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SUFFICIENT CONDITIONS FOR THE OSCILLATION OF ODD-ORDER NEUTRAL DYNAMIC EQUATIONS

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Abstract. In this paper, we study oscillation and asymptotic behaviour of odd-order delay dynamic equations. We first state an oscillation test for odd-order nonneutral equations, then by comparison we provide sufficient conditions for all solutions of neutral equations to be oscillatory or tend to zero depending on two main ranges of the neutral coefficient.

1. Introduction

In this paper, we will study the oscillation of solutions to the higher-order delay dynamic equations of the form

$$\big[x(t)+A(t)x\big(\alpha(t)\big)\big]^{\Delta^{n}}+B(t)x\big(\beta(t)\big)=0\quad\text{for } t\in[t_{0},\infty)_{\mathbb{T}},\tag{1.1}$$

where $n\in\mathbb{N}$, $\mathbb{T}$ is a time scale unbounded above, $t_0\in\mathbb{T}$, $A\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, $B\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,\infty)_{\mathbb{R}})$, and $\alpha,\beta\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ are unbounded nondecreasing functions such that $\alpha(t),\beta(t)\leq t$ for all $t\in[t_0,\infty)_{\mathbb{T}}$. We also assume when necessary that $\alpha\in\mathrm{C}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ has an inverse $\alpha^{-1}\in\mathrm{C}([\alpha(t_0),\infty)_{\mathbb{T}},[t_0,\infty)_{\mathbb{T}})$. We will confine our attention to the following ranges of the coefficient A.

(R1) $A\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[0,1]_{\mathbb{R}})$ with $\limsup_{t\to\infty}A(t)<1$.

(R2) $A\in\mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},[-1,0]_{\mathbb{R}})$ with $\liminf_{t\to\infty}A(t)>-1$.

The oscillation problem for dynamic equations on time scales has attracted a lot of attention immediately after the discovery of time scale calculus. Although there are several such works in the literature, the majority is focuses on second-order equations. An important reason for this is due to lack of technical inequalities which combines higher-order derivatives with lower-order ones. In this paper, we will generalize and improve the technique by Das [7] employed for differential equations. For some works on higher-order dynamic equations we refer the readers to [8–14, 17, 18, 20, 22, 23]. We will be giving comparison tests for the oscillation of higher-order delay dynamic equations with first-order delay dynamic equations. As we will be making comparison with first-order delay dynamic equations, we find useful to cite the following references [2, 3, 6, 16, 19, 24], where the authors study oscillation of all solutions of first-order delay dynamic equations.

To give an exact definition of a solution of delay dynamic equation (1.1), we need to define $t_{-1}:=\min\{\alpha(t_0),\beta(t_0)\}$.

Definition 1.1 (Solution). *A function $x:[t_{-1},\infty)_{\mathbb{T}}\to\mathbb{R}$, which is rd-continuous on $[t_{-1},t_0]_{\mathbb{T}}$ and $x+A\cdot x\circ\alpha$ is n-times rd-continuously Δ-differentiable on $[t_0,\infty)$, is called a solution of (1.1) provided that it satisfies the functional delay equation (1.1) identically on $[t_0,\infty)$.*

It can be shown as in [15] that (1.1) admits a unique solution, which exists on the entire interval $[t_{-1},\infty)_{\mathbb{T}}$, when an initial function $\varphi:[t_{-1},t_0]_{\mathbb{T}}\to\mathbb{R}$, which is n-times rd-continuously Δ-differentiable, is provided. More precisely, we mean in the equation that $x^{\Delta^{j}}(t)=\varphi^{\Delta^{j}}(t)$ for $t\in[t_{-1},t_0]_{\mathbb{T}}$ and $j=0,1,\cdots,n$.

Definition 1.2 (Oscillation). *A solution x of (1.1) is called nonoscillatory if there exists $s\in[t_0,\infty)_{\mathbb{T}}$ such that x is either positive or negative on $[s,\infty)$. Otherwise, the solution is said to oscillate (or is called oscillatory).*

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This paper is constructed in the following setting. § 2 includes some fundamental results on time scale polynomials and nonoscillatory functions (§ 2.1). In this section (§ 2.2), we also quote some results from the recent paper [17]. In § 3.1, we provide some comparison theorems on the qualitative behaviour of higher-order delay dynamic equations when the neutral term is absent (i.e., $A(t)\equiv 0$ for $t\in[t_0,\infty)_{\mathbb{T}}$), and in § 3.2, we extend these results to higher-order neutral delay dynamic equations together with some illustrative general examples.

2. Auxiliary lemmas

In this section, we will form the background for the proof of our main result.

2.1. Time scales.

This section is dedicated to some general results on time scales.

In the sequel, we introduce the definition of the generalized polynomials on time scales (see [1, Lemma 5] and/or [4, § 1.6]) $\mathrm{h}_k\in\mathrm{C}(\mathbb{T}\times\mathbb{T},\mathbb{R})$ as follows:

$$\mathrm{h}_k(t,s):=\begin{cases}1, & k=0\\ \displaystyle\int_s^t \mathrm{h}_{k-1}(\eta,s)\Delta\eta, & k\in\mathbb{N}\end{cases}\quad\text{for } s,t\in\mathbb{T}. \tag{2.1}$$

Note that, for all $s,t\in\mathbb{T}$ and all $k\in\mathbb{N}_0$, the function h_k satisfies

$$\mathrm{h}_k^{\Delta_1}(t,s)=\begin{cases}0, & k=0\\ \mathrm{h}_{k-1}(t,s), & k\in\mathbb{N}\end{cases}\quad\text{for } s,t\in\mathbb{T}. \tag{2.2}$$

In particular, for $\mathbb{T}=\mathbb{Z}$, we have $\mathrm{h}_k(t,s)=(t-s)^{(k)}/k!$ for all $s,t\in\mathbb{Z}$ and $k\in\mathbb{N}_0$, where ${}^{(\cdot)}$ is the usual factorial function, and for $\mathbb{T}=\mathbb{R}$, we have $\mathrm{h}_k(t,s)=(t-s)^k/k!$ for all $s,t\in\mathbb{R}$ and $k\in\mathbb{N}_0$.

Property 2.1 ([14, Property 1])**.** *By using induction and (2.1), it is easy to see for all $k\in\mathbb{N}_0$ that $\mathrm{h}_k(\cdot,s)\geq 0$ on $[s,\infty)_{\mathbb{T}}$ and $(-1)^k\mathrm{h}_k(\cdot,s)\geq 0$ on $(-\infty,s]_{\mathbb{T}}$. In view of (2.2), for all $k\in\mathbb{N}$, $\mathrm{h}_k(\cdot,s)$ is increasing on $[s,\infty)_{\mathbb{T}}$, and $(-1)^k\mathrm{h}_k(\cdot,s)$ is decreasing on $(-\infty,s]_{\mathbb{T}}$.*

Below, we give two lemmas related to the generalized polynomials on time scales.

Lemma 2.1 (Taylor's formula [4, Theorem 1.113])**.** *If $n\in\mathbb{N}$, $s\in\mathbb{T}$ and $f\in\mathrm{C}^n_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, then*

$$f(t)=\sum_{k=0}^{n-1}\mathrm{h}_k(t,s)f^{\Delta^k}(s)+\int_s^t\mathrm{h}_{n-1}\big(t,\sigma(\eta)\big)f^{\Delta^n}(\eta)\Delta\eta\quad\textit{for } t\in\mathbb{T}.$$

Lemma 2.2 ([17, Lemma 2])**.** *If $k\in\mathbb{N}$, $\ell\in\mathbb{N}_0$ and $s\in\mathbb{T}$, then*

$$\mathrm{h}_{k+\ell}(t,s)=\int_s^t\mathrm{h}_{k-1}\big(t,\sigma(\eta)\big)\mathrm{h}_\ell(\eta,s)\Delta\eta\quad\textit{for } t\in\mathbb{T}.$$

As an immediate consequence of Lemma 2.2, we can give the following alternative definition of the generalized polynomials:

$$\mathrm{h}_k(t,s):=\begin{cases}1, & k=0\\ \displaystyle\int_s^t \mathrm{h}_{k-1}\big(t,\sigma(\eta)\big)\Delta\eta, & k\in\mathbb{N}\end{cases}\quad\text{for } s,t\in\mathbb{T}. \tag{2.3}$$

The following is the main tool for studying qualitative properties of higher-order dynamic equations.

Lemma 2.3 (Kiguradze's lemma [1, Theorem 5])**.** *Let $\sup\mathbb{T}=\infty$, $n\in\mathbb{N}$ and $f\in\mathrm{C}^n_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$. Suppose that either $f^{\Delta^n}\geq 0(\not\equiv 0)$ or $f^{\Delta^n}\leq 0(\not\equiv 0)$ on $[t_0,\infty)_{\mathbb{T}}$. Then, there exist $s\in[t_0,\infty)_{\mathbb{T}}$ and $m\in[0,n)_{\mathbb{Z}}$ such that $(-1)^{n-m}f^{\Delta^n}(t)\geq 0$ for all $t\in[s,\infty)_{\mathbb{T}}$. Moreover, the following assertions hold.*

(i) $f^{\Delta^k}(t)>0$ holds for all $t\in[s,\infty)_{\mathbb{T}}$ and all $k\in[0,m)_{\mathbb{Z}}$.
(ii) $(-1)^{m+k}f^{\Delta^k}(t)>0$ holds for all $t\in[s,\infty)_{\mathbb{T}}$ and all $k\in[m,n)_{\mathbb{Z}}$.

A basic result on higher-order derivatives of a function is quoted below.

Lemma 2.4 ([1, Lemma 7])**.** *If $\sup\mathbb{T}=\infty$, $n\in\mathbb{N}$ and $f\in\mathrm{C}^n_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, then the following conditions are true.*

(i) $\liminf_{t\to\infty}f^{\Delta^n}(t)>0$ implies $\lim_{t\to\infty}f^{\Delta^k}(t)=\infty$ for all $k\in[0,n)_{\mathbb{Z}}$.

(ii) $\limsup_{t\to\infty} f^{\Delta^n}(t) < 0$ *implies* $\lim_{t\to\infty} f^{\Delta^k}(t) = -\infty$ *for all* $k \in [0,n)_{\mathbb{Z}}$.

The following can be easily obtained from the previous result.

Corollary 2.1 ([10, Corollary 2.10])**.** *If* $\sup \mathbb{T} = \infty$*,* $n \in \mathbb{N}$ *and* $f \in \mathrm{C}^n_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+_0)$*, then*

$$\lim_{t\to\infty} f^{\Delta^k}(t) = 0, \ \forall k \in (m,n)_{\mathbb{Z}},$$

where $m \in [0,n)_{\mathbb{Z}}$ *is the key number in Kiguradze's lemma.*

The following result is the key tool of this paper.

Lemma 2.5 (Cf. [7, Lemma 1])**.** *Assume that* $\sup \mathbb{T} = \infty$*,* $n \in \mathbb{N}$ *be odd and* $f \in \mathrm{C}^n_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+_0)$ *with* $f^{\Delta^n} \leq 0 (\not\equiv 0)$ *on* $[t_0,\infty)_{\mathbb{T}}$*. If Kiguradze's lemma holds with* $m \in [0,n)_{\mathbb{Z}}$*, then*

$$f(r) \geq \mathrm{h}_m(r,s)\mathrm{h}_{n-m-1}(r,t) f^{\Delta^{n-1}}(t) \quad \textit{for all } t \in [r,\infty)_{\mathbb{T}}, \tag{2.4}$$

where $r \in [s,\infty)_{\mathbb{T}}$.

Proof. By Lemma 2.3 (i), $r \in [s,\infty)_{\mathbb{T}}$ implies $f^{\Delta^j}(r) > 0$ for all $j \in [0,m)_{\mathbb{Z}}$ and $f^{\Delta^{m+1}}(r) \leq 0$, where $m \in [0,n)_{\mathbb{Z}}$ is even. Using Property 2.1, Lemma 2.2 and Taylor's formula, we get

$$\begin{aligned} f(r) &= \sum_{k=0}^{m-1} \mathrm{h}_k(r,s) f^{\Delta^k}(s) + \int_s^r \mathrm{h}_{m-1}\big(r,\sigma(\eta)\big) f^{\Delta^m}(\eta)\Delta\eta \\ &\geq \left(\int_s^r \mathrm{h}_{m-1}\big(r,\sigma(\eta)\big)\Delta\eta\right) f^{\Delta^m}(t) = \mathrm{h}_m(r,s) f^{\Delta^m}(r) \end{aligned} \tag{2.5}$$

for all $r \in [s,\infty)_{\mathbb{T}}$. By Lemma 2.3 (i), $t \in [r,\infty)_{\mathbb{T}}$ implies $(-1)^{m+k} f^{\Delta^k}(t) > 0$ for all $k \in [m,n)_{\mathbb{Z}}$ and $f^{\Delta^n}(t) \leq 0$. Using Property 2.1, (2.3) and Taylor's formula, we get

$$\begin{aligned} f^{\Delta^m}(r) &= \sum_{k=0}^{n-m-1} \mathrm{h}_k(r,t) f^{\Delta^{m+k}}(t) + \int_t^r \mathrm{h}_{n-m-2}\big(r,\sigma(\eta)\big) f^{\Delta^{n-1}}(\eta)\Delta\eta \\ &= \sum_{k=0}^{n-m-1} (-1)^k \mathrm{h}_k(r,t)(-1)^{m+k} f^{\Delta^{m+k}}(t) + \int_r^t (-1)^{n-m-2}\mathrm{h}_{n-m-2}\big(r,\sigma(\eta)\big) f^{\Delta^{n-1}}(\eta)\Delta\eta \\ &\geq \left(\int_r^t (-1)^{n-m-2}\mathrm{h}_{n-m-2}\big(r,\sigma(\eta)\big)\Delta\eta\right) f^{\Delta^{n-1}}(t) \\ &= \left(\int_t^r \mathrm{h}_{n-m-2}\big(r,\sigma(\eta)\big)\Delta\eta\right) f^{\Delta^{n-1}}(t) = \mathrm{h}_{n-m-1}(r,t) f^{\Delta^{n-1}}(t) \end{aligned} \tag{2.6}$$

for all $t \in [r,\infty)_{\mathbb{T}}$. Using (2.5) and (2.6), we arrive at (2.4), which completes the proof. ☐

2.2. **Recent results.** Below, we quote two fundamental results from [17], which will be required in the sequel.

Theorem 2.2 ([17, Theorem 2 (ii)])**.** *Assume that (R1) holds and* $n \in \mathbb{N}$ *is odd. If* (1.1) *has a nonoscillatory solution* x *with* $\limsup_{t\to\infty} |x(t)| > 0$*, then so does*

$$x^{\Delta^n}(t) + \big[1 - A\big(\alpha(t)\big)\big] B(t) x\big(\beta(t)\big) = 0 \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

Theorem 2.3 ([17, Theorem 3])**.** *Assume that (R2) holds and* $n \in \mathbb{N}$*. If* (1.1) *has a nonoscillatory solution* x *with* $\limsup_{t\to\infty} |x(t)| > 0$*, then so does*

$$x^{\Delta^n}(t) + B(t) x\big(\beta(t)\big) = 0 \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}.$$

3. Main results

Now, we can study oscillation and asymptotic behaviour of higher-order delay dynamic equations.

3.1. **Nonneutral equations.** In this section, we will focus on the higher-order delay dynamic equation

$$x^{\Delta^{n}}(t)+B(t)x\big(\beta(t)\big)=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.1}$$

Our first comparison result is the following, which presents results on oscillation.

Theorem 3.1. *Assume that $n\in\mathbb{N}$ is odd with $n\geq 3$, and there exist two functions $\delta,\gamma\in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ satisfying $\delta(t)\leq\beta(t)\leq\gamma(t)\leq t$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ such that*

$$x^{\Delta}(t)+B(t)\min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=even}}\Big\{\mathrm{h}_k\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-k-1}\big(\beta(t),\gamma(t)\big)\Big\}x\big(\gamma(t)\big)=0\quad \textit{for } t\in[t_0,\infty)_{\mathbb{T}} \tag{3.2}$$

is oscillatory. Then, every solution of (3.1) is oscillatory.

Proof. Assume the contrary that x is an eventually positive solution of (3.1). Then, there exists $t_1\in[t_0,\infty)_{\mathbb{T}}$ such that $x(t)$, $x\big(\beta(t)\big)>0$ for all $t\in[t_1,\infty)_{\mathbb{T}}$. By Kiguradze's lemma, there exist $t_2\in[t_1,\infty)_{\mathbb{T}}$ and an even integer $m\in[0,n)_{\mathbb{Z}}$ such that for all $t\in[t_2,\infty)_{\mathbb{T}}$, we have $x^{\Delta^{k}}(t)>0$ for all $k\in[0,m)_{\mathbb{Z}}$ and $(-1)^{m+k}x^{\Delta^{k}}(t)>0$ for all $k\in[m,n)_{\mathbb{Z}}$. In particular, we have $x^{\Delta^{n-1}}>0$ on $[t_2,\infty)_{\mathbb{T}}$. It follows from Lemma 2.5 that

$$x\big(\beta(t)\big)\geq \mathrm{h}_m\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-m-1}\big(\beta(t),\gamma(t)\big)x^{\Delta^{n-1}}\big(\gamma(t)\big)\quad\text{for all } t\in[t_3,\infty)_{\mathbb{T}}, \tag{3.3}$$

where $t_3\in[t_2,\infty)_{\mathbb{T}}$ satisfies $\delta(t)\geq t_2$ for all $t\in[t_3,\infty)_{\mathbb{T}}$. Substituting (3.3) into (3.2), we see that $x^{\Delta^{n-1}}$ is an eventually positive solution of

$$y^{\Delta}(t)+B(t)\mathrm{h}_m\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-m-1}\big(\beta(t),\gamma(t)\big)\big\}y\big(\gamma(t)\big)\leq 0\quad\text{for all } t\in[t_3,\infty)_{\mathbb{T}}$$

or

$$y^{\Delta}(t)+B(t)\min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=\text{even}}}\Big\{\mathrm{h}_k\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-k-1}\big(\beta(t),\gamma(t)\big)\Big\}y\big(\gamma(t)\big)\leq 0\quad\text{for all } t\in[t_3,\infty)_{\mathbb{T}}. \tag{3.4}$$

By [5, Theorem 3.1 and Corollary 4.2], (3.2) admits an eventually positive solution too. This is a contradiction and the proof is complete. ☐

Depending on the oscillation test to be applied for the first-order delay equation, the functions δ and γ can be chosen appropriately as it is illustrated by an example below. Before we proceed, let us recall a well-known result due to Myškis [21], which ensures that the first-order delay differential equation

$$x'(t)+B(t)x\big(\beta(t)\big)=0\quad\text{for } t\in[t_0,\infty)_{\mathbb{R}}$$

is oscillatory if

$$\limsup_{t\to\infty}\big(t-\beta(t)\big)<\infty\quad\text{and}\quad\liminf_{t\to\infty}\big(t-\beta(t)\big)\liminf_{t\to\infty}B(t)>\frac{1}{\mathrm{e}}.$$

Now, we can give a simple example to explain how to choose appropriately the functions δ and γ.

Example 3.1. *Let $\mathbb{T}=\mathbb{R}$, and consider the following odd-order delay differential equation*

$$x^{(n)}(t)+B(t)x(t-\beta_0)=0\quad\textit{for } t\in[t_0,\infty)_{\mathbb{R}}, \tag{3.5}$$

where $n\geq 3$ is an odd integer, $\beta_0\in\mathbb{R}^{+}$ and $B\in\mathrm{C}([t_0,\infty)_{\mathbb{R}},\mathbb{R}_0^{+})$. Choosing $\delta(t)=t-\delta_0$ and $\gamma(t)=t-\gamma_0$, where $\delta_0\geq\beta_0\geq\gamma_0$, we get

$$\begin{aligned}\min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=even}}\left\{\frac{\big((t-\beta_0)-(t-\delta_0)\big)^{k}}{k!}\frac{\big((t-\beta_0)-(t-\gamma_0)\big)^{n-k-1}}{(n-k-1)!}\right\}&=\min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=even}}\left\{\frac{(\delta_0-\beta_0)^{k}(\beta_0-\gamma_0)^{n-k-1}}{k!(n-k-1)!}\right\}\\ &=\frac{\big(\min\{\delta_0-\beta_0,\beta_0-\gamma_0\}\big)^{n-1}}{(n-1)!}.\end{aligned}$$

The last expression is maximized if $\delta_0-\beta_0=\beta_0-\gamma_0$. So, by letting $\delta_0=\beta_0+\alpha$ and $\gamma_0=\beta_0-\alpha$, where $\beta_0\geq\alpha\geq 0$, we obtain the dynamic equation

$$x'(t)+\frac{B(t)\alpha^{n-1}}{(n-1)!}x\big(t-(\beta_0-\alpha)\big)=0\quad\textit{for } t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.6}$$

Due to the result by Myškis (quoted above), every solution of ((3.6) and hence) (3.5) is oscillatory if

$$\liminf_{t\to\infty} \frac{B(t)\alpha^{n-1}(\beta_0-\alpha)}{(n-1)!} > \frac{1}{\mathrm{e}}. \tag{3.7}$$

Now, define the function

$$f(\lambda) = \lambda^{n-1}(\beta_0-\lambda) \quad \text{for } \beta_0 \geq \lambda \geq 0.$$

We compute that $f'(\lambda) = \lambda^{n-2}\big(\beta_0(n-1)-n\lambda\big)$. *Then,*

$$f'\Big(\frac{n-1}{n}\beta_0\Big) = 0 \quad \text{and} \quad f''\Big(\frac{n-1}{n}\beta_0\Big) = -n\Big(\frac{n-1}{n}\beta_0\Big)^{n-2} < 0,$$

which yields

$$\max_{0\leq\lambda\leq\beta_0}\big\{f(\lambda)\big\} = f\Big(\frac{n-1}{n}\beta_0\Big) = \frac{(n-1)^{n-1}}{n^n}\beta_0^n.$$

Therefore, the oscillation condition (3.7) is optimized as

$$\liminf_{t\to\infty} B(t) > \frac{n!}{\mathrm{e}\beta_0^n}\Big(\frac{n}{n-1}\Big)^{n-1}.$$

3.2. **Neutral equations.** In the previous section, we have stated some oscillation conditions for nonneutral equations, however, for neutral equations this phrase replaces with the so-called "almost oscillation", i.e., every solution oscillates or tends to zero at infinity. Our main tools in this section will be the comparison tests Theorem 2.2 and Theorem 2.3, and the oscillation test Theorem 3.1.

Our first result for neutral equations investigates (1.1) when the neutral coefficient is in the range (R1).

Theorem 3.2. *Assume that $n \in \mathbb{N}$ is odd with $n \geq 3$ and (R1) holds. Further, assume that there exist two functions $\delta, \gamma \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ satisfying $\delta(t) \leq \beta(t) \leq \gamma(t) \leq t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$x^{\Delta}(t) + [1-A\big(\beta(t)\big)]B(t) \min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=even}} \Big\{\mathrm{h}_k\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-k-1}\big(\beta(t),\gamma(t)\big)\Big\} x\big(\gamma(t)\big) = 0 \quad \text{for } [t_0,\infty)_{\mathbb{T}} \tag{3.8}$$

is oscillatory. Then every solution of (1.1) is oscillatory or tends to zero asymptotically.

Proof. The proof follows from Theorem 2.2 and Theorem 3.1. □

As an immediate consequence of the theorem above, we can give the following corollary by combining Theorem 3.2 by [2, Theorem 1] and [19, Theorem 2].

Corollary 3.1. *Assume that $n \in \mathbb{N}$ is odd with $n \geq 3$ and (R1) holds. If there exist two functions $\delta, \gamma \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ satisfying $\delta(t) \leq \beta(t) \leq \gamma(t) \leq t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ such that*

$$\liminf_{t\to\infty} \inf_{\substack{-\lambda\varphi\in\mathcal{R}^+([\gamma(t),t)_{\mathbb{T}})\\ \lambda>0}} \left\{\frac{1}{\lambda \mathrm{e}_{-\lambda\varphi}\big(t,\gamma(t)\big)}\right\} > 1$$

or

$$\liminf_{t\to\infty} \int_{\gamma(t)}^{t} \varphi(\eta)\Delta\eta > M \quad \text{and} \quad \limsup_{t\to\infty} \int_{\gamma(t)}^{\sigma(t)} \varphi(\eta)\Delta\eta > 1 - \Big(1-\sqrt{1-M}\Big)^2,$$

where

$$\varphi(t) := [1-A\big(\beta(t)\big)]B(t) \min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=even}} \Big\{\mathrm{h}_k\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-k-1}\big(\beta(t),\gamma(t)\big)\Big\} \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}},$$

then every solution of (1.1) oscillates or tends to zero asymptotically.

Our next result treats (1.1) when the neutral coefficient is in the range (R2).

Theorem 3.3. *Assume that $n \in \mathbb{N}$ is odd with $n \geq 3$ and (R2) holds. Further, assume that there exist two functions $\delta, \gamma \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ satisfying $\delta(t) \leq \beta(t) \leq \gamma(t) \leq t$ for all $t \in [t_0,\infty)_{\mathbb{T}}$ such that (3.2) is oscillatory. Then, every solution of (1.1) is oscillatory or tends to zero asymptotically.*

Proof. The proof follows from Theorem 2.3 and Theorem 3.1. □

Corollary 3.2. *Assume that $n \in \mathbb{N}$ is odd with $n \geq 3$ and (R2) holds. If there exist two functions $\delta, \gamma \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfying $\delta(t) \leq \beta(t) \leq \gamma(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$\liminf_{t\to\infty} \inf_{\substack{-\lambda\psi\in\mathcal{R}^{+}([\gamma(t),t)_{\mathbb{T}})\\ \lambda>0}} \left\{\frac{1}{\lambda \mathrm{e}_{-\lambda\psi}\big(t,\gamma(t)\big)}\right\} > 1$$

or

$$\liminf_{t\to\infty} \int_{\gamma(t)}^{t} \psi(\eta)\Delta\eta > M \quad \text{and} \quad \limsup_{t\to\infty} \int_{\gamma(t)}^{\sigma(t)} \psi(\eta)\Delta\eta > 1 - \Big(1 - \sqrt{1-M}\Big)^2,$$

where

$$\psi(t) := B(t) \min_{\substack{k\in[0,n)_{\mathbb{Z}}\\ k=even}} \Big\{\mathrm{h}_k\big(\beta(t),\delta(t)\big)\mathrm{h}_{n-k-1}\big(\beta(t),\gamma(t)\big)\Big\} \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}},$$

then every solution of (1.1) oscillates or tends to zero asymptotically.

Consider now the following additional assumptions

$$\int_{t_0}^{\infty} B(\eta)\Delta\eta = \infty \tag{3.9}$$

and

$$\limsup_{t\to\infty} A(t) < 0 \tag{3.10}$$

under which we will state an oscillation test for (1.1).

Theorem 3.4. *Assume that $n \in \mathbb{N}$ is odd with $n \geq 3$, (3.9) and (R2) hold with (3.10). Further, assume that there exist $k_0 \in \mathbb{N}$ and a function $\gamma \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfying $\beta(t) \leq \gamma(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$x^{\Delta}(t) + \sum_{\ell=0}^{k_0-1}\left[\prod_{\nu=0}^{\ell-1} A\big(\alpha^{\nu}(t)\big)\right] B(t)\mathrm{h}_{n-1}\big(\beta(t),\gamma(t)\big)x\big(\gamma(t)\big) = 0 \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{3.11}$$

is oscillatory. Then, every solution of (1.1) is oscillatory.

Proof. Assume the contrary that x is an eventually positive solution of (1.1). Then, there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)$, $x\big(\alpha(t)\big)$, $x\big(\beta(t)\big) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. Set

$$y(t) := x(t) + A(t)x\big(\alpha(t)\big) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}. \tag{3.12}$$

Therefore, we have

$$y^{\Delta^n}(t) + B(t)x\big(\beta(t)\big) = 0 \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}} \tag{3.13}$$

showing that for $j \in [0, n)_{\mathbb{Z}}$ the functions y^{Δ^j} are monotonic on $[t_2, \infty)_{\mathbb{T}}$ for some $t_2 \in [t_1, \infty)_{\mathbb{T}}$.

(C1) Let $y > 0$ on $[t_2, \infty)_{\mathbb{T}}$. By Kiguradze's lemma, we learn that there exist $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and an even integer $m \in [0, n)_{\mathbb{Z}}$ such that $t \in [t_2, \infty)_{\mathbb{T}}$ implies $y^{\Delta^k}(t) > 0$ for all $k \in [0, m)_{\mathbb{Z}}$ and $(-1)^{m+k}y^{\Delta^k}(t) > 0$ for all $k \in [m, n)_{\mathbb{Z}}$.

   (a) Let $m \in [2, n)_{\mathbb{Z}}$, i.e., $y^{\Delta} > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Therefore, $\lim_{t\to\infty} y(t) > 0$. We also have $x \geq y$ on $[t_2, \infty)_{\mathbb{T}}$, i.e., $\liminf_{t\to\infty} x(t) > 0$. This implies by (3.9) that $\lim_{t\to\infty} y^{\Delta^{n-1}}(t) = -\infty$, which yields $\lim_{t\to\infty} y(t) = -\infty$.

   (b) Let $m = 0$, i.e., $y^{\Delta} < 0$ on $[t_2, \infty)_{\mathbb{T}}$. Then, by recursively substituting x into (3.12), we get for $k \in \mathbb{N}$ and all $t \in [t_3, \infty)_{\mathbb{T}}$ that

$$\begin{aligned} x(t) &= \sum_{\ell=0}^{k-1}\left[\prod_{\nu=0}^{\ell-1} A\big(\alpha^{\nu}(t)\big)\right] y\big(\alpha^{\ell}(t)\big) + \left[\prod_{\nu=0}^{k-1} A\big(\alpha^{\nu}(t)\big)\right] x\big(\alpha^{k}(t)\big) \\ &\geq \sum_{\ell=0}^{k-1}\left[\prod_{\nu=0}^{\ell-1} A\big(\alpha^{\nu}(t)\big)\right] y(t), \end{aligned}$$

   where $\alpha^k(t) \geq t_2$ for all $t \in [t_3, \infty)_{\mathbb{T}}$. The rest of the proof follows from the proofs of [17, Theorem 3] and Theorem 3.1.

(C2) Let $y < 0$ on $[t_2, \infty)_{\mathbb{T}}$. In this case, we see that $x(t) \leq A(t)x\big(\alpha(t)\big)$ for all $t \in [t_2, \infty)_{\mathbb{T}}$, which implies boundedness of the function x. Therefore, y is bounded too. Applying Kiguradze's lemma for the function $(-y)$, we learn that there exist $t_2 \in [t_1, \infty)_{\mathbb{T}}$ and an odd integer $m \in [0, n)_{\mathbb{Z}}$ such that $t \in [t_2, \infty)_{\mathbb{T}}$ implies $y^{\Delta^k}(t) < 0$ for all $k \in [0, m)_{\mathbb{Z}}$ and $(-1)^{m+k}y^{\Delta^k}(t) < 0$ for all $k \in [m, n)_{\mathbb{Z}}$. In particular, we have $y^{\Delta} < 0$ on $[t_2, \infty)_{\mathbb{T}}$, i.e., $\lim_{t\to\infty} y(t) < 0$. Then, we obtain

$$y(t) \geq -A(t)x\big(\alpha(t)\big) \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}},$$

which yields

$$x(t) \geq -\frac{y\big(\alpha^{-1}(t)\big)}{A\big(\alpha^{-1}(t)\big)} \quad \text{for all } t \in [t_2, \infty)_{\mathbb{T}}.$$

Thus, we have $\liminf_{t\to\infty} x(t) > 0$ by (R2). Proceeding as in the case (C1a), we obtain $\lim_{t\to\infty} y(t) = -\infty$, which contradicts the boundedness of y.

Thus, the proof is complete. □

Corollary 3.3. *Assume that $n \in \mathbb{N}$ is odd with $n \geq 3$, (3.9) and (R2) hold. If there exist $k_0 \in \mathbb{N}$ and a function $\gamma \in \mathrm{C}_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfying $\beta(t) \leq \gamma(t) \leq t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$\liminf_{t\to\infty} \inf_{\substack{-\lambda\psi_{k_0} \in \mathcal{R}^+([\gamma(t),t)_{\mathbb{T}}) \\ \lambda > 0}} \left\{ \frac{1}{\lambda \mathrm{e}_{-\lambda\psi_{k_0}}\big(t, \gamma(t)\big)} \right\} > 1$$

or

$$\liminf_{t\to\infty} \int_{\gamma(t)}^{t} \psi_{k_0}(\eta)\Delta\eta > M \quad \text{and} \quad \limsup_{t\to\infty} \int_{\gamma(t)}^{\sigma(t)} \psi_{k_0}(\eta)\Delta\eta > 1 - \Big(1 - \sqrt{1 - M}\Big)^2,$$

where

$$\psi_k(t) := \sum_{\ell=0}^{k-1} \left[\prod_{\nu=0}^{\ell-1} A\big(\alpha^{\nu}(t)\big) \right] B(t)\mathrm{h}_{n-1}\big(\beta(t), \gamma(t)\big) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}} \text{ and } k \in \mathbb{N},$$

then every solution of (1.1) oscillates.

Remark 3.1. *If A is a constant function, i.e., $A(t) \equiv -a_0$, where $a_0 \in (0, 1)_{\mathbb{R}}$, then ψ_{k_0} in Corollary 3.3 can be replaced by*

$$\frac{B(t)}{1 - a_0}\mathrm{h}_{n-1}\big(\beta(t), \gamma(t)\big) \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}.$$

References

[1] R. P. Agarwal and M. Bohner, Basic calculus on time scales and some of its applications, Results Math. 35 (1999), no. 1-2, 3–22.

[2] M. Bohner, Some oscillation criteria for first order delay dynamic equations, Far East J. Appl. Math. 18 (2005), no. 3, 289–304.

[3] M. Bohner, B. Karpuz and Ö. Öcalan, Iterated oscillation criteria for delay dynamic equations of first order, Adv. Difference Equ. 2008 (2008), Art. ID. 458687.

[4] M. Bohner and A. C. Peterson, Dynamic Equations on Time Scales An Introduction with Applications, Birkhäuser Boston, Inc., Boston, MA, 2001.

[5] E. Braverman and B. Karpuz, Nonoscillation of first-order dynamic equations with several delays, Adv. Difference Equ. 2010 (2010), Art. ID. 873459.

[6] Y. Şahiner and I. P. Stavroulakis, Oscillations of first order delay dynamic equations, Dynam. Systems Appl. 15 (2006), no. 3-4, 645–655.

[7] P. Das, Oscillation criteria for odd order neutral equations, J. Math. Anal. Appl.,188 (1994), no. 1, 245–257.

[8] L. Erbe, R. Mert, A. C. Peterson, and A. Zafer, Oscillation of even order nonlinear delay dynamic equations on time scales, Czechoslovak Math. J. 63 (2013), no. 138(1), 265–279.

[9] L. H. Erbe, G. Hovhannisyan, and A. C. Peterson, Asymptotic behavior of n-th order dynamic equations, Nonlinear Dyn. Syst. Theory 12 (2012), no. 1, 63–80.

[10] L. H. Erbe, B. Karpuz, and A. C. Peterson, Kamenev-type oscillation criteria for higher-order neutral delay dynamic equations, Int. J. Difference Equ. 6 (2011), no. 1, 1–16.

[11] S. R. Grace, R. P. Agarwal, and A. Zafer, Oscillation of higher order nonlinear dynamic equations on time scales, Adv. Difference Equ. 2012 (2012), Art. ID. 67.

[12] S. R. Grace, R. Mert, and A. Zafer, Oscillatory behavior of higher-order neutral type dynamic equations, Electron. J. Qual. Theory Differ. Equ. 2013 (2013), Art. ID. 29.

[13] B. Karpuz, Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations, Appl. Math. Comput. 215 (2009), no. 6, 2174–2183.

[14] B. Karpuz, Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients, Electron. J. Qual. Theory Differ. Equ. 2009 (2009), Art. ID. 34.

[15] B. Karpuz, Existence and uniqueness of solutions to systems of delay dynamic equations on time scales, Int. J. Math. Comput. 10 (2011), no. M11, 48–58.

[16] B. Karpuz, Li type oscillation theorem for delay dynamic equations, Math. Methods Appl. Sci. 36 (2013), no. 9, 993–1002.

[17] B. Karpuz, Sufficient conditions for the oscillation and asymptotic behaviour of higher-order dynamic equations of neutral type, Appl. Math. Comput. 221 (2013), 453–462.

[18] B. Karpuz, Comparison tests for the asymptotic behaviour of higher-order dynamic equations of neutral type, Forum Math. 27(2015), no. 5, 2759–2773.

[19] B. Karpuz and Ö. Öcalan, New oscillation tests and some refinements for first-order delay dynamic equations, Turkish J. Math. 40 (2016), no. 4, 850–863.

[20] R. Mert, Oscillation of higher-order neutral dynamic equations on time scales, Adv. Difference Equ. 2012 (2012), Art. ID. 68.

[21] A. D. Myškis, Linear homogeneous differential equations of the first order with retarded argument, Uspehi Matem. Nauk (N.S.) 5 (1950), no. 2(36), 160–162.

[22] X. Wu, T. X. Sun, H. J. Xi, and C. H. Chen, Kamenev-type oscillation criteria for higher-order nonlinear dynamic equations on time scales, Adv. Difference Equ. 2013 (2013), Art. ID. 248.

[23] J. Yang, S. Liu, and X. K. Hou, Oscillation and existence of nonoscillatory solutions of forced higher-order neutral dynamic systems on time scales, Pure Appl. Math. (Xi'an) 25 (2009), no. 4, 665–670.

[24] B. G. Zhang and X. H. Deng, Oscillation of delay differential equations on time scales, Math. Comput. Modelling 36 (2002), no. 11-13, 1307–1318.

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