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SOME IMPROVEMENTS OF CONFORMABLE FRACTIONAL INTEGRAL INEQUALITIES

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ABSTRACT. In this study, we wish to set up and present some new conformable fractional integral inequalities of the Gronwall type which have a great variety of implementation area in differential and integral equations.

1. Introduction & Preliminaries

In light of recent events in theory of differential and integral equations, it is becoming extremely difficult to ignore the existence of integral inequalities which help to determine of bounds on unknown functions. For example, Gronwall and Pachpatte have great contribution in the literature [19], [20], [5], [6]. Together with this contributions, Gronwall inequality has been extended and applied in a number of context. However, in non-integer order of models the bound provided by the above authors are not feasible.

Additionally non-integer order calculus called fractional calculus has a number of fields of application such as control theory, computational analysis and engineering [12], see also [13]. Thus a number of new definitions have been introduced in academia to provide the best method for fractional calculus. For instance in more recent times a new local, limit-based definition of a conformable derivative has been introduced in [1], [4], [10], with several follow-up papers [2], [3], [7]- [9], [11], [14]- [18].

In this research, we presented conformable fractional version of some significant integral inequalities with the help of the Katugampola conformable fractional calculus. In detail, Katugampola conformable derivatives for $\alpha \in (0,1]$ and $t \in [0,\infty)$ given by

$$D^{\alpha}(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}, \ D^{\alpha}(f)(0) = \lim_{t \to 0} D^{\alpha}(f)(t), \tag{1.1}$$

provided the limits exist (for detail see, [10]). If f is fully differentiable at t, then

$$D^{\alpha}(f)(t) = t^{1-\alpha} \frac{df}{dt}(t). \tag{1.2}$$

A function f is $\alpha-$differentiable at a point $t \geq 0$ if the limit in (1.1) exists and is finite. This definition yields the following results;

Theorem 1.1. *Let $\alpha \in (0,1]$ and f, g be $\alpha-$differentiable at a point $t > 0$. Then*
i. $D^{\alpha}(af + bg) = aD^{\alpha}(f) + bD^{\alpha}(g)$, *for all* $a, b \in \mathbb{R}$,
ii. $D^{\alpha}(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$,
iii. $D^{\alpha}(fg) = fD^{\alpha}(g) + gD^{\alpha}(f)$,
iv. $D^{\alpha}\left(\frac{f}{g}\right) = \frac{fD^{\alpha}(g) - gD^{\alpha}(f)}{g^2}$
v. $D^{\alpha}(t^n) = nt^{n-\alpha}$ *for all* $n \in \mathbb{R}$
vi. $D^{\alpha}(f \circ g)(t) = f'(g(t))\, D^{\alpha}(g)(t)$ *for f is differentiable at $g(t)$.*

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Definition 1.1 (Conformable fractional integral)**.** *Let $\alpha \in (0,1]$ and $0 \leq a < b$. A function $f : [a,b] \to \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)\, d_\alpha x := \int_a^b f(x)\, x^{\alpha-1} dx$$

exists and is finite. All α-fractional integrable on $[a,b]$ is indicated by $L_\alpha^1([a,b])$

Remark 1.1.

$$I_\alpha^a(f)(t) = I_1^a\left(t^{\alpha-1}f\right) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx,$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

We will also use the following important results, which can be derived from the results above.

Lemma 1.1. *Let the conformable differential operator D^α be given as in (1.1), where $\alpha \in (0,1]$ and $t \geq 0$, and assume the functions f and g are α-differentiable as needed. Then*

i. $D^\alpha(\ln t) = t^{-\alpha}$ *for* $t > 0$

ii. $D^\alpha\left[\int_a^t f(t,s)\, d_\alpha s\right] = f(t,t) + \int_a^t D^\alpha [f(t,s)]\, d_\alpha s$

iii. $\int_a^b f(x)\, D^\alpha(g)(x)\, d_\alpha x = fg|_a^b - \int_a^b g(x)\, D^\alpha(f)(x)\, d_\alpha x.$

In this paper, by using the Katugampola type conformable fractional calculus, we introduced retarded Gronwall-Bellman and Bihari like conformable fractional integrals inequalities.

2. Main Findings & Cumulative Results

In this article, all the functions which appear in the inequalities are assumed to be real-valued and all the integrals involved exist on the respective domains of their definitions, and $C(M,S)$ and $C^1(M,S)$ denote the class of all continuous functions and the first order conformable derivative, respectively, defined on set M with range in the set S. Additionally, R denotes the set of real numbers such that $\mathbb{R}^+ = [0,\infty)$, $\mathbb{R}^1 = [1,\infty)$ and $\mathbb{Q} = [0,T)$ are the given subset of $\mathbb{R}$.

Theorem 2.1. *[14] Let $k, y, x \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is nondecreasing with $r(t) \leq t$ for $t \geq 0$. If $u \in C(\mathbb{R}^+, \mathbb{R}^+)$ satisfies*

$$u(t) \leq k(t) + y(t) \int_0^{r(t)} x(s) u(s) d_\alpha s, \quad t \geq 0, \tag{2.1}$$

then

$$u(t) \leq k(t) + y(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)y(s)d_\alpha s} x(r(\tau))\, k(r(\tau)) D^\alpha r(\tau) d_\alpha \tau, \quad t \geq 0. \tag{2.2}$$

Theorem 2.2. *Let $u, c, x, h, y \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and assume that r is non-decreasing with $r(t) \leq t$ for $t \geq 0$. Let $w(t,u)$ be a positive , continuous, monotonic, non-decreasing, sub-additive and sub-multiplicative function for $u > 0$ for each fixed t. Let the function $k(t) > 0$ and $\Psi(t) \geq 0$ be a non-decreasing in t and continuous on $[0,\infty)$. $\Psi(0) = 0$ and suppose further that the inequality*

$$u(t) \leq k(t) + c(t) \int_0^{r(t)} x(s) u(s) d_\alpha s + h(t) \Psi\left[\int_0^{r(t)} y(s) w(s, u(s)) d_\alpha s\right] \tag{2.3}$$

is satisfied for all $t > 0$. Then

$$\begin{aligned} u(t) \leq \Bigg[k(t) + h(t)\Psi\Bigg(\mathcal{G}^{-1}\Bigg(\mathcal{G}\left[\int_0^t y(s) w(s, k(s)m(s)) D^\alpha r(s) d_\alpha s\right] \\ + \int_0^t [y(s) w(s, h(s)m(s)) D^\alpha r(s)] d_\alpha s\Bigg)\Bigg)\Bigg] m(t) \end{aligned} \tag{2.4}$$

where

$$m(t) = 1 + c(t) \int_0^t e^{\int_{r(\tau)}^{r(t)} x(s)c(s)d_\alpha s} x(r(\tau)) D^\alpha r(\tau) d_\alpha \tau \tag{2.5}$$

and $\mathcal{G}^{-1}$ *is inverse of* $\mathcal{G}$ *such that*

$$\mathcal{G}(\xi) =: \int_1^\xi \frac{1}{w(s,\Psi(s))} d_\alpha s, \quad \xi \geq 0,$$

and

$$\mathcal{G}\left[\int_0^t y(s)w(s,k(s)m(s))D^\alpha r(s)d_\alpha s\right] + \int_0^t [y(s)w(s,h(s)m(s))D^\alpha r(s)]d_\alpha s \in Dom(\mathcal{G}^{-1}), \quad \forall t \geq 0.$$

Proof. Let define

$$z(t) = \int_0^{r(t)} y(s)w(s,u(s))d_\alpha s. \tag{2.6}$$

So $z(0) = 0$, then

$$u(t) \leq [k(t) + h(t)\Psi(z(t))] + c(t)\int_0^{r(t)} x(s)u(s)d_\alpha s. \tag{2.7}$$

As $[k(t) + h(t)\Psi(z(t))]$ is positive, monotonic, non-decreasing, continuous function over $[0,\infty)$, we can apply the Theorem 2.1, that is

$$u(t) \leq [k(t) + h(t)\Psi(z(t))]\, m(t) \tag{2.8}$$

where $m(t)$ defined in 2.5. Then if we take the conformable fractional derivative of equation 2.6, we obtain

$$\begin{aligned} D^\alpha z(t) &= y(r(t))w(r(t),u(r(t)))D^\alpha r(t) \\ &\leq y(t)w(t,u(t))D^\alpha r(t) \\ &\leq y(t)w(t,[k(t)+h(t)\Psi(z(t))]\, m(t))D^\alpha r(t) \\ &\leq y(t)w(t,k(t)m(t))D^\alpha r(t) + y(t)w(t,h(t)m(t))w(t,\Psi(z(t)))D^\alpha r(t) \end{aligned}$$

hence

$$\frac{D^\alpha z(t)}{w(t,\Psi(z(t)))} \leq \frac{y(t)w(t,k(t)m(t))}{w(t,\Psi(z(t)))} D^\alpha r(t) + y(t)w(t,h(t)m(t))D^\alpha r(t). \tag{2.9}$$

Then using the definition of $\mathcal{G}$, we get

$$\mathcal{G}(z(t)) \leq \mathcal{G}\left[\int_0^t y(s)w(s,k(s)m(s))D^\alpha r(s)d_\alpha s\right] + \int_0^t [y(s)w(s,h(s)m(s))D^\alpha r(s)]d_\alpha s. \tag{2.10}$$

Hence

$$z(t) \leq \mathcal{G}^{-1}\left(\mathcal{G}\left[\int_0^t y(s)w(s,k(s)m(s))D^\alpha r(s)d_\alpha s\right] + \int_0^t [y(s)w(s,h(s)m(s))D^\alpha r(s)]d_\alpha s\right). \tag{2.11}$$

If we combine the equation 2.8 and 2.11, we get the desired bound. □

Theorem 2.3. *Let* $u, x \in C(\mathbb{R}^+, \mathbb{R}^+)$, $r \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ *and assume that* r *is non-decreasing with* $r(t) \leq t$ *for* $t \geq 0$, *for which the inequality*

$$D^\alpha u(t) \leq p + \int_0^{r(t)} x(s)D^\alpha u^q(s)[u(s) + D^\alpha u(s)]d_\alpha s \tag{2.12}$$

holds, where p *is a positive constant and* $0 < q < 1$. *If*

$$[1 - q(p + u(0))^q \int_0^{r(t)} x(s)e^{qs}d_\alpha s] > 0, \quad t \geq 0, \tag{2.13}$$

then

$$D^\alpha u(t) \leq (p^\beta + \beta\int_0^t x(s)\Omega(s)d_\alpha s)^{1/\beta}, \quad t \geq 0. \tag{2.14}$$

where $q + \beta = 1$,

$$\Omega(t) = \frac{(u(0)+p)e^t}{[1 - q(u(0)+p)^q \int_0^t x(s)e^{qs}d_\alpha s]^{1/q}}. \tag{2.15}$$

Proof. Let denote the right hand side of equation 2.12 by $z(t)$, that is

$$z(t) = p + \int_0^{r(t)} x(s) D^\alpha u^q(s)[u(s) + D^\alpha u(s)] d_\alpha s. \tag{2.16}$$

Here $z(0) = p$ and $D^\alpha u(t) \leq z(t)$. If we integrate both sides of $D^\alpha u(t) \leq z(t)$ according to rules of conformable fractional calculus, we get

$$u(t) \leq u(0) + \int_0^t z(s) d_\alpha s. \tag{2.17}$$

Then if we take conformable fractional derivative of equation 2.16, we obtain

$$D^\alpha z(t) \leq x(r(t)) D^\alpha u^q(r(t))[u(r(t)) + D^\alpha u(r(t))] D^\alpha r(t). \tag{2.18}$$

After simple manipulation, we get

$$D^\alpha z(t) \leq x(t) z^q(t)[u(0) + z(t) + \int_0^t z(s) d_\alpha s] D^\alpha r(t) \tag{2.19}$$

Let define

$$w(t) = u(0) + z(t) + \int_0^t z(s) d_\alpha s. \tag{2.20}$$

Here $w(0) = u(0) + p$. Then by taking both sides of conformable fractional derivative of above expression and using $D^\alpha z(t) \leq x(t) z^q(t) w(t) D^\alpha r(t)$ and $z(t) < w(t)$, we get

$$\begin{aligned} D^\alpha w(t) &= D^\alpha z(t) + z(t) \\ &\leq x(t) z^q(t) w(t) D^\alpha r(t) + w(t) \\ &\leq x(t) w^{q+1}(t) D^\alpha r(t) + w(t). \end{aligned}$$

So we have

$$w(t) \leq \Omega(t), \quad t \geq 0, \tag{2.21}$$

where

$$\Omega(t) = \frac{(u(0)+p)e^t}{[1 - q(u(0)+p)^q \int_0^t x(s) e^{qs} d_\alpha s]^{1/q}} \tag{2.22}$$

If we substitute 2.22 into $D^\alpha z(t) \leq x(t) z^q(t) w(t) D^\alpha r(t)$, we get

$$D^\alpha z(t) \leq x(t) z^q(t) \Omega(t) D^\alpha r(t). \tag{2.23}$$

which implies the estimation for $z(t)$ such that,

$$z(t) \leq (p^\beta + \beta \int_0^t x(s) \Omega(s) d_\alpha s)^{1/\beta}, \quad t \geq 0. \tag{2.24}$$

If we combine the equation 2.24 and $D^\alpha u(t) \leq z(t)$, we get the desired result. □

3. Concluding Remark

In this study we established the explicit bounds on retarded integral inequalities with the help of conformable fractional calculus. We take the advantage of Katugampola type conformable fractional derivatives and integrals.

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