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MODIFIED HOMOTOPY ANALYSIS METHOD FOR NONLINEAR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, a combined form of natural transform with homotopy analysis method is proposed to solve nonlinear fractional partial differential equations. This method is called the fractional homotopy analysis natural transform method (FHANTM). The FHANTM can easily be applied to many problems and is capable of reducing the size of computational work. The fractional derivative is described in the Caputo sense. The results show that the FHANTM is an appropriate method for solving nonlinear fractional partial differentia equation.

1. introduction

The natural transform is a transform defined by an integral like all other transformations defined by integrals, such as the Laplace transform as well, as the Sumudu transform, where we find only used in solving of linear differential equations. This transform it was defined by Z. H. Khan and W. A. Khan [1] in 2008 and it has been used by many researchers in the resolution of linear differential equations ([2], [3], [4], [5]). But with the presence of some methods, such as the homotopy analysis method (HAM) ([6], [7], [8]) that used in the solution of linear and nonlinear differential equations. Then, with the advent of the compositions of this method with the natural transform, lead to facilitating the resolution of nonlinear fractional partial differential equations. The objective of this study is to combine two powerful methods, the first method is "homotopy analysis method", the second is called "the natural transform method", the fractional derivative is described in the Caputo sense, thus, we get the modified method "fractional homotopy analysis natural transform method" (FHANTM), and we apply this modified method to solve somme exemples of nonlinear fractional partial differential equations.

The present paper has been organized as follows: In Section 2 some basic definitions and properties of natural transform. In section 3 we will propose an analysis of the modified method. In section 4 we present three examples explaining how to apply the proposed method (FHANTM). Finally, the conclusion follows.

2. Basic definitions

In this section, we give some basic definitions and properties of fractional calculus, natral transform and natural transform of fractional derivatives which are used further in this paper.

2.1. **Fractional calculus.** There are several definitions of a fractional derivative of order $\alpha \geqslant 0$ (see [9], [10], [11]). The most commonly used definitions are the Riemann–Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. *Let $\Omega = [a, b]$ $(-\infty < a < b < +\infty)$ be a finite interval on the real axis $\mathbb{R}$. The Riemann–Liouville fractional integral $I^{\alpha}_{0+}f$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ is defined by*

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$$\begin{aligned}(I_{0+}^{\alpha}f)(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-\alpha}}, \quad t>0,\ \alpha>0, \\ (I_{0+}^{0}f)(t) &= f(t).\end{aligned} \tag{2.1}$$

Here $\Gamma(\cdot)$ is the gamma function.

Theorem 2.1. *Let $\alpha \geqslant 0$ and let $n=[\alpha]+1$. If $f(t)\in AC^n[a,b]$, then the Caputo fractional derivative $({}^cD_{0+}^{\alpha}f)(t)$ exist almost evrywhere on $[a,b]$.*

If $\alpha \notin \mathbb{N}$, $({}^cD_{0+}^{\alpha}f)(t)$ is represented by

$$({}^cD_{0+}^{\alpha}f)(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha-n+1}}, \tag{2.2}$$

where $D=\frac{d}{dx}$ and $n=[\alpha]+1$.

Proof. (see [10]). □

Remark 2.1. *In this paper, we consider the time-fractional derivative in the Caputo's sense. When $\alpha\in\mathbb{R}^+$, the time-fractional derivative is defined as*

$$\begin{aligned}({}^cD_t^{\alpha}u)(x,t) &= \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} \\ &= \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}, & m-1<\alpha<m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\end{cases}\end{aligned} \tag{2.3}$$

where $m\in\mathbb{N}^*$.

2.2. **Definitions of the N-transform.** We give some basic definitions and properties of the N-Transform which are used further in this paper (see [1], [12], [13]). When the real function $f(t)>0$ and $f(t)=0$ for $t<0$ is sectionwise continuous, exponential order and defined in the set

$$A=\left\{f(t): \exists M,\quad k_1,k_2>0,\quad |f(t)|<Me^{\frac{|t|}{k_j}},\ if\ t\in(-1)^j\times[0,\ \infty)\right\}.$$

Definition 2.2. *[12] The N-transform of the function $f(t)>0$ and $f(t)=0$ for $t<0$ is defined by*

$$\mathbb{N}^+[f(t)] = R(s,u) = \int_0^{\infty} e^{-st}f(ut)dt; s>o, u>o. \tag{2.4}$$

Where s and u are the transform variables. The original function $f(t)$ in (2.4) is called the inverse transform or inverse of $R(s,u)$ and it is defined by

$$\mathbb{N}^{-1}\{R(s,u)\} = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{st}{u}}R(s,u)ds, \tag{2.5}$$

2.2.1. *N-Transform of fractional derivatives.*

Proposition 2.1. *If $R(s,u)$ is the N-Transform of the function $f(t)$, then the N-Transform of fractional integral of order α is defined by*

$$\mathbb{N}^+\left[(I_{0+}^{\alpha}f)(t)\right] = \frac{s^{\alpha}}{u^{\alpha}}R(s,u). \tag{2.6}$$

Proposition 2.2. *If $R(s,u)$ is the N-Transform of the function $f(t)$, then the N-Transform of fractional derivative of order α is defined as*

$$\mathbb{N}^+\left[({}^cD_{0+}^{\alpha}f)(t)\right] = \frac{s^{\alpha}}{u^{\alpha}}R(s,u) - \sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}f^{(k)}(0). \tag{2.7}$$

2.2.2. *Somme properties of the N-transform.* Here are some properties of the N-Transform:

1. If $\mathbb{N}^+\{f(t)\} = R(s,u)$ then, $\mathbb{N}^+\{f(at)\} = \frac{1}{a}R(s,u)$.

2. Generalised N-Transform For any value of n the generalised N-Transform of function $f(t) > 0$ is defined by

$$\mathbb{N}^+\{f(t)\} = R(s,u) = \sum_{n=0}^{\infty} \frac{n! a_n u^n}{s^{n+1}}. \tag{2.8}$$

3. N-Transform of derivative If $f^{(n)}(t)$ is the nth derivative of function $f(t)$, then its N-Transform is given by

$$\mathbb{N}^+\left\{f^{(n)}(t)\right\} = R_n(s,u) = \frac{s^n}{u^n}R(s,u) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{u^{n-k}} f^{(k)}(0). \tag{2.9}$$

For $n = 1$ and $n = 2$, (2.9) gives the N-Transform of first and second derivatives of $f(t)$

$$\mathbb{N}^+\left\{f^{'}(t)\right\} = R_1(s,u) = \frac{s}{u}R(s,u) - \frac{1}{u}f(0). \tag{2.10}$$

$$\mathbb{N}^+\left\{f^{''}(t)\right\} = R_2(s,u) = \frac{s^2}{u^2}R(s,u) - \frac{s}{u^2}f(0) - \frac{1}{u}f^{'}(0). \tag{2.11}$$

4. N-Transform of integral

$\mathbb{N}^+\{f(t)\} = R(s,u)$ then $\mathbb{N}^+\left\{\int_0^t f(r)dr\right\} = \frac{u}{s}R(s,u)$.

5. The function $f(t)$ in set A is multiplied with shift function t^n then

$$\mathbb{N}^+\{t^n f(t)\} = \frac{s^n}{u^n}\frac{d^n}{du^n}u^n R(s,u). \tag{2.12}$$

And

$$\mathbb{N}^+\left\{\frac{t^\alpha}{\Gamma(\alpha+1)}\right\} = \frac{u^\alpha}{s^{\alpha+1}},\ \alpha \geqslant 0.$$

3. Fractional Homotopy Analysis N-Transform Method (FHANTM)

To illustrate the basic idea of this method, we consider a general nonlinear nonhomogeneous time-fractional partial differential equation

$${}^cD_t^\alpha U(x,t) + LU(x,t) + RU(x,t) = g(x,t), \tag{3.1}$$

where $m = 1, 2, \ldots,$ and the initial conditions

$$\left.\frac{\partial^{m-1}U(x,t)}{\partial t^{m-1}}\right|_{t=0} = f_{m-1}(x), \quad m = 1, 2, \ldots, \tag{3.2}$$

where ${}^cD_t^\alpha U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, L is the linear differential operator, R represents the general nonlinear differential operator, and $g(x,t)$ is the source term.

Applying the N-Transform (denoted in this paper by $\mathbb{N}^+$) on both sides of (3.1), we get

$$\mathbb{N}^+\left[{}^cD_t^\alpha U(x,t)\right] + \mathbb{N}^+\left[LU(x,t) + RU(x,t) - g(x,t)\right] = 0. \tag{3.3}$$

Using the property of the N-Transform, we have the following form

$$\mathbb{N}^+\left[U(x,t)\right] - \frac{u^\alpha}{s^\alpha}\sum_{k=0}^{n-1}\frac{s^{\alpha-(k+1)}}{u^{\alpha-k}}U^{(k)}(x,0) + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[LU(x,t) + RU(x,t) - g(x,t)\right] = 0 \tag{3.4}$$

Or

$$\mathbb{N}^{+}\left[U(x,t)\right]-\sum_{k=0}^{n-1}\frac{u^{k}}{s^{k+1}}U^{(k)}(x,0)+\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[LU(x,t)+RU(x,t)-g(x,t)\right]=0. \tag{3.5}$$

Define the nonlinear operator

$$\begin{aligned} R[\phi(x,t;p)] &= \mathbb{N}^{+}\left[\phi(x,t;p)\right]-\sum_{k=0}^{n-1}\frac{u^{k}}{s^{k+1}}\phi^{(k)}(x,0,p) \\ &+\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[L\phi(x,t;p)+R\phi(x,t;p)-g(x,t;p)\right] \end{aligned} \tag{3.6}$$

By means of homotopy analysis method [6], we construct the so-called the zero-order deformation equation

$$(1-p)\mathbb{N}^{+}[\phi(x,t;p)-U_{0}(x,t)]=phH(x,t)R[\phi(x,t;p)], \tag{3.7}$$

where p is an embedding parameter and $p\in[0,\,1]$, $H(x;t)\neq 0$ is an auxiliary function, $h\neq 0$ is an auxiliary parameter, $\mathbb{N}^{+}$ is an auxiliary linear N-Transform operator. When $p=0$ and $p=1$, we have

$$\begin{cases} \phi(x,t;0)=U_{0}(x,t), \\ \phi(x,t;1)=U(x,t). \end{cases} \tag{3.8}$$

When p increases from 0 to 1, the $\phi(x,t,p)$ various from $U_0(x,t)$ to $U(x,t)$. Expanding $\phi(x,t;p)$ in Taylor series with respect to p, we have

$$\phi(x,t;p)=U_{0}(x,t)+\sum_{m=1}^{+\infty}U_{m}(x,t)p^{m}, \tag{3.9}$$

where

$$U_{m}(x,t)=\frac{1}{m!}\frac{\partial^{m}\phi(x,t;p)}{\partial p^{m}}\mid_{p=0}. \tag{3.10}$$

When $p=1$, the (3.9) becomes

$$U(x,t)=U_{0}(x,t)+\sum_{m=1}^{+\infty}U_{m}(x,t). \tag{3.11}$$

We define the vectors

$$\overrightarrow{U}_{n}=\{U_{0}(x,t),U_{1}(x,t),U_{2}(x,t),\ldots,U_{n}(x,t)\}. \tag{3.12}$$

Differentiating (3.7) $m-$times with respect to p, then setting $p=0$ and finally dividing them by $m!$, we obtain the so-called mth-order deformation equation

$$\mathbb{N}^{+}[U_{m}(x,t)-\chi_{m}U_{m-1}(x,t)]=hpH(x,t)\Re_{m}(\overrightarrow{U}_{m-1}(x,t)), \tag{3.13}$$

where

$$\Re_{m}(\overrightarrow{U}_{m-1}(x,t))=\frac{1}{(m-1)!}\frac{\partial^{m-1}R[\phi(x,t;p)]}{\partial p^{m-1}}\mid_{p=0}, \tag{3.14}$$

and

$$\chi_{m}=\begin{cases} 0, & m\leqslant 1, \\ 1, & m>1. \end{cases}$$

Applying the inverse N-Transform on both sides of (3.13), we have

$$U_{m}(x,t)=\chi_{m}U_{m-1}(x,t)+\mathbb{N}^{-1}\left[hpH(x,t)\Re_{m}(\overrightarrow{U}_{m-1}(x,t))\right]. \tag{3.15}$$

The mth deformation equation (3.15) is a linear which can be easily solved. So, the solution of (3.1) can be written into the following form

$$U(x,t)=\sum_{m=0}^{M}U_m(x,t), \tag{3.16}$$

when $M\to\infty$, we can obtain an accurate approximation solution of (3.1).

4. Application of the FHANTM

In this section, we apply the homotopy analysis method (HAM) coupled with the N-Transform for solving somme exemples of nonlinear time-fractional partial differential equations.

Example 4.1. *Consider the following nonlinear time-fractional KdV equation*

$$^cD_t^\alpha U-3(U^2)_x+U_{xxx}=0,\quad 0<\alpha\leqslant 1, \tag{4.1}$$

with the initial condition

$$U(x,0)=6x. \tag{4.2}$$

Applying the N-Transform on both sides of (4.1), we get

$$\mathbb{N}^+\left[U\right]-\frac{1}{s}U(x,0)+\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[-3(U^2)_x+U_{xxx}\right]=0. \tag{4.3}$$

From (4.3) and the initial condition (4.2), we have

$$\mathbb{N}^+\left[U\right]-\frac{1}{s}6x-\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[3(U^2)_x-U_{xxx}\right]=0. \tag{4.4}$$

We take the nonlinear part as

$$R[\phi(x,t,p)]=\mathbb{N}^+\left[\phi\right]-\frac{1}{s}6x-\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[3(\phi^2)_x-\phi_{xxx}\right]. \tag{4.5}$$

We construct the so-called the zero-order deformation equation with assumption $H(x;t)=1$, we have

$$(1-p)\mathbb{N}^+[\phi(x,t;p)-U_0(x,t)]=phR[\phi(x,t;p)]. \tag{4.6}$$

When $p=0$ and $p=1$, we can obtain

$$\begin{cases}\phi(x,t;0)=U_0(x,t),\\ \phi(x,t;1)=U(x,t).\end{cases}$$

Therefore, we have the mth order deformation equation

$$\mathbb{N}^+[U_m(x,t)-\chi_mU_{m-1}(x,t)]=h\Re_m(\overrightarrow{U}_{m-1}(x,t)). \tag{4.7}$$

Operating the inverse N-Transform operator on both sides of (4.7), we get

$$U_m(x,t)=\chi_mU_{m-1}(x,t)+\mathbb{N}^{-1}[h\Re_m(\overrightarrow{U}_{m-1}(x,t))]. \tag{4.8}$$

From (4.8), we have

$$\begin{aligned}U_1(x,t)&=h\mathbb{N}^{-1}[\ \Re_1(\overrightarrow{U}_0(x,t))],\\ U_2(x,t)&=U_1+h\mathbb{N}^{-1}[\ \Re_2(\overrightarrow{U}_1(x,t))],\\ U_3(x,t)&=U_2+h\mathbb{N}^{-1}[\ \Re_3(\overrightarrow{U}_2(x,t))],\\ &\vdots\end{aligned} \tag{4.9}$$

where

$$\begin{aligned}\Re_1(\overrightarrow{U}_0(x,t))&=\mathbb{N}^+\left[U_0\right]-\frac{1}{s}6x-\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[3(U_0^2)_x-(U_0)_{xxx}\right],\\ \Re_2(\overrightarrow{U}_1(x,t))&=\mathbb{N}^+\left[U_1\right]-\frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[3(2U_0U_1)_x-(U_1)_{xxx}\right],\end{aligned} \tag{4.10}$$

$$\Re_3(\overrightarrow{U}_2(x,t)) = \mathbb{N}^+[U_2] - \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[3(2U_0U_2+U_1^2)_x - (U_2)_{xxx}\right],$$
$$\vdots$$

Using the initial condition (4.2), the iteration formulas (4.9) and (4.10), we obtain

$$\begin{aligned}
U_0(x,t) &= 6x,\\
U_1(x,t) &= (-h)(6x)\frac{36}{\Gamma(\alpha+1)}t^\alpha,\\
U_2(x,t) &= (-h)(1+h)(6x)\frac{36}{\Gamma(\alpha+1)}t^\alpha + 2h^2(6x)\frac{(36)^2}{\Gamma(2\alpha+1)}t^{2\alpha},\\
U_3(x,t) &= (-h)(1+h)^2(6x)\frac{36}{\Gamma(\alpha+1)}t^\alpha + 4(h^2+h^3)(6x)\frac{(36)^2}{\Gamma(2\alpha+1)}t^{2\alpha}\\
&\quad +(-h^3)(6x)(36)^3\left[\frac{4}{\Gamma(2\alpha+1)}+\frac{1}{\Gamma^2(\alpha+1)}\right]\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha},\\
&\ \ \vdots
\end{aligned}\tag{4.11}$$

The other components of the FHANTM can be determined in a similar way. Finally, the approximate solution of (4.1) in a series form is

$$\begin{aligned}
U(x,t) &= 6x + (-h)(3+3h+h^2)(6x)\frac{36}{\Gamma(\alpha+1)}t^\alpha + (6h^2+4h^3)(6x)\frac{(36)^2}{\Gamma(2\alpha+1)}t^{2\alpha}\\
&\quad +(-h^3)(6x)(36)^3\left[\frac{4}{\Gamma(2\alpha+1)}+\frac{1}{\Gamma^2(\alpha+1)}\right]\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}+\cdots
\end{aligned}\tag{4.12}$$

When $h=-1$, the approximate solution of (4.1), is given by

$$\begin{aligned}
U(x,t) &= 6x\left[1+\frac{36}{\Gamma(\alpha+1)}t^\alpha + 2\frac{(36)^2}{\Gamma(2\alpha+1)}t^{2\alpha}\right.\\
&\quad \left. + (36)^3\left(\frac{4}{\Gamma(2\alpha+1)}+\frac{1}{\Gamma^2(\alpha+1)}\right)\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)}t^{3\alpha}+\cdots\right],
\end{aligned}\tag{4.13}$$

and when $\alpha=1$, we obtain

$$U(x,t) = 6x\left[1+36t+(36t)^2+(36t)^3+\cdots\right].\tag{4.14}$$

That gives

$$U(x,t) = \frac{6x}{1-36t},\quad |36t|<1,\tag{4.15}$$

which is an exact solution to the nonlinear KdV equation as presented in [14].

Example 4.2. *Consider the following nonlinear time-fractional partial differential equation*

$${}^cD_t^\alpha U - 2\frac{x^2}{t}UU_x = 0,\ t>0,\quad 1<\alpha\leqslant 2,\tag{4.16}$$

with the initial conditions

$$U(x,0)=0,\quad U_t(x,0)=x.\tag{4.17}$$

Applying the N-Transform on both sides of (4.16), we get

$$\mathbb{N}^+[U] - \frac{1}{s}U(x,0) - \frac{u}{s^2}U_t(x,0) + \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[-2\frac{x^2}{t}UU_x\right] = 0.\tag{4.18}$$

From (4.18) and the initial conditions (4.17), we have

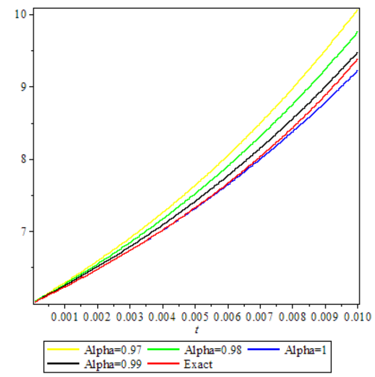

FIGURE 1. Shows the exact solution and approximate solutions of (4.1) for different values of α when $x = 1$. We note that the graph has changed its position according to the values of α, if the value of α is closer to 1, we see that the graph corresponding to this value takes the position closest to the graph of the exact solution.

$$\mathbb{N}^{+}\left[U\right]-\frac{u}{s^{2}}x-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[2\frac{x^{2}}{t}UU_{x}\right]=0. \tag{4.19}$$

We take the nonlinear part as

$$R[\phi(x,t,p)]=\mathbb{N}^{+}\left[\phi\right]-\frac{u}{s^{2}}x-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[2\frac{x^{2}}{t}\phi\phi_{x}\right]. \tag{4.20}$$

We construct the so-called the zero-order deformation equation with assumption $H(x;t) = 1$, we have

$$(1-p)\mathbb{N}^{+}[\phi(x,t;p)-U_{0}(x,t)]=phR[\phi(x,t;p)]. \tag{4.21}$$

When $p = 0$ and $p = 1$, we can obtain

$$\begin{cases}\phi(x,t;0)=U_{0}(x,t),\\ \phi(x,t;1)=U(x,t).\end{cases}$$

Therefore, we have the mth order deformation equation

$$\mathbb{N}^{+}[U_{m}(x,t)-\chi_{m}U_{m-1}(x,t)]=h\Re_{m}(\overrightarrow{U}_{m-1}(x,t)). \tag{4.22}$$

Operating the inverse N-Transform operator on both sides of (4.22), we get

$$U_{m}(x,t)=\chi_{m}U_{m-1}(x,t)+\mathbb{N}^{-1}[h\Re_{m}(\overrightarrow{U}_{m-1}(x,t))]. \tag{4.23}$$

From the formula (4.23), we have

$$\begin{aligned}U_{1}(x,t)&=h\mathbb{N}^{-1}[\Re_{1}(\overrightarrow{U}_{0}(x,t))],\\ U_{2}(x,t)&=U_{1}+h\mathbb{N}^{-1}[\Re_{2}(\overrightarrow{U}_{1}(x,t))],\\ U_{3}(x,t)&=U_{2}+h\mathbb{N}^{-1}[\Re_{3}(\overrightarrow{U}_{2}(x,t))],\\ &\vdots\end{aligned} \tag{4.24}$$

where

$$\begin{aligned}\Re_{1}(\overrightarrow{U}_{0}(x,t))&=\mathbb{N}^{+}\left[U_{0}\right]-x\frac{u}{s^{2}}-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[2\frac{x^{2}}{t}U_{0}U_{0x}\right],\\ \Re_{2}(\overrightarrow{U}_{1}(x,t))&=\mathbb{N}^{+}\left[U_{1}\right]-\frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[2\frac{x^{2}}{t}(U_{0}U_{1x}+U_{1}U_{0x})\right],\end{aligned} \tag{4.25}$$

$$\Re_3(\overrightarrow{U}_2(x,t)) = \mathbb{N}^+[U_2] - \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[2\frac{x^2}{t}(U_0U_{2x} + U_1U_{1x} + U_2U_{0x})\right],$$
$$\vdots$$

and so on. Consequently, while taking the initial conditions (4.17), and according to (4.24) and (4.25), the first few components of the Homotopy Analysis N-Transform Method of (4.16) are derived as follows

$$\begin{aligned}
U_0(x,t) &= xt,\\
U_1(x,t) &= (-h)\frac{2x^3}{\Gamma(\alpha+2)}t^{\alpha+1},\\
U_2(x,t) &= (-h)(1+h)\frac{2x^3}{\Gamma(\alpha+2)}t^{\alpha+1} + h^2\frac{16x^5}{\Gamma(2\alpha+2)}t^{2\alpha+1},\\
U_3(x,t) &= (-h)(1+h)^2\frac{2x^3}{\Gamma(\alpha+2)}t^{\alpha+1} + h^2(1+h)\frac{32x^5}{\Gamma(2\alpha+2)}t^{2\alpha+1}\\
&\quad +(-h^3)\left[\frac{32\times 6}{\Gamma(2\alpha+2)} + \frac{24}{\Gamma^2(\alpha+2)}\right]\frac{\Gamma(2\alpha+2)}{\Gamma(3\alpha+2)}x^7t^{3\alpha+1},\\
&\ \ \vdots
\end{aligned} \tag{4.26}$$

The other components of the (FHANTM) can be determined in a similar way. Finally, the approximate solution of (4.16) in a series form is

$$\begin{aligned}
U(x,t) &= xt + (-h)\left[2+h+(1+h)^2\right]\frac{2x^3}{\Gamma(\alpha+2)}t^{\alpha+1} + h^2(3+2h)\frac{16x^5}{\Gamma(2\alpha+2)}t^{2\alpha+1} \\
&\quad +(-h^3)\left[\frac{32\times 6}{\Gamma(2\alpha+2)} + \frac{24}{\Gamma^2(\alpha+2)}\right]\frac{\Gamma(2\alpha+2)}{\Gamma(3\alpha+2)}x^7t^{3\alpha+1} + \cdots
\end{aligned} \tag{4.27}$$

When $h = -1$, the approximate solution of (4.16) is given by

$$\begin{aligned}
U(x,t) &= xt + \frac{2x^3}{\Gamma(\alpha+2)}t^{\alpha+1} + \frac{16x^5}{\Gamma(2\alpha+2)}t^{2\alpha+1}\\
&\quad + \left[\frac{32\times 6}{\Gamma(2\alpha+2)} + \frac{24}{\Gamma^2(\alpha+2)}\right]\frac{\Gamma(2\alpha+2)}{\Gamma(3\alpha+2)}x^7t^{3\alpha+1} + \cdots .
\end{aligned} \tag{4.28}$$

When $\alpha = 2$, the (4.28) becomes

$$U(x,t) = xt + \frac{1}{3}(xt)^3 + \frac{2}{15}(xt)^5 + \frac{17}{315}(xt)^7 + \cdots . \tag{4.29}$$

On the other hand, we see that the development of the function $U(x,t) = \tan(xt)$ according to the Taylor series in the vicinity of $t = 0$ is given by

$$U(x,t) = xt + \frac{1}{3}(xt)^3 + \frac{2}{15}(xt)^5 + \frac{17}{315}(xt)^7 + 0(\frac{t}{x})^8. \tag{4.30}$$

Therefore, we conclude that

$$U(x,t) = \tan(xt), \tag{4.31}$$

which is the exact solution of (4.16) in the case $\alpha = 2$.

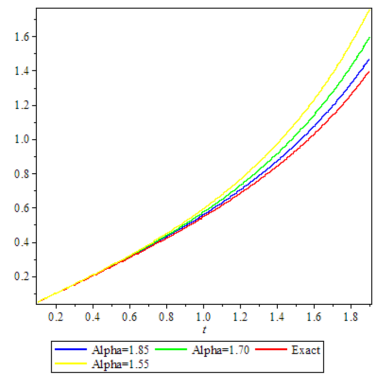

FIGURE 2. Shows the exact solution and approximate solutions of (4.16) for different values of α when $x = 1/2$. We note that the graph has changed its position according to the values of α, if the value of α is closer to 2, we see that the graph corresponding to this value takes the position closest to the graph of the exact solution.

Example 4.3. *Finaly, we consider the nonlinear time-fractional partial differential equation*

$$ {}^{c}D_t^{\alpha}U - \frac{3}{8}\left[(U_{xx})^2\right]_x = \frac{3}{2}t, \quad 2 < \alpha \leqslant 3, \tag{4.32}$$

with the initial conditions

$$U(x,0) = -\frac{1}{2}x^2, \quad U_t(x,0) = \frac{1}{3}x^3, \quad U_{tt}(x,0) = 0. \tag{4.33}$$

Applying the N-Transform on both sides of (4.32), we get

$$\mathbb{N}^+[U] - \frac{1}{s}U(x,0) - \frac{u}{s^2}U_t(x,0) + \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[-\frac{3}{8}\left[(U_{xx})^2\right]_x\right] = \frac{3}{2}\frac{u^{\alpha+1}}{s^{\alpha+2}}. \tag{4.34}$$

From (4.34) and the initial conditions (4.33), we have

$$\mathbb{N}^+[U] + \frac{1}{s}\frac{x^2}{2} - \frac{u}{s^2}\frac{x^3}{3} - \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[\frac{3}{8}\left[(U_{xx})^2\right]_x\right] = \frac{3}{2}\frac{u^{\alpha+1}}{s^{\alpha+2}}. \tag{4.35}$$

We take the nonlinear part as

$$R[\phi(x,t,p)] = \mathbb{N}^+[\phi] + \frac{1}{s}\frac{x^2}{2} - \frac{u}{s^2}\frac{x^3}{3} - \frac{3}{2}\frac{u^{\alpha+1}}{s^{\alpha+2}} - \frac{u^{\alpha}}{s^{\alpha}}\mathbb{N}^+\left[\frac{3}{8}\left[(\phi_{xx})^2\right]_x\right]. \tag{4.36}$$

We construct the so-called the zero-order deformation equation with assumption $H(x;t) = 1$, we have

$$(1-p)\mathbb{N}^+[\phi(x,t;p) - U_0(x,t)] = phR[\phi(x,t;p)]. \tag{4.37}$$

When $p = 0$ and $p = 1$, we can obtain

$$\begin{cases} \phi(x,t;0) = U_0(x,t), \\ \phi(x,t;1) = U(x,t). \end{cases}$$

Therefore, we have the mth order deformation equation

$$\mathbb{N}^+[U_m(x,t) - \chi_m U_{m-1}(x,t)] = h\Re_m(\overrightarrow{U}_{m-1}(x,t)). \tag{4.38}$$

Operating the inverse N-Transform operator on both sides of (4.38), we get

$$U_m(x,t) = \chi_m U_{m-1}(x,t) + \mathbb{N}^{-1}[h\Re_m(\overrightarrow{U}_{m-1}(x,t))]. \tag{4.39}$$

From the formula (4.23), we have

$$\begin{aligned} U_1(x,t) &= h\mathbb{N}^{-1}[\Re_1(\overrightarrow{U}_0(x,t))], \\ U_2(x,t) &= U_1 + h\mathbb{N}^{-1}[\Re_2(\overrightarrow{U}_1(x,t))], \\ U_3(x,t) &= U_2 + h\mathbb{N}^{-1}[\ \Re_3(\overrightarrow{U}_2(x,t))], \\ &\vdots \end{aligned} \tag{4.40}$$

where

$$\begin{aligned} \Re_1(\overrightarrow{U}_0(x,t)) &= \mathbb{N}^+\left[U_0\right] + \frac{1}{s}\frac{x^2}{2} - \frac{u}{s^2}\frac{x^3}{3} - \frac{3}{2}\frac{u^{\alpha+1}}{s^{\alpha+2}} - \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{3}{8}\left[(U_{0xx})^2\right]_x\right], \\ \Re_2(\overrightarrow{U}_1(x,t)) &= \mathbb{N}^+\left[U_1\right] - \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{3}{8}(U_0U_{1xx} + U_1U_{0xx})_x\right], \\ \Re_3(\overrightarrow{U}_2(x,t)) &= \mathbb{N}^+\left[U_2\right] - \frac{u^\alpha}{s^\alpha}\mathbb{N}^+\left[\frac{3}{8}(U_0U_{2xx} + U_1U_{1xx} + U_2U_{0xx})_x\right], \\ &\vdots \end{aligned} \tag{4.41}$$

Consequently, while taking (4.33), and according to (4.40) and (4.41), the first few components of the Homotopy Analysis N-Transform Method of (4.32), are derived as follows

$$\begin{aligned} U_0(x,t) &= -\frac{1}{2}x^2 + \frac{1}{3}x^3t, \\ U_1(x,t) &= (-h)6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}, \\ U_2(x,t) &= (-h)(1+h)6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}, \\ U_3(x,t) &= (-h)(1+h)^2 6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}, \\ &\vdots \\ U_n(x,t) &= (-h)(1+h)^{n-1}6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \end{aligned} \tag{4.42}$$

The other components of the (FNTHAM) can be determined in a similar way. Finally, the approximate solution of (4.32) in a series form is given by

$$\begin{aligned} U(x,t) &= -\frac{1}{2}x^2 + \frac{1}{3}x^3t + (-h)6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + (-h)(1+h)6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} \\ &\quad + (-h)(1+h)^2 6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)} + \cdots + (-h)(1+h)^{n-1}6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}. \end{aligned} \tag{4.43}$$

When $h=-1$, we obtain the exact solution of (4.32)

$$U(x,t) = -\frac{1}{2}x^2 + \frac{1}{3}x^3t + 6x\frac{t^{\alpha+2}}{\Gamma(\alpha+3)}. \tag{4.44}$$

We note that from the term U_2, all terms are equal to zero.
When $\alpha = 3$, the (4.44) becomes

$$U(x,t) = -\frac{1}{2}x^2 + \frac{1}{3}x^3t + \frac{1}{20}xt^5, \tag{4.45}$$

which is the exact solution of (4.32) in the case $\alpha = 3$.

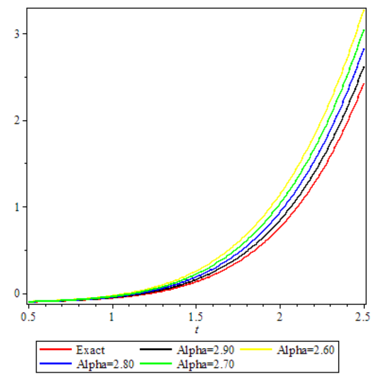

FIGURE 3. Shows the exact solution and approximate solutions of (4.32) for different values of α when $x = 1/2$. We note that the graph has changed its position according to the values of α, if the value of α is closer to 3, we see that the graph corresponding to this value takes the position closest to the graph of the exact solution.

5. Conclusion

The coupling of homotopy analysis method (HAM) and the natural transform method proved very effective to solve nonlinear fractional partial differential equations. The proposed algorithm provides the solution in a series form that converges rapidly to the exact solution if it exists. The advantage of this method is its ability to combine two powerful methods for obtaining exact or approximate solutions for nonlinear fractional partial differential equations. From the obtained results, it is clear that the FHANTM yields very accurate approximate solutions using only a few iterates.

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