

ON GENERALIZED INEQUALITIES OF HERMITE-HADAMARD TYPE FOR CONVEX FUNCTIONS

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ABSTRACT. In this paper, new integral inequalities of Hermite-Hadamard type are developed for n -times differentiable convex functions. Also a parallel development is made base on concavity.

1. INTRODUCTION

A function $f : [a, b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex if whenever $x, y \in [a, b]$ and $t \in [0, 1]$, the following inequality holds:

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

We say that f is concave if $(-f)$ is convex. This definition has its origins in Jensen's results from [6] and has opened up the most extended, useful and multi-disciplinary domain of mathematics, namely, convex analysis. Convex curves and convex bodies have appeared in mathematical literature since antiquity and there are many important results related to them.

On November 22, 1881, Hermite (1822-1901) sent a letter to the Journal Mathesis. This letter was published in Mathesis 3 (1883, p: 82) and in this letter an inequality presented which is well-known in the literature as Hermite-Hadamard integral inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \quad (1.1)$$

where $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I of a real numbers and $a, b \in I$ with $a < b$. If the function f is concave, the inequality in (1.1) is reversed.

The inequalities (1.1) have become an important cornerstone in mathematical anlysis and optimization. Many uses of these inequalities have been discovered in a variety of settings. Moreover , many inequalities of special means can be obtained for a particular choice of the function f . Due to the rich geometrical significance of Hermite-Hadamard's inequality, there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example ([4], [7]- [11], [13], [14]- [19]) and the references therein.

In 2000, Cerone *et. al.* (see [3]) proved the following generalization for n -times differentiable functions.

Theorem 1.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping such that the derivative $f^{(n-1)}$ ($n \geq 1$) is absolutely continuous on $[a, b]$. Then*

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{k=0}^{n-1} \frac{1}{(k+1)!} \left[(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b) \right] \\ &\quad + \frac{1}{n!} \int_a^b (x-t)^n f^{(n)}(t)dt, \end{aligned}$$

for all $x \in [a, b]$.

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For other recent results concerning the n -times differentiable functions see [1]- [3], [5], [7], [12], [17], [18] where further references are given.

In [8], Kavurmacı *et. al.* obtained the following theorems.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{(x-a)^2}{(b-a)} \left[\frac{2|f'(a)| + |f'(x)|}{6} \right] + \frac{(b-x)^2}{(b-a)} \left[\frac{|f'(x)| + 2|f'(b)|}{6} \right] \end{aligned} \quad (1.2)$$

for each $x \in [a, b]$.

Theorem 1.3. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$ and for some fixed $q > 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2} \right)^{\frac{1}{q}} \\ & \quad \times \left[\frac{(x-a)^2 [|f'(a)|^q + |f'(x)|^q]^{\frac{1}{q}} + (b-x)^2 [|f'(x)|^q + |f'(b)|^q]^{\frac{1}{q}}}{b-a} \right] \end{aligned} \quad (1.3)$$

and $q = \frac{p}{p-1}$.

Theorem 1.4. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$ is convex on $[a, b]$ and for some fixed $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a} \int_a^b f(u)du \right| \\ & \leq \frac{1}{2} \left(\frac{1}{3} \right)^{\frac{1}{q}} \\ & \quad \times \left[\frac{(x-a)^2 [|f'(x)|^q + 2|f'(a)|^q]^{\frac{1}{q}} + (b-x)^2 [|f'(x)|^q + 2|f'(b)|^q]^{\frac{1}{q}}}{b-a} \right] \end{aligned} \quad (1.4)$$

for each $x \in [a, b]$.

The main purpose of the present paper is to establish several new inequalities for n -times differentiable mappings that are connected with the celebrated Hermite-Hadamard integral inequality.

2. MAIN RESULTS

Lemma 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be n -times differentiable functions. If $f^{(n)} \in L[a, b]$, then*

$$\begin{aligned} \int_a^b f(t)dt &= \frac{\sum_{k=0}^{n-1} (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \\ &+ (-1)^n \frac{(x-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n)}(tx + (1-t)a)dt \\ &+ (-1)^n \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n)}(tx + (1-t)b)dt \end{aligned} \quad (2.1)$$

where $x \in [a, b]$ and n natural number, $n \geq 1$.

Proof. The proof is by mathematical induction.

The case $n = 1$ is [[8], Lemma 1].

Assume that (2.1) holds for "n" and let us prove it for "n + 1". That is, we have to prove the equality

$$\begin{aligned} \int_a^b f(t)dt &= \sum_{k=0}^n \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \\ &+ (-1)^{n+1} \frac{(x-a)^{n+2}}{(n+1)!} \int_0^1 (t-1)^{n+1} f^{(n+1)}(tx + (1-t)a) dt \\ &+ (-1)^{n+1} \frac{(b-x)^{n+2}}{(n+1)!} \int_0^1 (1-t)^{n+1} f^{(n+1)}(tx + (1-t)b) dt \end{aligned} \quad (2.2)$$

where $x \in [a, b]$.

Then, we can write

$$\begin{aligned} I &= \frac{(x-a)^{n+2}}{(n+1)!} \int_0^1 (t-1)^{n+1} f^{(n+1)}(tx + (1-t)a) dt \\ &+ \frac{(b-x)^{n+2}}{(n+1)!} \int_0^1 (1-t)^{n+1} f^{(n+1)}(tx + (1-t)b) dt \end{aligned}$$

and integrating by parts gives

$$\begin{aligned} I &= \frac{(x-a)^{n+2}}{(n+1)!} \left\{ (t-1)^{n+1} \frac{f^{(n)}(tx + (1-t)a)}{x-a} \Big|_0^1 \right. \\ &\quad \left. - \frac{m+1}{x-a} \int_0^1 (t-1)^n f^{(n)}(tx + (1-t)a) dt \right\} \\ &+ \frac{(b-x)^{n+2}}{(n+1)!} \left\{ (1-t)^{n+1} \frac{f^{(n)}(tx + (1-t)b)}{x-b} \Big|_0^1 \right. \\ &\quad \left. + \frac{m+1}{x-b} \int_0^1 (1-t)^n f^{(n)}(tx + (1-t)b) dt \right\} \\ &= (-1)^{n+2} \frac{(x-a)^{n+1}}{(n+1)!} f^{(n)}(a) - \frac{(x-a)^{n+1}}{n!} \int_0^1 (t-1)^n f^{(n)}(tx + (1-t)a) dt \\ &+ \frac{(b-x)^{n+1}}{(n+1)!} f^{(n)}(b) - \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n f^{(n)}(tx + (1-t)b) dt. \end{aligned}$$

Now, using the mathematical induction hypothesis, we get

$$\begin{aligned} \frac{1}{(-1)^n} \int_a^b f(t)dt &= \frac{1}{(-1)^n} \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \\ &+ (-1)^{n+2} \frac{(x-a)^{n+1}}{(n+1)!} f^{(n)}(a) + \frac{(b-x)^{n+1}}{(n+1)!} f^{(n)}(b) - I. \end{aligned} \quad (2.3)$$

Multiplying the both sides of (2.3) by $(-1)^n$, we obtain

$$\begin{aligned}
 \int_a^b f(t)dt &= \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \\
 &\quad + \frac{(x-a)^{n+1}}{(n+1)!}f^{(n)}(a) + (-1)^n \frac{(b-x)^{n+1}}{(n+1)!}f^{(n)}(b) \\
 &\quad - (-1)^n \left\{ \frac{(x-a)^{n+2}}{(n+1)!} \int_0^1 (t-1)^{n+1}f^{(n+1)}(tx+(1-t)a)dt \right. \\
 &\quad \quad \left. + \frac{(b-x)^{n+2}}{(n+1)!} \int_0^1 (1-t)^{n+1}f^{(n+1)}(tx+(1-t)b)dt \right\} \\
 &= \sum_{k=0}^n \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \\
 &\quad + (-1)^{n+1} \frac{(x-a)^{n+2}}{(n+1)!} \int_0^1 (t-1)^{n+1}f^{(n+1)}(tx+(1-t)a)dt \\
 &\quad + (-1)^{n+1} \frac{(b-x)^{n+2}}{(n+1)!} \int_0^1 (1-t)^{n+1}f^{(n+1)}(tx+(1-t)b)dt.
 \end{aligned}$$

Thus, the identity (2.2) and the lemma is proved. \square

Theorem 2.1. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable function, $a, b \in I$ and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|$ ($n \geq 1$) is convex on $[a, b]$, then we have*

$$\begin{aligned}
 &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \right| \tag{2.4} \\
 &\leq \frac{(x-a)^{n+1}}{(n+2)!} \left\{ (n+1) |f^{(n)}(a)| + |f^{(n)}(x)| \right\} \\
 &\quad + \frac{(b-x)^{n+1}}{(n+2)!} \left\{ |f^{(n)}(x)| + (n+1) |f^{(n)}(b)| \right\}
 \end{aligned}$$

where $x \in [a, b]$.

Proof. From Lemma 2.1 and using the properties of modulus, we can write

$$\begin{aligned}
 &\left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \right| \\
 &\leq \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n |f^{(n)}(tx+(1-t)a)| dt \\
 &\quad + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n |f^{(n)}(tx+(1-t)b)| dt.
 \end{aligned}$$

Since $|f^{(n)}|$ is convex on $[a, b]$, it follows that

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
& \leq \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n \left[t |f^{(n)}(x)| + (1-t) |f^{(n)}(a)| \right] dt \\
& \quad + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left[t |f^{(n)}(x)| + (1-t) |f^{(n)}(b)| \right] dt \\
& = \frac{(x-a)^{n+1}}{(n+2)!} \left\{ (n+1) |f^{(n)}(a)| + |f^{(n)}(x)| \right\} \\
& \quad + \frac{(b-x)^{n+1}}{(n+2)!} \left\{ |f^{(n)}(x)| + (n+1) |f^{(n)}(b)| \right\}.
\end{aligned}$$

This completes the proof. \square

Remark 2.1. In the inequality (2.4), if we choose $n = 1$, then we have the inequality (1.2).

Corollary 2.1. In the inequality (2.4), if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we have

$$\begin{aligned}
& \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
& \leq \frac{(b-a)^2}{192} \left\{ 3 |f''(a)| + 2 \left| f'' \left(\frac{a+b}{2} \right) \right| + 3 |f''(b)| \right\} \\
& \leq \frac{(b-a)^2}{48} \{ |f''(a)| + |f''(b)| \}.
\end{aligned}$$

Theorem 2.2. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable function, $x \in [a, b]$ and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ ($n \geq 1$) is convex on $[a, b]$, then we the following inequality:

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \tag{2.5} \\
& \leq \left(\frac{1}{np+1} \right)^{\frac{1}{p}} \left\{ \frac{(x-a)^{n+1}}{n!} \left[\frac{|f^{(n)}(a)|^q + |f^{(n)}(x)|^q}{2} \right]^{\frac{1}{q}} \right. \\
& \quad \left. + \frac{(b-x)^{n+1}}{n!} \left[\frac{|f^{(n)}(x)|^q + |f^{(n)}(b)|^q}{2} \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Using Lemma 2.1 and Hölder integral inequality, we obtain

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
& \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 (1-t)^{np} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f^{(n)}(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^{np} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f^{(n)}(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f^{(n)}|^q$ is convex on $[a, b]$, then

$$\begin{aligned}
 & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
 & \leq \frac{(x-a)^{n+1}}{n!} \left(\frac{1}{np+1} \right)^{\frac{1}{p}} \left(\int_0^1 [t |f^{(n)}(x)|^q + (1-t) |f^{(n)}(a)|^q] dt \right)^{\frac{1}{q}} \\
 & \quad + \frac{(b-x)^{n+1}}{n!} \left(\frac{1}{np+1} \right)^{\frac{1}{p}} \left(\int_0^1 [t |f^{(n)}(x)|^q + (1-t) |f^{(n)}(b)|^q] dt \right)^{\frac{1}{q}} \\
 & = \left(\frac{1}{np+1} \right)^{\frac{1}{p}} \left\{ \frac{(x-a)^{n+1}}{n!} \left[\frac{|f^{(n)}(a)|^q + |f^{(n)}(x)|^q}{2} \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \frac{(b-x)^{n+1}}{n!} \left[\frac{|f^{(n)}(x)|^q + |f^{(n)}(b)|^q}{2} \right]^{\frac{1}{q}} \right\}
 \end{aligned}$$

which completes the proof. \square

Remark 2.2. In Theorem 2.2, if we choose $n = 1$, then we have the inequality (1.3).

Corollary 2.2. In Theorem 2.2, if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we obtain

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\
 & \leq \frac{(b-a)^2}{16} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \\
 & \quad \times \left\{ \left[\frac{|f''(a)|^q + |f''(\frac{a+b}{2})|^q}{2} \right]^{\frac{1}{q}} + \left[\frac{|f''(\frac{a+b}{2})|^q + |f''(b)|^q}{2} \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Theorem 2.3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable function and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is convex on $[a, b]$, then we get

$$\begin{aligned}
 & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \tag{2.6} \\
 & \leq \left(\frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\
 & \quad \times \left\{ \frac{(x-a)^{n+1}}{n!} \left[\frac{1}{p+2} |f^{(n)}(a)|^q + \frac{1}{(p+1)(p+2)} |f^{(n)}(x)|^q \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \frac{(b-x)^{n+1}}{n!} \left[\frac{1}{(p+1)(p+2)} |f^{(n)}(x)|^q + \frac{1}{p+2} |f^{(n)}(b)|^q \right]^{\frac{1}{q}} \right\}
 \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ and $x \in [a, b]$.

Proof. From Lemma 2.1 and using the properties of modulus, we get

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
& \leq \frac{(x-a)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(tx + (1-t)a) \right| dt \\
& \quad + \frac{(b-x)^{n+1}}{n!} \int_0^1 (1-t)^n \left| f^{(n)}(tx + (1-t)b) \right| dt \\
& = \frac{(x-a)^{n+1}}{n!} \int_0^1 \frac{(1-t)^n (1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}} \left| f^{(n)}(tx + (1-t)a) \right| dt \\
& \quad + \frac{(b-x)^{n+1}}{n!} \int_0^1 \frac{(1-t)^n (1-t)^{\frac{p}{q}}}{(1-t)^{\frac{p}{q}}} \left| f^{(n)}(tx + (1-t)b) \right| dt.
\end{aligned}$$

Using the Hölder integral inequality, we can write

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
& \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 \left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}} \right]^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^p \left| f^{(n)}(tx + (1-t)a) \right|^q dt \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 \left[\frac{(1-t)^n}{(1-t)^{\frac{p}{q}}} \right]^{\frac{q}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^p \left| f^{(n)}(tx + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}.
\end{aligned}$$

Since $|f^{(n)}|^q$ is convex on $[a, b]$, then

$$\begin{aligned}
& \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
& \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 (1-t)^{\frac{nq-p}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^p \left[t \left| f^{(n)}(x) \right|^q + (1-t) \left| f^{(n)}(a) \right|^q \right] dt \right)^{\frac{1}{q}} \\
& \quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^{\frac{nq-p}{q-1}} dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^p \left[t \left| f^{(n)}(x) \right|^q + (1-t) \left| f^{(n)}(b) \right|^q \right] dt \right)^{\frac{1}{q}} \\
& = \left(\frac{q-1}{nq+q-p-1} \right)^{1-\frac{1}{q}} \\
& \quad \times \left\{ \frac{(x-a)^{n+1}}{n!} \left[\frac{1}{p+2} \left| f^{(n)}(a) \right|^q + \frac{1}{(p+1)(p+2)} \left| f^{(n)}(x) \right|^q \right]^{\frac{1}{q}} \right. \\
& \quad \left. + \frac{(b-x)^{n+1}}{n!} \left[\frac{1}{(p+1)(p+2)} \left| f^{(n)}(x) \right|^q + \frac{1}{p+2} \left| f^{(n)}(b) \right|^q \right]^{\frac{1}{q}} \right\}
\end{aligned}$$

which completes the proof of the theorem. \square

Corollary 2.3. *In Theorem 2.3, if we choose $n = 1$, we have*

$$\begin{aligned}
 & \left| \frac{(x-a)f(a) + (b-x)f(b)}{(b-a)} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
 & \leq \left(\frac{q-1}{2q-p-1} \right)^{1-\frac{1}{q}} \\
 & \quad \times \left\{ \frac{(x-a)^2}{(b-a)} \left[\frac{1}{p+2} |f'(a)|^q + \frac{1}{(p+1)(p+2)} |f'(x)|^q \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \frac{(b-x)^2}{(b-a)} \left[\frac{1}{(p+1)(p+2)} |f'(x)|^q + \frac{1}{p+2} |f'(b)|^q \right]^{\frac{1}{q}} \right\}.
 \end{aligned} \tag{2.7}$$

Corollary 2.4. *In the inequality (2.7), if we choose $x = \frac{a+b}{2}$, then we obtain*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
 & \leq \frac{b-a}{4} \left(\frac{q-1}{2q-p-1} \right)^{1-\frac{1}{q}} \\
 & \quad \times \left\{ \left[\frac{1}{p+2} |f'(a)|^q + \frac{1}{(p+1)(p+2)} \left| f' \left(\frac{a+b}{2} \right) \right|^q \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left[\frac{1}{(p+1)(p+2)} \left| f' \left(\frac{a+b}{2} \right) \right|^q + \frac{1}{p+2} |f'(b)|^q \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Corollary 2.5. *In Theorem 2.3, if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we have*

$$\begin{aligned}
 & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\
 & \leq \frac{(b-a)^2}{16} \left(\frac{q-1}{3q-p-1} \right)^{1-\frac{1}{q}} \\
 & \quad \times \left\{ \left[\frac{1}{p+2} |f''(a)|^q + \frac{1}{(p+1)(p+2)} \left| f'' \left(\frac{a+b}{2} \right) \right|^q \right]^{\frac{1}{q}} \right. \\
 & \quad \left. + \left[\frac{1}{(p+1)(p+2)} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{1}{p+2} |f''(b)|^q \right]^{\frac{1}{q}} \right\}.
 \end{aligned}$$

Theorem 2.4. *For $n \geq 1$, let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable function and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is convex on $[a, b]$, for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned}
 & \left| \int_a^b f(t)dt - \frac{\sum_{k=0}^{n-1} (x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\
 & \leq \frac{(x-a)^{n+1}}{(n+1)!} \left[\frac{(n+1) |f^{(n)}(a)|^q + |f^{(n)}(x)|^q}{(n+2)} \right]^{\frac{1}{q}} \\
 & \quad + \frac{(b-x)^{n+1}}{(n+1)!} \left[\frac{|f^{(n)}(x)|^q + (n+1) |f^{(n)}(b)|^q}{(n+2)} \right]^{\frac{1}{q}}.
 \end{aligned} \tag{2.8}$$

Proof. From Lemma 2.1 and using the well known Power-mean integral inequality, we have

$$\begin{aligned} & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\ & \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 (1-t)^n dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n |f^{(n)}(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^n dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n |f^{(n)}(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned}$$

Since $|f^{(n)}|^q$ is convex on $[a, b]$, for $q \geq 1$, then we obtain

$$\begin{aligned} & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \\ & \leq \frac{(x-a)^{n+1}}{n!} \left(\frac{1}{n+1} \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n [t |f^{(n)}(x)|^q + (1-t) |f^{(n)}(a)|^q] dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^{n+1}}{n!} \left(\frac{1}{n+1} \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n [t |f^{(n)}(x)|^q + (1-t) |f^{(n)}(b)|^q] dt \right)^{\frac{1}{q}} \\ & = \frac{(x-a)^{n+1}}{(n+1)!} \left[\frac{(n+1) |f^{(n)}(a)|^q + |f^{(n)}(x)|^q}{(n+2)} \right]^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^{n+1}}{(n+1)!} \left[\frac{|f^{(n)}(x)|^q + (n+1) |f^{(n)}(b)|^q}{(n+2)} \right]^{\frac{1}{q}}. \end{aligned}$$

Hence, the proof of the theorem is completed. \square

Remark 2.3. In Theorem 2.4, if we choose $n = 1$, we obtain the inequality (1.4).

Corollary 2.6. In the inequality (2.8) if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we get

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \frac{(b-a)^2}{48} \left\{ \left[\frac{3 |f''(a)|^q + |f''(\frac{a+b}{2})|^q}{4} \right]^{\frac{1}{q}} + \left[\frac{|f''(\frac{a+b}{2})|^q + 3 |f''(b)|^q}{4} \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Theorem 2.5. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable function and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is concave on $[a, b]$, then we obtain

$$\begin{aligned} & \left| \int_a^b f(t) dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1} f^{(k)}(a) + (-1)^k (b-x)^{k+1} f^{(k)}(b)}{(k+1)!} \right| \tag{2.9} \\ & \leq \left(\frac{1}{np+1} \right)^{\frac{1}{p}} \left\{ \frac{(x-a)^{n+1}}{n!} \left| f^{(n)} \left(\frac{a+x}{2} \right) \right| + \frac{(b-x)^{n+1}}{n!} \left| f^{(n)} \left(\frac{x+b}{2} \right) \right| \right\} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1 and Hölder integral inequality, we obtain

$$\begin{aligned} & \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \right| \\ & \leq \frac{(x-a)^{n+1}}{n!} \left\{ \left(\int_0^1 (1-t)^{np} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f^{(n)}(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right. \\ & \quad \left. + \left(\int_0^1 (1-t)^{np} dt \right)^{\frac{1}{p}} \left(\int_0^1 |f^{(n)}(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}} \right\}. \end{aligned} \quad (2.10)$$

Since $|f^{(n)}|^q$ is concave on $[a, b]$, we can write the following inequalities via Jensen inequality:

$$\begin{aligned} \int_0^1 |f^{(n)}(tx + (1-t)a)|^q dt &= \int_0^1 t^0 |f^{(n)}(tx + (1-t)a)|^q dt \\ &\leq \left(\int_0^1 t^0 dt \right) \left| f^{(n)} \left(\frac{\int_0^1 (tx + (1-t)a) dt}{\int_0^1 t^0 dt} \right) \right|^q \\ &= \left| f^{(n)} \left(\frac{a+x}{2} \right) \right|^q \end{aligned} \quad (2.11)$$

and similarly

$$\int_0^1 |f^{(n)}(tx + (1-t)b)|^q dt \leq \left| f^{(n)} \left(\frac{x+b}{2} \right) \right|^q. \quad (2.12)$$

Thus, if we use (2.11) and (2.12) in the inequality (2.10), we obtain the inequality of (2.9). This completes the proof. \square

Corollary 2.7. *In the inequality (2.9) if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we obtain*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \\ & \leq \frac{(b-a)^2}{16} \left(\frac{1}{2p+1} \right)^{\frac{1}{p}} \left\{ \left| f'' \left(\frac{3a+b}{4} \right) \right| + \left| f'' \left(\frac{a+3b}{4} \right) \right| \right\}. \end{aligned}$$

Theorem 2.6. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be n -times differentiable function and $a < b$. If $f^{(n)} \in L[a, b]$ and $|f^{(n)}|^q$ is concave on $[a, b]$, for $q \geq 1$, then the following inequality holds:*

$$\begin{aligned} & \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \right| \\ & \leq \frac{(x-a)^{n+1}}{(n+1)!} \left| f^{(n)} \left(\frac{(n+1)a+x}{n+2} \right) \right| + \frac{(b-x)^{n+1}}{(n+1)!} \left| f^{(n)} \left(\frac{x+(n+1)b}{n+2} \right) \right|. \end{aligned} \quad (2.13)$$

Proof. From Lemma 2.1 and using the well known Power-mean inequality, we have

$$\begin{aligned} & \left| \int_a^b f(t)dt - \sum_{k=0}^{n-1} \frac{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)}{(k+1)!} \right| \\ & \leq \frac{(x-a)^{n+1}}{n!} \left(\int_0^1 (1-t)^n dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n |f^{(n)}(tx + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{(b-x)^{n+1}}{n!} \left(\int_0^1 (1-t)^n dt \right)^{1-\frac{1}{q}} \left(\int_0^1 (1-t)^n |f^{(n)}(tx + (1-t)b)|^q dt \right)^{\frac{1}{q}}. \end{aligned} \quad (2.14)$$

Using the Jensen inequality, we can write

$$\begin{aligned}
 & \int_0^1 (1-t)^n \left| f^{(n)}(tx + (1-t)a) \right|^q dt \\
 & \leq \left(\int_0^1 (1-t)^n dt \right) \left| f^{(n)} \left(\frac{\int_0^1 (1-t)^n (tx + (1-t)a) dt}{\int_0^1 (1-t)^n dt} \right) \right|^q \\
 & = \left(\frac{1}{n+1} \right) \left| f^{(n)} \left(\frac{(n+1)a + x}{n+2} \right) \right|^q
 \end{aligned} \tag{2.15}$$

and similarly

$$\int_0^1 (1-t)^n \left| f^{(n)}(tx + (1-t)b) \right|^q dt \leq \left(\frac{1}{n+1} \right) \left| f^{(n)} \left(\frac{x + (n+1)b}{n+2} \right) \right|^q. \tag{2.16}$$

Thus, if we use (2.15) and (2.16) in the inequality (2.14), we obtain the inequality of (2.13). The proof of the theorem is completed. \square

Corollary 2.8. *In Theorem 2.6, if we choose $n = 2$, $x = \frac{a+b}{2}$ and $f'(x) = f'(a+b-x)$ (that is, f' symmetric function), then we have*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{(b-a)^2}{48} \left\{ \left| f'' \left(\frac{7a+b}{8} \right) \right| + \left| f'' \left(\frac{a+7b}{8} \right) \right| \right\}.$$

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