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SOME NEW OSTROWSKI TYPE INEQUALITIES VIA FRACTIONAL INTEGRALS

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Abstract. We have found a new version of well known Ostrowski inequality in a very simple and antique way via Riemann-Liouville fractional integrals. Also some related results have been derived.

1. Introduction

Ostrowski Type Inequalities Via Riemann-Liouville Fractional Integrals

Ostrowski Inequality. In 1938, the following celebrated inequality was established by Ostrowski [11].

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$ where I is an interval in $\mathbb{R}$, be a mapping differentiable in I°, the interior of I and $a, b \in I^\circ$, $a < b$. If $|f'(t)| \leq M$, for all $t \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right](b-a)M, \tag{1.1}$$

for $x \in [a, b]$.

It is well known as Ostrowski inequality and its consideration by a lot of mathematicians reflects importance and motivation.

In fact Ostrowski inequality plays a vital role while studying the error bounds of different numerical quadrature rules for example mid point's, trapezoidal's, Simpson's and other generalized Riemann type. It also motivated the researchers to find its refinements, generalizations, extensions and their applications (see, [1–5, 12] and references therein).

Riemann-Liouville Fractional Integral Operators. Fractional calculus deals with the study of integral and differential operators of non-integral order. Many mathematicians like Liouville, Riemann and Weyl made major contributions to the theory of fractional calculus. The study on the fractional calculus continued with contributions from Fourier, Abel, Lacroix, Leibniz, Grunwald and Letnikov, (for details see, [6, 8, 10]). Riemann-Liouville fractional integral operator is the first formulation of an integral operator of non-integral order.

Definition 1.1. *[14] Let $f \in L_1[a, b]$. Then the Riemann-Liouville fractional integrals of f of order $\alpha > 0$ with $a \geq 0$ is defined by*

$$I_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \ x > a$$

and

$$I_{b_-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \ \ x < b.$$

In fact these formulations of fractional integral operators have been established due to Letnikov [9], Sonin [13] and then by Laurent [7].

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Fractional Ostrowski Type Inequalities. Remaining within the assumptions of Ostrowski inequality following more general inequality is observed.

Theorem 1.2. *Under the assumptions of Theorem 1.1 we have*

$$\left|f(x)\left((b-x)^{\beta}+(x-a)^{\alpha}\right)-\left(\Gamma(\beta+1)I_{b-}^{\beta}f(x)+\Gamma(\alpha+1)I_{a+}^{\alpha}f(x)\right)\right| \le M\left(\frac{\beta}{\beta+1}(b-x)^{\beta+1}+\frac{\alpha}{\alpha+1}(x-a)^{\alpha+1}\right),\ x\in[a,b] \tag{1.2}$$

where $\alpha,\beta>0$.

Proof. For $t\in[a,x],\alpha>0$ we have

$$(x-t)^{\alpha}\le(x-a)^{\alpha}. \tag{1.3}$$

Under given condition on f' and by (1.3) we have

$$\int_a^x(M-f'(t))(x-t)^{\alpha}dt\le(x-a)^{\alpha}\int_a^x(M-f'(t))dt$$

and

$$\int_a^x(M+f'(t))(x-t)^{\alpha}dt\le(x-a)^{\alpha}\int_a^x(M+f'(t))dt.$$

Integrating and simplifying the calculations we obtain the following inequalities

$$f(x)(x-a)^{\alpha}-\Gamma(\alpha+1)I_{a+}^{\alpha}f(x)\le\frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}$$

and

$$\Gamma(\alpha+1)I_{a+}^{\alpha}f(x)-f(x)(x-a)^{\alpha}\le\frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}.$$

Above inequalities result the following inequality

$$|f(x)(x-a)^{\alpha}-\Gamma(\alpha+1)I_{a+}^{\alpha}f(x)|\le\frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}. \tag{1.4}$$

Now on the other hand for $t\in[x,b],\beta>0$ we have

$$(t-x)^{\beta}\le(b-x)^{\beta}. \tag{1.5}$$

Under given condition on f' and by (1.5) we have

$$\int_x^b(M-f'(t))(t-x)^{\beta}dt\le(b-x)^{\beta}\int_x^b(M-f'(t))dt$$

and

$$\int_x^b(M+f'(t))(t-x)^{\beta}dt\le(b-x)^{\beta}\int_x^b(M+f'(t))dt.$$

Integrating and simplifying the calculations we obtain the following inequalities

$$f(x)(b-x)^{\beta}-\Gamma(\beta+1)I_{b-}^{\beta}f(x)\le\frac{M\beta}{\beta+1}(b-x)^{\beta+1}$$

and

$$\Gamma(\beta+1)I_{b-}^{\beta}f(x)-f(x)(b-x)^{\beta}\le\frac{M\beta}{\beta+1}(b-x)^{\beta+1}.$$

Above inequalities result the following inequality

$$\left|f(x)(b-x)^{\beta}-\Gamma(\beta+1)I_{b-}^{\beta}f(x)\right|\le\frac{M\beta}{\beta+1}(b-x)^{\beta+1}. \tag{1.6}$$

By adding (1.4) and (1.6) we get (1.2). □

The following more general result for a differentiable function which is bounded below as well as bounded above holds.

Theorem 1.3. *Let $f : I \longrightarrow \mathbb{R}$ where I is an interval in $\mathbb{R}$, be a mapping differentiable in I°, the interior of I and $a, b \in I^\circ$, $a < b$. If $m < f'(t) \leq M$ for all $t \in [a, b]$, then we have*

$$\left((x-a)^\alpha - (b-x)^\beta\right) f(x) - \Big(\Gamma(\alpha+1) I^\alpha_{a+} f(x) - \Gamma(\beta+1) I^\beta_{b-} f(x)\Big)$$
$$\leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1} - \frac{m\beta}{\beta+1}(b-x)^{\beta+1},\ x \in [a,b]$$

and

$$\left((b-x)^\beta - (x-a)^\alpha\right) f(x) + \Big(\Gamma(\alpha+1) I^\alpha_{a+} f(x) - \Gamma(\beta+1) I^\beta_{b-} f(x)\Big)$$
$$\leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1} - \frac{m\alpha}{\alpha+1}(x-a)^{\alpha+1},\ x \in [a,b],$$

where $\alpha, \beta > 0$

Proof. Proof is on the same lines just after comparing conditions on derivative of f, of the proof of Theorem 1.2, let we left it for the reader. □

In the following we have obtained a related result to fractional Ostrowski inequality (1.2).

Theorem 1.4. *Under the assumptions of Theorem 1.1 we have*

$$\left|\left((b-x)^\beta f(b) + (x-a)^\alpha f(a)\right) - \Big(\Gamma(\beta+1) I^\beta_{x+} f(b) + \Gamma(\alpha+1) I^\alpha_{x-} f(a)\Big)\right| \tag{1.7}$$
$$\leq M\left(\frac{\beta}{\beta+1}(b-x)^{\beta+1} + \frac{\alpha}{\alpha+1}(x-a)^{\alpha+1}\right),\ x \in [a,b]$$

where $\alpha, \beta > 0$.

Proof. For $t \in [a, x], \alpha > 0$ we have

$$(t-a)^\alpha \leq (x-a)^\alpha. \tag{1.8}$$

Under given condition on f' and by (1.8) we have

$$\int_a^x (M - f'(t))(t-a)^\alpha dt \leq (x-a)^\alpha \int_a^x (M - f'(t)) dt$$

and

$$\int_a^x (M + f'(t))(t-a)^\alpha dt \leq (x-a)^\alpha \int_a^x (M + f'(t)) dt.$$

Integrating and simplifying the calculations we obtain the following inequalities

$$\Gamma(\alpha+1) I^\alpha_{x-} f(a) - f(a)(x-a)^\alpha \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}$$

and

$$f(a)(x-a)^\alpha - \Gamma(\alpha+1) I^\alpha_{x-} f(a) \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}.$$

Above inequalities result the following inequality

$$|f(a)(x-a)^\alpha - \Gamma(\alpha+1) I^\alpha_{x-} f(a)| \leq \frac{M\alpha}{\alpha+1}(x-a)^{\alpha+1}. \tag{1.9}$$

Now on the other hand for $t \in [x, b], \beta > 0$ we have

$$(b-t)^\beta \leq (b-x)^\beta. \tag{1.10}$$

Under given condition on f' and by (1.10) we have

$$\int_x^b (M - f'(t))(b-t)^\beta dt \leq (b-x)^\beta \int_x^b (M - f'(t)) dt$$

and

$$\int_x^b (M + f'(t))(b-t)^\beta dt \leq (b-x)^\beta \int_x^b (M + f'(t)) dt.$$

Integrating and simplifying the calculations we obtain the following inequalities

$$f(b)(b-x)^\beta - \Gamma(\beta+1) I^\beta_{x+} f(b) \leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}$$

and

$$\Gamma(\beta+1)I^{\beta}_{x^+}f(b)-f(b)(b-x)^{\beta}\leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}.$$

Above inequalities result the following inequality

$$\left|f(b)(b-x)^{\beta}-\Gamma(\beta+1)I^{\beta}_{x^+}f(b)\right|\leq \frac{M\beta}{\beta+1}(b-x)^{\beta+1}. \tag{1.11}$$

By adding (1.9) and (1.11) we get (1.7). □

Some Implications. Following implications have been observed.

Corollary 1.1. *If β takes value α in (1.2), then we leads the following fractional Ostrowski inequality*

$$\begin{aligned}&\left|f(x)\left((b-x)^{\alpha}+(x-a)^{\alpha}\right)-\Gamma(\alpha+1)\left(I^{\alpha}_{b^-}f(x)+I^{\alpha}_{a^+}f(x)\right)\right|\\ &\leq M\frac{\alpha}{\alpha+1}\left((b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right),\ x\in[a,b],\end{aligned}$$

where $\alpha>0$.

Corollary 1.2. *If β and α simultaneously take value 1, then we lead to the Ostrowski inequality (1.1).*

Corollary 1.3. *If β takes value α in Theorem 1.4, then we lead to the following inequality*

$$\begin{aligned}&\left|\left((b-x)^{\alpha}f(b)+(x-a)^{\alpha}f(a)\right)-\Gamma(\alpha+1)\left(I^{\alpha}_{x^+}f(b)+I^{\alpha}_{x^-}f(a)\right)\right|\\ &\leq \frac{M\alpha}{\alpha+1}\left((b-x)^{\alpha+1}+(x-a)^{\alpha+1}\right),\ x\in[a,b],\end{aligned}$$

where $\alpha>0$.

Remark 1.1. *Following the steps of the proof of Theorem 1.2 line by line with $\alpha=\beta=1$, an alternative proof of the Ostrowski inequality is followed (see, [5]).*

Remark 1.2. *If m is replaced with $-M$ in Theorem 1.3, then with some rearrangements one can get Theorem 1.2.*

Remark 1.3. *A more general form of Theorem 1.4 like Theorem 1.3 for a differentiable function which is bounded below as well as bounded above holds which we leave for reader.*

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References

[1] P. Cerone, S. S. Dragomir, Midpoint-type rules from an inequalities point of view, handbook of analytic-computational methods in applied mathematics, Editor: G. Anastassiou, CRC Press, New York, 2000.

[2] S. S. Dragomir, Ostrowski-type inequalities for Lebesgue integral: A survey of recent results, Aust. J. Math. Anal. Appl., 14 (1) (2017), 1-287.

[3] S. S. Dragomir, T. M. Rassias, (Eds.) Ostrowski-type inequalities and applications in numerical integration, Kluwer academic publishers, Dordrecht, Boston, London, 2002.

[4] S. S. Dragomir, S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and some numerical quadrature rules, Comput. Math. Appl., 33 (1997), 15-20.

[5] G. Farid, Straightforward proofs of Ostrowski inequality and some related results, Int. J. Anal. 2016 (2016), Article ID 3918483.

[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and applications of fractional differential equations, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.

[7] H. Laurent, Sur le calcul des derivees a indicies quelconques, Nouv. Annales de Mathematiques., 3 (3) (1884), 240-252.

[8] M. Lazarević, Advanced topics on applications of fractional calculus on control problems, System stability and modeling, WSEAS Press, 2014.

[9] A. V. Letnikov, Theory of differentiation with arbitray pointer (Russian), Matem. Sbornik., 3 (1868), 1-66.

[10] K. Miller, B. Ross, An introduction to the fractional calculus and fractional differential equations, John, Wiley and sons Inc, New York, 1993.

[11] A. Ostrowski, Über die Absolutabweichung einer dierentierbaren Funktion von ihren Integralmittelwert, Comment. Math. Helv., 10 (1938), 226–227.

[12] X. Qiaoling, Z. Jian, L. Wenjun, A new generalization of Ostrowski-type inequality involving functions of two independent variables, Comput. Math. Appl., 60 (2010), 2219–2224.

[13] N. Y. Sonin, On differentiation with arbitray index, Moscow Matem. Sbornik., 6 (1) (1869), 1-38.
[14] Z. Tomovski, R. Hiller, H. M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler function, Integral Transforms Spec. Funct., 21 (11) (2010), 797-814.

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