International Journal of Analysis and Applications
ISSN 2291-8639
Volume 4, Number 2 (2014), 192-200
http://www.etamaths.com

CONVERGENCE OF A MODIFIED MULTI-STEP ITERATIVE SCHEME FOR P- NEARLY UNIFORMLY L-LIPSCHITZIAN ASYMPTOTICALLY PSEUDOCONTRATIVE MAPPINGS

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Abstract. In this paper, it is shown that the modified multi-step iteration converges strongly to the common fixed point of a finite family of nearly uniformly L- Lipschitzian asymptotically pseudocontractive mappings. The main result is an improvement and extension of well known results in the literature.

1. Preliminary.

Let E be a real Banach space and let E^* be its dual space. The normalized duality mappping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in X^* : < x, f > = \|x\|\|f\|\},$$

where $< .,. >$ denotes the generalized duality pairing and $\|x\| = \|f\|$. We shall denote the single-valued normalized duality pairing by j. J satisfies the following properties:
(1) J is an odd mapping, i.e $J(-x) = -J(x)$.
(2) J is positive homogeneous , i.e for any number $\lambda > 0$, $J(\lambda x) = \lambda J(x)$.
(3) J is bounded, i.e. for any subset A of E $J(A)$ is a bounded subset of E^*.
(4) If E is smooth (or E^* is strictly convex), Then J is singled-valued.
Consistent with Goebel and Kirk [4] we give the following definitions,
Let K be a nonempty closed convex subset of E and $T : K \to K$ be a map.

Definition 1.1 A mapping T is said to be asymptotically nonexpansive if for each $x, y \in K$

$$< T^n x - T^n y > \leq k_n \|x - y\|,$$

2010 *Mathematics Subject Classification.* 47H10.
Key words and phrases. Modified multi-step iteration process, asymptotically pseudocontrative maps, uniformly L- Lipschitzian maps, nearly uniformly L-Lipschitzian maps, nearly asymptotically pseudocontrative maps, Banach spaces.

$\forall n \geq 0$, where the sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$.

Definition 1.2 The mapping is said to be uniformly L- Lipschitzian if there exists a constant $L \geq 0$ such that

$$\|T^n - T^n y\| \leq L\|x - y\|,$$

for any $x, y \in K$ and $\forall n \geq 0$.
We give the definition of asymptotically pseudocontractive as in [14].
Definition 1.3[14] The mapping T is said to be asymptotically pseudocontractive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and for any $x, y \in K$ there exist $j(x - y) \in J(x - y)$ such that

$$< T^n x - T^n y, j(x - y) > \leq k_n \|x - y\|^2, \ \forall n \geq 0.$$

Remark 1.4 Every asymptotically nonexpansive mapping is both asymptotically pseudocontractive and uniformly L- Lipschitzian. The converse is not true in general. Clearly, every operator which is asymptotically pseudocontractive in general may not admit a fixed point. The existence of fixed point result for asymptotically pseudocontractive maps depend on the space, nature of subset and further properties of the operators (see, [1]).
Infact, asymptotically nonexpansive and asymptotically pseudocontractive were first introduced by Goebel and Kirk [4] and Schu [15] respectively. Since then many authors have studied several iterative process for asymptotically nonexpansive and asymptotically pseudocontractive in both Hilbert and Banach spaces (see, [7, 8, 12]).
Schu[15] proved the convergence of Mann[6] iterative sequence to the fixed point of uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in the setting of Hilbert space. In 2001, Chang kextended the work of Schu to the setting of real uniformly smooth Banach spaces. Also, Ofoedu [11] extended Theorem 1.2 of Chang[1] to the setting of arbitrary real Banach spaces and dropped the boundedness assumption. He proved the following theorem:

Theorem 1.5[11] Let E be a real Banach space, K be a nonempty closed convex subspace of E and $T : K \to K$ uniformly L-Lipschitzian and asymptotically pseudocontractive mappings with a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty), k_n \to 1$ and $x^* \in F(T)$. Let the sequence $\{\alpha_n\}$ be a sequence in [0,1] satisfying the following conditions:
(a-1) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(a-2) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
(a-3) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_0 \in K$, let $\{x_n\}_{n\geq 0}^{\infty}$ be an iterative sequence defined by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \tag{1.1}$$

for all $n \geq 0$. If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0$ such that

$$< T^n x - x^*, j(x - x^*) > \leq k_n \|x - x^*\|^2 - \Phi(\|x - x^*\|)$$

$\forall\ n \geq 0$. Then
(1) $\{x_n\}_{n \geq 0}^{\infty}$ is bounded;
(2) $\{x_n\}_{n \geq 0}^{\infty}$ converges strongly to $x^* \in F(T)$.
Further more, Chang Cho and Kim [2] improved on the theorem above by extending the parameters and modifying the iterative procedure. Infact, they proved the following theorem:
Theorem 1.6[2] Let E be a real Banach space, K be a nonempty closed and convex subspace of E and $T : K \to K$ uniformly L-Lipschitzian and asymptotically pseudocontractive mappings with a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$,
$k_n \to 1$ and $x^* \in F(T)$. Let $\{a_n\}, \{b_n\}$ and$\{c_n\}$ be a real sequences in [0,1] satisfying the following conditions:
(a-1) $a_n + b_n + c_n = 1$;
(a-2) $\sum_{n=0}^{\infty} (b_n + c_n) = \infty$;
(a-3) $\sum_{n=0}^{\infty} (b_n + c_n)^2 < \infty$;
(a-4) $\sum_{n=0}^{\infty} (b_n + c_n)(k_n - 1) < \infty$;
(a-5) $\sum_{n=0}^{\infty} c_n < \infty$.
For any $x_0 \in K$, let $\{x_n\}_{n \geq 0}^{\infty}$ be a sequence in K iteratively defined by

$$x_{n+1} = a_n x_n + b_n T^n x_n + c_n u_n,$$

for all $n \geq 0$. where $\{u_n\}$ is a bounded sequence in K. Suppose that there exists a strictly increasing continuous function $\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0$ such that

$$< T^n x - x^*, j(x - x^*) > \leq k_n \|x - x^*\|^2 - \Phi(\|x - x^*\|),$$

$\forall\ n \geq 0$. Then
(1) $\{x_n\}_{n \geq 0}^{\infty}$ is bounded;
(2) $\{x_n\}_{n \geq 0}^{\infty}$ converges strongly to $x^* \in F(T)$.

Another extension of the fixed point theory is the iterative processes for approximating fixed points of mappings. Several authors have studied and extended the Mann[6],Ishikawa[5], Noor[10] and multi-step[13] iterative process to evaluate the fixed point of uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in Hilbert and Banach spaces.(see[7,8,16,17]).

We remarked that in all these theorems above, for certain application the continuity assumption becomes a rather strong condition. In this direction, a natural question arises that whether there is any class of(not necessarily continuous) mapping more general than the class of asymptotically nonexpansive and asymptotically pseudocontractive (which has asymptotically nonexpansiveness)? Motivated by this inspired question, Sahu[14] introduced the classes of nearly contraction and nearly asymptotically nonexpansive mappings. He gave the following definition:
Definition 1.7 [14] Let K be a nonempty subset of a Banach space E andj sequence

$\{a_n\}$ in $[0, \infty)$ with $a_n \to 0$. A mapping $T : K \to K$ is called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in N$, there exists a constant $k_n \geq 0$ such that :

$$< T^n x - T^n y > \leq k_n(\|x - y\| + a_n)$$

for all $x, y \in K$.

Remark 1.8 It is important to note that Lipschitzian mappings are always continuous but nearly Lipschitzian mappings need not be continuous. The class of nearly asymptotically nonexpansive mappings contains the class of asymptotically nonexpasive mappings and is contained in the class of mappings of asymptotically nonexpansive type. Hence, according to Sahu[14] nearly asymptotically pseudocontractive mapping is a generalisation of asymptotically pseudocontractive mapping.
Example 1.9[16] Let $E = R$ and $T : K \to K$ be defined by:

$$T(x) = \begin{cases} \frac{x}{2} \ x \in [0, 1); \\ \\ 0 \ x = 1. \end{cases}$$

Then T is a discontinuous mapping which is not Lipschitzian, but nearly $\frac{1}{2}$ Lipschitzian with sequence $\{\frac{1}{2^n}\}$.
Sahu[14] proved some theorems in an attempt to develop asymptotically fixed point theory for a more general class of demicontinuous nearly Lipschirzian mappings in Banach spaces. And later extended this theorem to uniformly convex Banach spaces. He proved the following theorem:
Theorem 1.10[14, Theorem 3.8] Let C be a nonempty closed convex subset of a uniformly convex Banach space X and $T : C \to$ a demicontinuos nearly Lipschitzian mapping with sequences $\{(a_n, \eta(T^n))\}$ such that $\lim_{n\to\infty} \eta(T^n) \leq 1$. Then the following statements are equivalent:
(a) T has a fixed point;
(b) there exists a bounded sequence $\{T^n x_0\}$ in C;
(c) there exists a bounded sequence $\{y^n\}$ in C such that
$\lim_{m\to\infty}(\lim_{n\to\infty} \|y_n - T^m y_n\|) = 0$.
Very recently, Thakur[16] stated and proved the following theorem:
Theorem 1.11[16] Let E be a real Banach space, K be a nonempty closed convex subset of E and $T_i : K \to K, i = 1, 2$ be two asymptotically generalised Φ-hemicontractive nearly uniformly L_i Lipschitzian mappings with sequence $\{a_n\}$ and $F(T_1) \cap F(T_2) \neq \phi$ where $F(T_i)$ is the set of fixed point of T_i in K. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in [0, 1] satisfying the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
(iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;
(iv) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.
Let $\{x_n\}$ be a sequence in K generated from arbitrary $x_1 \in K$ by

$$X_{n+1} = (1 - \alpha_n)x_n + \alpha {T_1}^n y_n,$$
$$y_n = (1 - \beta_n)x_n + \beta {T_2}^n x_n, \ n \in N. \quad (1.2)$$

Then, $\{x_n\}$ converges strongly to $x^* \in F(T_1) \cap F(T_2)$.

The purpose of this sequel is to improve Theorem 1.10 and Theorem 1.11 by extending to a finite family of nearly p- uniformly L- Lipschitzian asymptotically pseudocontractive mappings.

2. Main Results

The following concepts and Lemmas will be used.

Definition 2.1 [13]. Let $T_1, T_2, ..., T_i : K \to K$ be finite family of maps. For any given $x_1 \in K$, the multi-step iteration $\{x_n\}_{(n=1)}^{\infty} \subset K$ is defined by

$$\begin{aligned} x_{n+1} &= (1-b_n)x_n + b_n {T_1}^n {y_n}^1, \ n \geq 1 \\ {y_n}^i &= (1-{b_n}^i)x_n + {b_n}^i {T_i}^n {y_n}^{i+1}, \ i = 1,2,...,p-2, \\ {y_n}^{p-1} &= (1-{b_n}^{p-1})x_n + {b_n}^{p-1} {T_p}^n x_n, \ p \geq 2. \end{aligned} \tag{2.1}$$

Lemma 2.2 [9]. Let $J : E \to 2^{E*}$ be the normalized duality mapping. Then for any $x, y \in E$, we have

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >, \forall j(x+y) \in J(x,y)$$

Lemma 2.3 [3]. Let $\{d_n\}, \{e_n\}$ and $\{h_n\}$ be three positive real sequences and $\Phi : [0,\infty) \to [0,\infty)$ be a strictly increasing function with $\Phi(x) = 0 \Leftrightarrow x = 0$ satisfying the following inequality:

$$d_{n+1}^2 \leq {d_n}^2 - e_n \Phi(d_{n+1}) + h_n, \forall n \geq 0$$

where $e_n \in [0,1]$, with $\sum_{n=0}^{\infty} e_n = +\infty$ and $h_n = o(e_n)$. Then $\lim_{n\to\infty} d_n = 0$.

Theorem 2.4 Let K be a nonempty closed convex subset of a real Banach space E, and $T_i : K \to K, (i = 1,2,.....,p, \ p \geq 2)$ be a finite family of nearly L_i - uniformly Lipschitzian mappings with sequence $\{k_n\}_{n\geq 1} \subset [1,\infty)$,
$k_n \to 1$ and $\sum_{n\geq 1}(k_n - 1) < \infty$ such that $\cap_{i=1}^{p\geq 2} F(T_i) \neq \phi$. Let $\{b_n\}_{n\geq 1}, \{b_n^i\}_{n\geq 1}$ and $\{b_n^{i+1}\}_{n\geq 1}$ be the real sequences in [0,1] satisfying:
$(i) b_n, b_n^i, b_n^{i+1} \to 0$, as $n \to \infty, (i = 1,2,...,p-2)$.
$(ii) \sum_{n\geq 1} b_n = \infty$.
For any $x_1 \in K$, define $\{x_n\}_{n\geq 0}$ by the iterative process (2.1). Suppose there exists a strictly increasing function $\Phi : [0,\infty) \to [0,\infty), \Phi(0) = 0$ such that

$$< {T_i}^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|) \tag{2.2}$$

$\forall x \in K, (i = 1,2,....,p, \ p \geq 2)$. Then $\{x_n\}_{n\geq 0}$ converges strongly to $\rho \in \cap_{i=1}^{p\geq 2} F(T_i)$.

Proof: First we show that for any $n \geq 1, \{x_n\}$ is a bounded sequence. Since $T_1, T_2, ..., T_p, \ p \geq 2$ are nearly uniformly L_i - Lipschizian mappings, we have that, $\forall x, y \in K$

$$\|T_i^n x - T_i^n y\| \leq L_i(\|x-y\| + a_n), \ (i = 1,2,....,p, \ p \geq 2).$$

Setting $\max\{k_n : n \geq 1\} = k$ and $L = \max\{L_1, L_2, ..., L_p\}$.
There exists $x_1 \in K$ with $x_1 \neq Tx_1$ such that $a_0 = (k+L)\|x_1 - \rho\|^2 \in D(\Phi)$. If $\Phi(a) \to +\infty$ as $a \to +\infty$, then $a_0 \in D(\Phi)$. If $\sup\{\Phi(a) : a \in [0, \infty)\} = a_1 < +\infty$ with $a_1 < a_0$, then there exists a sequence $\{\tau_n\} \subset K$ such that $\tau_n \to \rho$ as $n \to \infty$ with $\tau_n \neq \rho$, thus there exist a positive integer n_0 such that $(k+L)\|x_1 - \rho\|^2 < \frac{a_1}{2}$ for all $n \geq n_0$. Redefining $x_1 = \tau_{n0}$,
$(k+L)\|x_1 - \rho\|^2 \in D(\Phi)$ and setting $D(\Phi) = \Phi^{-1}(a_0)$, we obtain $\|x_1 - \rho\| \leq D$. Also let $A_1 = \{x \in K : \|x_1 - \rho\| \leq D\}, A_2 = \{x \in K : \|x_1 - \rho\| \leq 2D\}$. Now, we show that $x_n \in A_1$ for any $n \geq 1$. If $n = 1$, then $x_1 \in A_1$. Assume that for some n, $x_n \in A_1$, we show that $x_{n+1} \in A_1$. Suppose x_{n+1} is not in A_1, then
$\|x_n - \rho\| > D$. Now we denote

$$\nu_0 = min\{\frac{D}{D(1+2L)}, \frac{\Phi(D)}{10D^2}, \frac{\Phi(D)}{10(2DL + 3DL^2)}\} \tag{2.3}$$

Since $b_n, b_n^i, b_n^{i+1}, k_{n-1} \to 0$, as $n \to \infty, (i = 1, 2, ..., p-2)$, without loss of generality, let $0 \leq b_n, b_n^i, b_n^{i+1}, k_{n-1} \leq \nu_0$ for any $n \geq 1$. Thus we have

$$\begin{aligned}
\|{y_n}^{p-1} - \rho\| &= \|(1 - {b_n}^{p-1})x_n + {b_n}^{p-1}{T_p}^n x_n - \rho\| \\
&= \|(x_n - \rho) - b_n^{p-1}(x_n - T_p^n x_n)\| \\
&\leq \|x_n - \rho\| + b_n^{p-1}(\|x_n - \rho\| + \|T_p^n x_n - \rho\|) \\
&\leq \|x_n - \rho\| + \nu_0(\|x_n - \rho\| + L(\|x_n - \rho\| + a_n)) \\
&\leq D + \nu_0(D + LD + La_n) \\
&< 2D. \\
\|{y_n}^{p-2} - \rho\| &= \|(1 - {b_n}^{p-2})x_n + {b_n}^{p-2}{T_{p-1}}^n y_n^{p-1} - \rho\| \\
&\leq \|x_n - \rho\| + b_n^{p-2}\|T_{p-1}^n y_n^{p-1} - x_n\| \\
&\leq \|x_n - \rho\| + \nu_0[L(\|y_n^{p-1} - \rho\| + a_n) + \|x_n - \rho\|)] \\
&\leq D + \nu_0(L(2D + a_n) + D) \\
&\leq 2D.
\end{aligned}$$

Recursively, we have $\|y_n^{i-1} - \rho\| \leq 2D$, Thus, $\|y_n^i - \rho\| \leq 2D$ for $i = 1, 2, ..., p-2$.

Also,

$$\begin{aligned}
\|x_{n+1} - {y_n}^i\| &\leq \|x_{n+1} - x_n\| + \|y_n - x_n\| \\
&\leq \|x_{n+1} - x_n\| + {b_n}^{p-2}\|T^n y_n^{p-1} - x_n\| \\
&\leq D(1+2L) + D(1+L) \\
&\leq D(2+3L).
\end{aligned}$$

But,

$$\begin{aligned}
\|x_n - T_i^n x_n\| &\leq \|x_n - \rho\| + \|T_i^n x_n - \rho\| \\
&\leq D + L(D + a_n) \\
&\leq D(1+L)
\end{aligned}$$

and

$$\begin{aligned}\|x_n - T_i^n y_n^i)\| &\leq \|x_n - \rho\| + \|T_i^n y_n^i - \rho\| \\ &\leq D + L(\|y_n^i - \rho\| + a_n) \\ &\leq D + L(2D + a_n) \\ &\leq D(1 + 2L).\end{aligned}$$

Then,

$$\begin{aligned}\|T^n x_{n+1} - T^n y^i{}_n\| &\leq L(\|x_{n+1} - y^i{}_n\| + a_n) \\ &\leq L[(\|y^i{}_n - x_n\| + \|x_{n+1} - x_n\| + a_n)] \\ &\leq b^i{}_n L((\|x_n - T_i^n x_n)\| + b_n\|x_n - T_i^n y_n^i\| + a_n) \\ &\leq b_n^i L((D(1+L) + b_n(D(1+2L)) + a_n \\ &\leq \nu_0(L(D(1+L) + D(1+2L) + a_n)) \\ &= \nu_0(2DL + 3DL^2) \\ &\leq \frac{\Phi(D)}{10D}.\end{aligned}$$

Applying Lemma(2.2) and the estimates above we obtain,

$$\begin{aligned}\|x_{n+1} - \rho\|^2 &= \|(1 - b_n)x_n + b_n T_1{}^n y_n^1 - \rho\|^2 \\ &= \|x_n - \rho - b_n(T_1^n y_n^1 - x_n)\|^2 \\ &\leq \|x_n - \rho\|^2 - 2b_n < T_1^n y_n^1 x_n, j(x_{n+1} - \rho) > \\ &= \|x_n - \rho\|^2 - 2b_n < T_1^n x_{n+1} - \rho, j(x_{n+1} - \rho) > \\ &\quad -2b_n < x_{n+1}, j(x_{n+1} - \rho) > \\ &\quad +\|2b_n < T_1^n y_n^1 - T_1^n x_{n+1}, j(x_{n+1} - \rho) > \\ &\quad +2b_n < x_{n+1} - x_n, j(x_{n+1} - \rho) > \\ &\leq \|x_n - \rho\|^2 + 2b_n(k_n\|x_n - \rho\|^2 - \Phi(\|x_{n+1} - \rho\|)) \\ &\quad -2b_n\|x_{n+1} - \rho\|^2 + 2b_n\|T_1^n y_n^1 - T_1^n x_{n+1}\|\|x_{n+1} - \rho\| \\ &\quad +2b_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ &= \|x_n - \rho\|^2 + 2b_n(k_n - 1)\|x_{n+1} - \rho\|^2 - 2b_n\Phi\|x_{n+1} - \rho\|) \\ &\quad +2b_n L(\|y_n^1 - x_{n+1}\| + a_n)\|x_{n+1} - \rho\| + 2b_n\|x_{n+1} - x_n\|\|x_{n+1} - \rho\| \\ &\leq \|x_n - \rho\|^2 + 2b_n(k_n - 1)\|x_n - \rho\|^2 - 2b_n\Phi(D) \\ &\quad +2b_n(\frac{\Phi(D)}{10D}) \times 2D + 2b_n(D(1+2L))) \times 2D \\ &\leq D^2 + 8b_n(k_n - 1)D^2 - 2b_n\Phi(D) \\ &\quad +D^2 + 8b_n(k_n - 1)D^2 - \frac{8}{5}b_n\Phi(D) + 4b_n D(D + 2L) \quad (2.4)\end{aligned}$$

Since $b_n \to 0$ and $(k_n - 1) \to 1$ as $n \to \infty$,then (2.4) becomes

$$\|x_n - \rho\|^2 \leq D^2$$

which is a contradiction. Hence, $x_{n+1} \in A_1$. Therefore, $\{x_n\}$ is bounded.

Next we prove that $\|x_n - \rho\| \to 0$ as $n \to \infty$. We have shown above that $\{\|x_n - \rho\|\}$ is a bounded sequence and so is $\{\|y_n^i - \rho\|\}$.
Let $R_0 = \sup_{n\geq}\{\|x_n - \rho\|\} + \sup\{\|y_n - \rho\|\}$.

But

$$\begin{aligned}\|x_{n+1} - y_n^i\| &\leq (\|y_n^i - x_n\| + \|x_{n+1} - x_n\|) \\ &\leq b_n^i(\|x_n - T_n^i x_n\|) + b_n\|x_n - T_i^n y_n^i\| \\ &\leq b_n^i(\|x_n - \rho\| + \|T_n^i x_n - \rho\|) + b_n(\|x_n - \rho + \|T_i^n y_n^i\|) \\ &\leq (b_n + b_n^i)M_0 + (b_n + b_n^i)(1 + L)M_0 \quad (2.5)\end{aligned}$$

Using Lemma(2.2),equations (2.4) and (2.5) we have

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &= \|(1-b_n)x_n + b_n T_1{}^n y_n^1 - \rho\|^2 \\
&= \|x_n - \rho - b_n(T_1^n y_n^1 - x_n)\|^2 \\
&\leq \|x_n-\rho\|^2 - 2b_n < T_1^n y_n^1 - \rho, j(x_{n+1}-\rho) > \\
&= \|x_n-\rho\|^2 - 2b_n < T_1^n x_{n+1} - \rho, j(x_{n+1}-\rho) > \\
&\quad -2b_n < x_{n+1}, j(x_{n+1}-\rho) > \\
&\quad +\|2b_n < T_1^n y_n^1 - T_1^n x_{n+1}, j(x_{n+1}-\rho) > \\
&\quad +2b_n < x_{n+1} - x_n, j(x_{n+1}-\rho) > \\
&\leq +2b_n(k_n)\|x_n-\rho\|^2 - \Phi(\|x_{n+1}-\rho\|)) \\
&\quad -2b_n\|x_{n+1}-\rho\|^2 + 2b_n\|T_1^n y_n^1 - T_1^n x_{n+1}\|\|x_{n+1}-\rho\| \\
&\quad +2b_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
&= \|x_n-\rho\|^2 + 2b_n(k_n-1)\|x_{n+1}-\rho\|^2 - 2b_n\Phi(x_{n+1}-\rho\|) \\
&\quad +2b_n L(\|y_n^1 - x_{n+1}\| + a_n)\|x_{n+1}-\rho\| + 2b_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\
&\leq \|x_n-\rho\|^2 + 2b_n(k_n-1)R_0^2 - 2b_n\Phi(\|x_{n+1}-\rho\|) \\
&\quad +2b_n L(b_n + b_n^i)R_0 + (b_n+b_n^i)(1+L)R_0) + a_n)R_0 + 4b_nR^2 \quad (2.6) \\
&\leq \|x_n-\rho\|^2 - b_n\Phi(\|x_{n+1}-\rho\|) + B_n
\end{aligned}$$

Where

$$\begin{aligned}
B_n &= 2b_n(k_n-1)R_0^2 + 2b_nL(b_n+b_n^i)R_0 \\
&\quad +(b_n+b_n^i)(1+L)R_0 + a_n)R_0 + 4b_nR^2.
\end{aligned}$$

Taking $d_n = \|x_n-\rho\|^2, e_n = b_n$, and $h_n = B_n$. Then (2.6) becomes

$$d_{n+1}^2 \leq d_n^2 - e_n\Phi(d_{n+1}) + h_n, \forall n \geq N_0.$$

Therefore, by Lemma 2.3, we obtain $\lim \to \infty\, d_n = 0$. Hence $x_n \to \rho$ as $n \to \infty$.This completes the proof. Also, using Lemma 2.3 and the conditions of the parameters, we obtain:
$d_n \to 0$ as $n \to \infty$. This ends the proof.

We make the following remarks: (1) Clearly,Is it possible to drop the continuity condition in Theorem 1.5 and extend to a finite family of nearly p- uniformly L- Lipschitzian asymptotically pseudocontractive mappings?
(2) we have dropped the continuity condition in Theorem 1.6 and show that the modified multi-step converges to the common fixed point of T?
Corollary 2.5 The result in Theorem 2.4 is also true for two and three nearly L_i - uniformly Lipschitzian asymptotically pseudocontractive mappings which extends the work of Sahu[14].

Corollary 2.6 Let K be a nonempty closed convex subset of a real Banach s-pace E, $T_i : K \to K, (i = 1, 2, ..., p,\ p \geq 2)$ be p uniformly L_i Lipschitzian mappings with sequence $\{k_n\}_{n\geq 0} \subset [1,\infty), k_n \to 1$ and $\sum_{n\geq 0}(k_n - 1) < \infty$ such that $\rho \in \cap_{i=1}^{p\geq 2} F(T_i) \neq \phi$. Let $\{b_n\}_{n\geq 0}, \{b_n^i\}_{n\geq 0}$ and $\{b_n^{i+1}\}_{n\geq 0}$ be the real sequences in [0,1] satisfying:

$(i) b_n, b_n^i, b_n^{i+1} \to 0$, as $n \to \infty, (i = 1, 2, ..., p - 2)$.
$(ii) \sum_{n \geq 0} b_n = \infty$.
For any $x_0 \in K$, define $\{x_n\}_{n \geq 0}$ by the iterative process (2.1). Suppose there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty), \Phi(0) = 0$ such that

$$< {T_i}^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|) \tag{2.7}$$

$\forall x \in K, \ (i = 1, 2, ..., p, \ p \geq 2)$. Then $\{x_n\}_{n \geq 0}$ converges strongly to $\rho \in \cap_{i=1}^{p \geq 2} F(T_i)$.

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