_4}^{s=t} - \int_{t_4}^{t} H_s^{\Delta}(t,s)w^{\sigma}(s)\Delta s,$$

we have

$$\int_{t_4}^{t} H(t,s)Q(s)\alpha(s)\Delta s \leq H(t,t_4)w(t_4) + \int_{t_4}^{t}\left(\left[H_s^{\Delta}(t,s) + H(t,s)\frac{\alpha_+^{\Delta}(s)}{\alpha^{\sigma}(s)}\right]w^{\sigma}(s)\right.$$
$$\left. -H(t,s)\frac{\alpha(s)\tau_1^{\Delta}(s)}{(\alpha^{\sigma}(s))^2(r\circ\tau_1)(s)L_3}[w^{\sigma}(s)]^2\right)\Delta s$$
$$\leq H(t,t_4)w(t_4) + \int_{t_4}^{t}\left(\frac{h(t,s)}{\alpha^{\sigma}(s)}\sqrt{H(t,s)}w^{\sigma}(s)\right.$$
$$\left. -H(t,s)\frac{\alpha(s)\tau_1^{\Delta}(s)}{(\alpha^{\sigma}(s))^2(r\circ\tau_1)(s)L_3}[w^{\sigma}(s)]^2\right)\Delta s$$
$$= H(t,t_4)w(t_4) + \int_{t_4}^{t}\left(\frac{[h(t,s)]^2(r\circ\tau_1)(s)L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right.$$
$$\left. -\left[\frac{h(t,s)}{2\alpha^{\sigma}(s)}\sqrt{\frac{(\alpha^{\sigma}(s))^2(r\circ\tau_1)(s)L_3}{\alpha(s)\tau_1^{\Delta}(s)}} - \sqrt{H(t,s)\frac{\alpha(s)\tau_1^{\Delta}(s)}{(\alpha^{\sigma}(s))^2(r\circ\tau_1)(s)L_3}}w^{\sigma}(s)\right]^2\right)\Delta s$$
$$\leq H(t,t_4)w(t_4) + \int_{t_4}^{t}\frac{[h(t,s)]^2(r\circ\tau_1)(s)L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\Delta s.$$

So, we get

$$\frac{1}{H(t,t_4)}\int_{t_4}^{t}\left(H(t,s)Q(s)\alpha(s) - \frac{[h(t,s)]^2(r\circ\tau_1)(s)L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right)\Delta s \leq w(t_4) < \infty,$$

which is a contradiction to (2.7). The proof is completed. □

3. Examples

Example 3.1. *Let $\mathbb{T}=\mathbb{R}$. Consider the equation*

$$\left(\frac{1}{t}\frac{x'(t)}{2+sin^2(x(t))}\right)' + t^{\frac{1}{3}}\frac{x'(t)}{2+sin^2(x(t))} + \frac{1}{t^2}x(t^{\frac{1}{5}}-3)(x(t^{\frac{1}{5}}-3)^2+4) + \frac{10}{t^{21}}\frac{x(t^2-3)}{x^2(t^2-3)+1} = 0, \quad (3.1)$$

for $t \geq t_0 := 4$
Here $r(t) = \frac{1}{t}, p(t) = t^{\frac{1}{3}}, q_1(t) = \frac{1}{t^2}, q_2(t) = \frac{10}{t^{21}}, \tau_1(t) = t^{\frac{1}{5}} - 3, \tau_2(t) = t^2 - 3,\ g(x(t),x'(t)) = \frac{x'(t)}{2+sin^2(x(t))},\ f_1(u) = u(u^2+4)$ and $f_2(u) = \frac{u}{u^2+1},\ \frac{f_1(u)}{u} = u^2+4 \geq 4 := L_1$ and $\frac{f_2(u)}{u} = \frac{1}{u^2+1} \leq 1 := L_2$ for $u \neq 0,\ | f_2(u) | \leq \frac{1}{2} := M,\ \frac{g(x(t),x'(t))}{x'(t)} \leq \frac{1}{2} := L_3$.

Thus, we obtain $\int_{t_0}^{\infty}\frac{1}{r(t)}dt = \int_{t_0}^{\infty} tdt = \infty, v(t) = t^{\frac{1}{10}} < t$ for $t \in [4,\infty)$ and $1-\mu(t)\frac{p(t)}{r(t)} = 1 > 0$ for $t \in [4,\infty)$.

Also, we get

$$\frac{q_1(t)L_1}{e_{\frac{-p}{r}(\sigma(t),t_0)}} - q_2(v(t))L_2v'(t) = \frac{4e^{(t-4)^{\frac{4}{3}}}}{t^2} - \frac{1}{t^3} = \frac{4te^{(t-4)^{\frac{4}{3}}}-1}{t^3} > 0 \quad for \quad t \in [4,\infty)$$

and

$$\int_t^{\infty}\left[\frac{1}{r(s)}\int_{v(s)}^{s} q_2(u)du\right]ds = \int_t^{\infty}\left[s\int_{s^{\frac{1}{10}}}^{s}\frac{10}{u^{21}}du\right]ds = \int_t^{\infty}\frac{s^{17}-1}{2s^{19}}ds < \infty \quad for \quad t \in [4,\infty).$$

Hence, we have

$$\int_T^{\infty}\left(\left[\frac{q_1(s)L_1}{e_{-\frac{p}{r}(\sigma(s),t_0)}} - q_2(v(s))L_2v'(s)\right]\alpha(s) - \frac{(\alpha_+'(s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^{\Delta}(s)}\right)ds$$
$$= \int_T^{\infty}\left(\frac{4se^{(s-4)^{\frac{4}{3}}}-1}{s^3}s^3 - \frac{3s^2\frac{1}{s^{\frac{1}{5}}-3}\frac{1}{2}}{4s^3\frac{1}{5s^{\frac{4}{5}}}}\right)ds$$

$$= \int_T^\infty \left(4se^{(s-4)^{\frac{4}{3}}} - 1 - \frac{15}{8(s^{\frac{2}{5}} - 3s^{\frac{1}{5}})}\right)ds$$
$$= \int_T^\infty \left(\frac{32(s^{\frac{7}{5}} - 3s^{\frac{6}{5}})e^{(s-4)^{\frac{4}{3}}} - 8(s^{\frac{2}{5}} - 3s^{\frac{1}{5}}) - 15}{8(s^{\frac{2}{5}} - 3s^{\frac{1}{5}})}\right)ds = \infty.$$

Therefore, according to Theorem 2.1, *every solution of* (3.1) *is oscillatory on* $[4, \infty)$.

Example 3.2. *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$. *Consider the equation*

$$\left(tx^\Delta(t)\right)^\Delta + \frac{1}{t^2}x^\Delta(t) + \frac{t+1}{2}x(\frac{t}{2})(x^2(\frac{t}{2}) + 2) + \frac{1}{2t^2}\frac{x(2t)}{2 + x^2(2t)} = 0, \tag{3.2}$$

for $t \in [t_0, \infty)_{\mathbb{T}}, t \geq t_0 := 2$
Here, $r(t) = t$, $p(t) = \frac{1}{t^2}$, $q_1(t) = \frac{t+1}{2}$, $q_2(t) = \frac{1}{2t^2}$, $\tau_1(t) = \frac{t}{2} < t$, $\tau_2(t) = 2t$, $g(x(t), x^\Delta(t)) = x^\Delta(t)$, $f_1(u) = u(u^2 + 2)$ *and* $f_2(u) = \frac{u}{u^2+2}$, $\frac{f_1(u)}{u} = u^2 + 2 \geq 2 := L_1$ *and* $\frac{f_2(u)}{u} = \frac{1}{u^2+2} \leq \frac{1}{2} := L_2$ *for* $u \neq 0$, $\mid f_2(u) \mid \leq \frac{1}{2} := M$, $\frac{g(x(t), x^\Delta(t))}{x^\Delta(t)} = 1$, $L_3 = 1$.

Therefore, we obtain $\int_{t_0}^\infty \frac{1}{r(t)}\Delta t = \int_2^\infty \frac{1}{t}\Delta t = \infty$, $v(t) = \frac{t}{4} < t, v^\Delta(t) = \frac{1}{4}$ *for* $t \in [2, \infty)_{\mathbb{T}}$ *and* $1 - \mu(t)\frac{p(t)}{r(t)} = 1 - t\frac{1}{t^3} = 1 - \frac{1}{t^2} > 0$ *for* $t \in [2, \infty)_{\mathbb{T}}$.

Also, we get

$$\frac{q_1(t)L_1}{e_{\frac{-p}{r}(\sigma(t), t_0)}} - q_2(v(t))L_2v^\Delta(t) > q_1(t)L_1 - q_2(v(t))L_2v^\Delta(t) = \frac{t^3 + t^2 - 1}{t^2} > 0 \quad for \quad t \in [2, \infty)_{\mathbb{T}}$$

and

$$\int_t^\infty \left[\frac{1}{r(s)}\int_{v(s)}^s q_2(u)\Delta u\right]\Delta s = \int_t^\infty \left[\frac{1}{s}\int_{\frac{s}{4}}^s \frac{1}{2u^2}\Delta u\right]\Delta s = \int_t^\infty \left[\frac{1}{s}\left(\frac{-1}{u}\right)\Big|_{\frac{s}{4}}^s \Delta u\right]\Delta s = \int_t^\infty \frac{3}{s^2}\Delta s < \infty.$$

Hence we have

$$\int_T^\infty \left(\left[\frac{q_1(s)L_1}{e_{-\frac{p}{r}(\sigma(s), t_0)}} - q_2(v(s))L_2v^\Delta(s)\right]\alpha(s) - \frac{(\alpha_+^\Delta(s))^2(r(\tau_1(s)))L_3}{4\alpha(s)\tau_1^\Delta(s)}\right)\Delta s$$
$$= \int_T^\infty \left(\left[\frac{s^3 + s^2 - 1}{s^2}\right]s - \frac{\frac{s}{2}}{4s\frac{1}{2}}\right)\Delta s$$
$$= \int_T^\infty \left(\frac{s^3 + s^2 - 1}{s} - \frac{1}{4}\right)\Delta s = \infty.$$

Thus, according to Theorem 2.1, *every solution of* (3.2) *is oscillatory on* $[2, \infty)_{\mathbb{T}}$.

References

[1] S. Hilger, Analysis on measure chains-A unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18-56.
[2] M. Bohner, A. Peterson, Dynamic Equations on time scales, An Introduction with Applications, Birkhäuser, Boston, 2001.
[3] M. Bohner and A. Peterson (Eds), Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[4] R.P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: a survey, J. Comput. and Appl. Math. 141 (1-2)(2002) 1-26.
[5] R.P. Agarwal, M. Bohner, S.H. Saker, Oscillation of second order delay dynamic equations, Can. Appl. Math. Q. 13 (2005) 1-18.
[6] H.A. Agwa, A.M.M. Khodier, H.A. Hassan, Oscillation of second-order nonlinear delay dynamic equations on time scales, Int. J. Differ. Equ. 2011 (2011) Article ID 863801.
[7] M. Bohner, Some oscillation criteria for first order delay dynamic equations, Far East J. Appl. Math. 18 (3)(2005) 289-304.
[8] M. Bohner, S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, Rocky Mountain J. Math. 34 (4) (2004) 1239-1254.
[9] D.X. Chen, P.X. Qu, Y.H. Lan, Oscillation of second-order nonlinear dynamic equations with positive and negative coefficients, Adv. Differ. Equ. 2013 (2013) Art. ID 168.
[10] L. Erbe, A. Peterson, S.H. Saker, Oscillation criteria for second-order nonlinear dynamic equations on time scales, J. Lond. Math. Soc. 67 (3) (2003) 701-714.
[11] L. Erbe, A. Peterson, S.H. Saker, Oscillation criteria for second-order nonlinear delay dynamic equations, J. Math. Anal. Appl. 333 (1) (2007) 505-522.

[12] L. Erbe, T. Hassan, A. Peterson, Oscillation of second order functional dynamic equations, Int. J. Differ. Equ. 5 (2) (2010) 175-193.
[13] T.S. Hassan, Oscillation criteria for half-linear dynamic equations on time scales, J. Math. Anal. Appl. 345 (1) (2008) 176-185.
[14] S.H. Saker, Oscillation criteria of second-order half-linear dynamic equations on time scales, J. Comput. Appl. Math. 177 (2) (2005) 375-387.
[15] S. Sun, Z. Han, C. Zhang, Oscillation of second-order delay dynamic equations on time scales, J. Appl. Math. Comput. 30 (1-2) (2009) 459-468.
[16] Y. Ṣahiner, Oscillation of second-order delay differential equations on time scales, Nonlinear Anal., Theory, Methods Appl. 63 (5-7) (2005) 1073-1080.
[17] M.T. Ṣenel, Oscillation theorems for the second order damped nonlinear dynamic equation on time scales, Contemp. Anal. Appl. Math. 1 (2) (2013) 167-180.
[18] Q. Zhang, Oscillation of second-order half-linear delay dynamic equations with damping on time scales, J. Comput. Appl. Math. 235 (5) (2011) 1180-1188.
[19] Q. Zhang, L. Gao, Oscillation of second-order nonlinear delay dynamic equations with damping on time scales, J. Appl. Math. Comput. 37 (1-2) (2011) 145-158.
[20] Q. Zhang, L. Gao, L. Wang, Oscillation of second-order nonlinear delay dynamic equations on time scales, Comput. Math. Appl. 61 (8) (2011) 2342-2348.

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