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# UNIFORM LACUNARY STATISTICAL CONVERGENCE ON TIME SCALES

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Abstract. We introduce $(\theta, m)$-uniform lacunary density of any set and $(\theta, m)$-uniform lacunary statistical convergence on an arbitrary time scale. Moreover, $(\theta, m)$-uniform strongly $p$-lacunary summability and some inclusion relations about these new concepts are also presented.

## 1. Introduction and preliminaries

The idea of statistical convergence goes back to the study of Zygmund [42] which was published in 1935. Statistical convergence of number sequences was formally introduced by Fast [13] and Steinhaus [40] independently in the same year. Over the years and under different names, statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory, number theory, approximation theory, measure theory, trigonometric series, turnpike theory and Banach spaces. Later on, it was further investigated from the sequence space point of view and linked with summability theory by Fridy [15], Connor [8], Maddox [23], Rath and Tripathy [33], Tripathy [37], Moricz [28], Belen and Mohiuddine [3], Braha et al. [5], Edely et al. [10], Mohiuddine et al . [26] and references therein.

The statistical convergence is related to the density of subsets of $\mathbb{N}$. The asymptotic density of a set $A \subset \mathbb{N}$ is defined by

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : k \in A\}\right|,$$

whenever the limit exists. Here, $|\{k \leq n : k \in A\}|$ indicates the number of elements of $A \subseteq \mathbb{N}$ not exceeding $n$. Any finite subset of $\mathbb{N}$ has zero asymptotic density and $\delta(A^c) = 1 - \delta(A)$. A sequence $(x_k)$ is statistically convergent [13] to a real number $L$ if for each $\varepsilon > 0$,

$$\delta(\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\}) = 0.$$

In this case, $S\text{-}\lim x_k = L$. The set of all statistically convergent sequences is denoted by $S$.

By a lacunary sequence $\theta = (k_r)$ $(r = 0, 1, 2, ...)$, where $k_0 = 0$, we shall mean an increasing sequence of non- negative integers with $k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by $\theta$ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$ where $q_r = \dfrac{k_r}{k_{r-1}}$ (see [14]). The space of all lacunary strongly convergent sequences $N_\theta$ was defined by Freedman et al. as follows

$$N_\theta = \left\{ x = (x_k) : \lim_{r\to\infty} \left( \frac{1}{h_r} \sum_{k\in I_r} |x_k - L| \right) = 0, \text{ for some } L \right\}.$$

To understand lacunary sequences, we need to consider below examples.

**Example 1.1.** *$\theta = (r^2)$ is a lacunary sequence. Let us check the above conditions. One can easily see that $0 < r^2 < (r+1)^2$. So, $\theta$ is an increasing sequence where $k_0 = 0$. Furthermore, $h_r \to \infty$ as $r$ goes to infinite as shown in following table:*

| $h_r = k_r - k_{r-1}$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $\dots$ | $h_{10}$ | $h_{11}$ | $\dots$ | $h_{100}$ | $\dots$ | $h_r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_r = r^2$ | *1* | *3* | *5* | *7* | $\to$ | *19* | *21* | $\to$ | *199* | $\to$ | $2r-1$ |

***Table 1.*** *$h_r \to \infty$ as $r$ goes to infinite.*

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**Example 1.2.** *$\theta=(r)$ is not a lacunary sequence. Although the first two conditions satisfy, $h_r$ does not go to infinite as $r\to\infty$ as seen in following table:*

| $h_r = k_r - k_{r-1}$ | $h_1$ | $h_2$ | $h_3$ | $h_4$ | $\dots$ | $h_{10}$ | $h_{11}$ | $\dots$ | $h_{100}$ | $\dots$ | $h_r$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k_r = r$ | 1 | 1 | 1 | 1 | $\to$ | 1 | 1 | $\to$ | 1 | $\to$ | 1 |

***Table 2.*** *$h_r\to 1$ as $r$ goes to infinite.*

Let $K\subset\mathbb{N}$. One defines the $\theta$-density [16] of a set $K$ by

$$\delta_\theta(K)=\lim_{r\to\infty}\frac{1}{h_r}\left|K\cap I_r\right|.$$

By using lacunary sequences, Fridy and Orhan [16] studied a related concept of convergence in which $\{k : k\le n\}$ is replaced by $\{k : k_{r-1}<k\le k_r\}$ for a lacunary sequence $\theta=\{k_r\}$ as follows: A real or complex sequence $(x_k)$ is lacunary statistically convergent to $L$ if for every $\varepsilon>0$,

$$\lim_{r\to\infty}\frac{1}{h_r}\left|\{k\in I_r :\ |x_k-L|\ge\varepsilon\}\right|=0.$$

In this case, $S_\theta\text{-}\lim x=L$. Lacunary statistical convergence and related notions were studied by many authors (see [7], [9], [11], [12], [17], [20], [24], [25], [27], [29]). Furthermore, Nuray and Aydın [31] introduced and studied *strongly lacunary summable* functions. Here, our aim is to move some notions and properties about lacunary sequence to time scale calculus. Before our new concepts, we recall the main features of the time scale theory.

A time scale $\mathbb{T}$ is an arbitrary, nonempty, closed subset of real numbers. The calculus of time scale was introduced by Hilger in his Ph.D. thesis supervised by Auldbach in 1988 (see [21], [22]). It allows to unify the usual differential and integral calculus for one variable. One can replace the range of definition $\mathbb{R}$ of the functions under consideration by an arbitrary time scale $\mathbb{T}$. Recently, time scale theory has been applied to different areas by many authors (see [4], [18], [19]). The followings notions are very important for this theory.

Forward jump operator $\sigma:\mathbb{T}\to\mathbb{T}$ and graininess function $\mu:\mathbb{T}\to[0,\infty)$ are defined by $\sigma(t)=\inf\{s\in\mathbb{T} :\ s>t\}$ and $\mu(t)=\sigma(t)-t$ for $t\in\mathbb{T}$, respectively. In this definition, we put $\inf\phi=\sup\mathbb{T}$, where $\phi$ is an empty set. A half closed interval on $\mathbb{T}$ is given by

$$[a,b)_{\mathbb{T}}=\{t\in\mathbb{T}\colon\ a\le t<b\}\ \text{ or }\ (a,b]_{\mathbb{T}}=\{t\in\mathbb{T}\colon\ a<t\le b\}.$$

Open and closed intervals can be defined similarly in [4], [19].

Let $A$ be the family of all left closed and right open intervals on $\mathbb{T}$ of the form $[a,b)_{\mathbb{T}}$ and $\widetilde{m}:A\to[0,\infty)$ be a set function on $A$ such that $\widetilde{m}([a,b)_{\mathbb{T}})=b-a$. Then, it is known that $\widetilde{m}$ is a countably additive measure on $A$. Now, the Caratheodory extension of the set function $\widetilde{m}$ associated with family $A$ is said to be the Lebesque $\Delta$-measure on $\mathbb{T}$ and is denoted by $\mu_\Delta$. In this case, it is known that if $a\in\mathbb{T}-\{\max\mathbb{T}\}$, then the single point set $\{a\}$ is $\Delta$-measurable and $\mu_\Delta(a)=\sigma(a)-a$. If $a,b\in\mathbb{T}$ and $a\le b$, then $\mu_\Delta((a,b)_{\mathbb{T}})=b-\sigma(a)$ and $\mu_\Delta([a,b)_{\mathbb{T}})=b-a$. If $a,b\in\mathbb{T}-\{\max\mathbb{T}\}$ and $a\le b$, then $\mu_\Delta((a,b]_{\mathbb{T}})=\sigma(b)-\sigma(a)$ and $\mu_\Delta([a,b])_{\mathbb{T}})=\sigma(b)-a$ (see [38]).

Statistical convergence is applied to time scales for different purposes by various authors in the literature. For instance, Seyyidoglu and Tan [35] gave some important notions such as $\Delta$-convergence, $\Delta$-Cauchy by using $\Delta$-density an investigate their relations on $\mathbb{T}$ and, in the recent past, they explained a generalization of statistical cluster and limit points [36]. Turan and Duman [38] introduced density and statistical convergence of $\Delta$-measurable real-valued functions defined on $\mathbb{T}$. Furthermore, Altin et al. [1] expressed $m$- and $(\lambda,m)$-uniform density of a set and $m$- and $(\lambda,m)$-uniform statistical convergence on $\mathbb{T}$. However, Yilmaz et al. [41] defined $\lambda$-statistical convergence on $\mathbb{T}$. Turan and Duman [39] defined lacunary sequence and lacunary statistical convergence on $\mathbb{T}$. Now, we give a generalization of their study in a different form where $\theta=\{k_{t-t_0+1}\}$ is a lacunary sequence on $\mathbb{T}$.

**Definition 1.1.** *Let $\Omega$ be a $\Delta$-measurable subset of $\mathbb{T}$ and $\theta$ be a lacunary sequence. Then, we define the set $\Omega(t,\theta)$ by*

$$\Omega(t,\theta)=\left\{s\in(k_{t-2t_0+1},k_{t-t_0+1}]_{\mathbb{T}} : s\in\Omega\right\},$$

*for $t\in\mathbb{T}$. In this case, the $\theta$-density of $\Omega$ on $\mathbb{T}$ is denoted by*

$$\delta_{\mathbb{T}}^{\theta}(\Omega)=\lim_{t\to\infty}\frac{\mu_\Delta(\Omega(t,\theta))}{\mu_\Delta((k_{t-2t_0},k_{t-t_0}]_{\mathbb{T}})},$$

*provided that the above limit exists.*

**Definition 1.2.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta$-measurable function and $\theta$ be a lacunary sequence. Then, $f$ is lacunary statistically convergent to a real number $L$ on $\mathbb{T}$ if*

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(s\in(k_{t-2t_0+1},k_{t-t_0+1}]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\right)}{\mu_\Delta\left((k_{t-2t_0},k_{t-t_0}]_{\mathbb{T}}\right)}=0,$$

*for $\forall\varepsilon>0$ and $t\in\mathbb{T}$. In this case, $s_{\mathbb{T}}^{\theta}\text{-}\lim_{t\to\infty}(f(t))=L$. The set of all lacunary statistical convergence functions on $\mathbb{T}$ will be denoted by $s_{\mathbb{T}}^{\theta}$. $(k_{t-2t_0+1},k_{t-t_0+1}]$ turns to $(k_{r-1},k_r]$ for $t=r$, $t_0=1$ and $\mathbb{T}=\mathbb{N}$. In this case, lacunary statistical convergence on time scales is reduced to classical lacunary statistical convergence which is defined by Fridy and Orhan [16].*

In this study, we will give notions of $(\theta,m)$-uniform lacunary density of an arbitrary set, $(\theta,m)$-uniform lacunary statistical convergence and some properties of $(\theta,m)$-lacunary statistically convergent sequences on an arbitrary time scale. Before this, we recall some concepts about uniform density and uniform statistical convergence in classical case to use in our main results. Uniformly density of an arbitrary set was introduced by Raimi [32] as follows:

**Definition 1.3.** *A subset $E\subset\mathbb{N}$ is uniformly dense if*

$$u(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_E(j+m)=a,$$

*or equivalently*

$$\lim_{n\to\infty}\frac{1}{n}\left|E\cap\{m+1,...,m+n\}\right|=a,$$

*uniformly in $m$, where $m=0,1,2,...$ and $\chi_E$ is characteristic function.*

Subsequently, uniformly density was studied by Baláž and Šalát [2]. Later, $m$-uniform statistical convergence is introduced by Nuray [30] in the following manner.

**Definition 1.4.** *Let $x=(x_k)$ be a real or complex valued sequence. If*

$$\lim_{n\to\infty}\frac{1}{n}\left|\{m\leq k<n+m:|x_k-L|\geq\varepsilon\}\right|=0,$$

*uniformly in $m$, $x=(x_k)$ is said to be $m$-uniform statistically convergent to $L$ for all $\varepsilon>0$.*

Based on Definition 1.4, we can generalize $m$-uniform statistical convergence to lacunary type sequences as follows:

**Definition 1.5.** *Let $K\subset\mathbb{N}$ and $\theta$ be a lacunary sequence. Then, we define the $(\theta,m)$-uniform density of $K$ by*

$$\delta_\theta^m(K)=\lim_{r\to\infty}\frac{1}{h_{r,m}}\left|\{k_{r-1+m}<k\leq k_{r+m}:k\in K\}\right|,$$

*uniformly in $m\geq0$, where $h_{r,m}=k_{r+m}-k_{r+m-1}$.*

**Definition 1.6.** *A sequence $x=(x_k)$ is said to be $(\theta,m)$-uniform lacunary statistically convergent to a real number $L$ if*

$$\lim_{r\to\infty}\frac{1}{h_{r,m}}\left|\{k_{r-1+m}<k\leq k_{r+m}:|x_k-L|\geq\varepsilon\}\right|=0,$$

*for all $\varepsilon>0$, uniformly in $m$.*

Above definitions are special cases of $\sigma$-statistical convergence and lacunary $\sigma$-statistical convergence [34]. In the next section, we shall define above notions on time scale $\mathbb{T}$.

## 2. Main results

In this section, we define and study the $(\theta, m)$-density, $(\theta, m)$-uniform lacunary statistical convergence and $(\theta, m)$-uniform strongly $p$-lacunary summability on $\mathbb{T}$, where $\theta = \{k_{t-t_0+m+1}\}$ is a lacunary sequence for $t \in \mathbb{T}$.

**Definition 2.1.** *Let $\Omega$ be a $\Delta$-measurable subset of $\mathbb{T}$ and $\theta$ be a lacunary sequence. Then, we can define the set $\Omega(t, \theta, m)$ by*

$$\Omega(t, \theta, m) = \{s \in (k_{t-2t_0+m+1}, k_{t-t_0+m+1}]_{\mathbb{T}} : s \in \Omega\},$$

*for $t \in \mathbb{T}$. In this case, $(\theta, m)$-density of $\Omega$ on $\mathbb{T}$ is defined by*

$$\delta_{\mathbb{T}}^{\theta, m}(\Omega) = \lim_{t \to \infty} \frac{\mu_\Delta(\Omega(t, \theta, m))}{\mu_\Delta((k_{t-2t_0+m}, k_{t-t_0+m}]_{\mathbb{T}})}, \tag{2.1}$$

*provided that the above limit exists.*

**Definition 2.2.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta$-measurable function and $\theta$ be a lacunary sequence. Then, $f$ is $(\theta, m)$-uniform lacunary statistically convergent to a real number $L$ on $\mathbb{T}$ if*

$$\lim_{t \to \infty} \frac{\mu_\Delta(s \in (k_{t-2t_0+m+1}, k_{t-t_0+m+1}]_{\mathbb{T}} : |f(s) - L| \geq \varepsilon)}{\mu_\Delta((k_{t-2t_0+m}, k_{t-t_0+m}]_{\mathbb{T}})} = 0, \tag{2.2}$$

*uniformly in $m$, for all $\varepsilon > 0$ and $t \in \mathbb{T}$. In this case, $s_{\mathbb{T}}^{\theta,m}$-$\lim_{t\to\infty}(f(t)) = L$. The set of all $(\theta, m)$-uniform lacunary statistically convergent functions on $\mathbb{T}$ will be denoted by $s_{\mathbb{T}}^{\theta,m}$.*

We remark that $(k_{t-2t_0+m+1}, k_{t-t_0+m+1}]$ turns to $(k_{r+m-1}, k_{r+m}]$ when $t = r$, $t_0 = 1$ and $\mathbb{T} = \mathbb{N}$. In this instance, $(\theta, m)$-uniform lacunary statistical convergence on time scales is reduced to classical $(\theta, m)$-uniform lacunary statistical convergence which is given by Definition 1.6. This shows that our results are generalizations of classical results.

**Proposition 2.1.** *Let $\theta$ be a lacunary sequence. If $f, g : \mathbb{T} \to \mathbb{R}$ with $s_{\mathbb{T}}^{\theta,m}$-$\lim_{t\to\infty} f(t) = L_1$ and $s_{\mathbb{T}}^{\theta,m}$-$\lim_{t\to\infty} g(t) = L_2$, then the following statements hold:*

(i) $s_{\mathbb{T}}^{\theta,m}$-$\lim_{t\to\infty}(f(t) + g(t)) = L_1 + L_2$,

(ii) $s_{\mathbb{T}}^{\theta,m}$-$\lim_{t\to\infty}(cf(t)) = cL_1$ $(c \in \mathbb{R})$.

However, $m$-uniform statistical convergence on $\mathbb{T}$ was first defined by Altin et al. [1] in the following way.

**Definition 2.3.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a $\Delta$-measurable function. Then, $f$ is $m$-uniform statistically convergent to a real number $L$ on $\mathbb{T}$ if*

$$\lim_{t \to \infty} \frac{\mu_\Delta(s \in [m + t_0 - 1, t + m) : |f(s) - L| \geq \varepsilon)}{\mu_\Delta([m + t_0 - 1, t + m)_{\mathbb{T}})} = 0,$$

*for all $\varepsilon > 0$ and uniformly in $m$. In this case, $s_{\mathbb{T}}^{m}$-$\lim_{t\to\infty}(f(t)) = L$. The set of all $m$-uniform statistically convergent functions on $\mathbb{T}$ is denoted by $s_{\mathbb{T}}^{m}$.*

Note that above Definition 2.3 is a generalization of Definition 1.4. Now we can give some inclusion relations between $s_{\mathbb{T}}^{m}$, $s_{\mathbb{T}}^{\theta,m}$ and $s_{\mathbb{T}}^{\theta}$.

**Theorem 2.1.** *Let $\theta = \{k_{t-t_0+m+1}\}$ be a lacunary sequence for $t \in \mathbb{T}$ uniformly in $m$. Then,*

(i) $s_{\mathbb{T}}^{\theta,m} \subset s_{\mathbb{T}}^{m}$ *if* $\limsup_t \left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \infty$,

(ii) $s_{\mathbb{T}}^{m} \subset s_{\mathbb{T}}^{\theta}$ *if* $\liminf_t \left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) > 1$,

(iii) $s_{\mathbb{T}}^{m} = s_{\mathbb{T}}^{\theta}$ *if* $1 < \liminf_t \left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \limsup_t \left(\frac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \infty$.

*Proof.* It can be proved by using a similar approach to Theorem 3.3 of [31]. □

The definition of strongly $p$-Cesàro summability on $\mathbb{T}$ was given by Turan and Duman [38] in the following manner.

**Definition 2.4.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta$-measurable function and $0<p<\infty$. Then, $f$ is strongly $p$-Cesàro summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\frac{1}{\mu_{\Delta}\left([t_0,t]_{\mathbb{T}}\right)}\int\limits_{[t_0,t]_{\mathbb{T}}}|f(s)-L|^{p}\,\Delta s=0.$$

*The set of all strongly $p$-Cesàro summable functions on $\mathbb{T}$ is denoted by $[W_p]_{\mathbb{T}}$.*

The measure theory on time scales was first constructed by Guseinov [19] and Lebesque $\Delta$-integral on time scales introduced by Cabada and Vivero [6]. Now, we introduce $m$-uniform strongly $p$-summablility and $(\theta,m)$-uniform strongly $p$-lacunary summability of a $\Delta$-measurable function and establish some results.

**Definition 2.5.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta$-measurable function and $0<p<\infty$. Then, $f$ is $m$-uniform strongly $p$-summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\frac{1}{\mu_{\Delta}\left([m+t_0-1,t+m)_{\mathbb{T}}\right)}\int\limits_{[m+t_0-1,t+m)_{\mathbb{T}}}|f(s)-L|^{p}\,\Delta s=0.$$

*In this case, $\left[W_p^m\right]_{\mathbb{T}}$-$\lim f(s)=L$. The set of all $m$-uniform strongly $p$-summable functions on $\mathbb{T}$ will be denoted by $\left[W_p^m\right]_{\mathbb{T}}$.*

**Definition 2.6.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta$-measurable function and let $\theta$ be a lacunary sequence. Assume also that $0<p<\infty$. Then, $f$ is $(\theta,m)$-uniform strongly $p$-lacunary summable on $\mathbb{T}$ if there exists some $L\in\mathbb{R}$ such that*

$$\lim_{t\to\infty}\frac{1}{\mu_{\Delta}\left((k_{t-2t_0+m},k_{t-t_0+m}]_{\mathbb{T}}\right)}\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}}|f(s)-L|^{p}\,\Delta s=0.$$

*In that case, $\left[W_{\theta p}^m\right]_{\mathbb{T}}$-$\lim f(s)=L$. The set of all $(\theta,m)$-uniform strongly $p$-lacunary summable functions on $\mathbb{T}$ will be denoted by $\left[W_{\theta p}^m\right]_{\mathbb{T}}$.*

**Lemma 2.1.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta$-measurable function, $\theta$ be a lacunary sequence and*

$$\Omega(t,\theta,m)=\left\{s\in(k_{t-2t_0+m+1},k_{t-t_0+m+1}]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\right\},$$

*for all $\varepsilon>0$. Thus, we have*

$$\mu_{\Delta}\left(\Omega(t,\theta,m)\right)\leq\frac{1}{\varepsilon}\int\limits_{\Omega(t,\theta,m)}|f(s)-L|\,\Delta s\leq\frac{1}{\varepsilon}\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}}|f(s)-L|\,\Delta s,$$

*uniformly in $m$.*

*Proof.* It can be proved by using similar way with in [38]. □

**Theorem 2.2.** *Let $f:\mathbb{T}\to\mathbb{R}$ be a $\Delta$-measurable function and let $\theta$ be a lacunary sequence. Asume also that $0<p<\infty$ and $L\in\mathbb{R}$. Then,*

(i) *If $f$ is $(\theta,m)$-uniform strongly $p$-lacunary summable to $L$, then $s_{\mathbb{T}}^{\theta,m}$-$\lim\limits_{t\to\infty}(f(t))=L$.*

(ii) *If $s_{\mathbb{T}}^{\theta,m}$-$\lim\limits_{t\to\infty}(f(t))=L$ and $f$ is a bounded function, then $f$ is $(\theta,m)$-uniform strongly $p$-lacunary summable to $L$.*

*Proof.* (i) Suppose $f$ is $(\theta,m)$-uniform strongly $p$-lacunary summable to $L$. For given $\varepsilon>0$, let

$$\Omega(t,\theta,m)=\left\{s\in(k_{t-2t_0+m+1},k_{t-t_0+m+1}]_{\mathbb{T}}:|f(s)-L|\geq\varepsilon\right\}$$

on $\mathbb{T}$. Then, it follows

$$\varepsilon^p \mu_\Delta\left(\Omega\left(t,\theta,m\right)\right) \leq \int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}} \left|f\left(s\right)-L\right|^p \Delta s.$$

from lemma 2.1. Dividing this inequality by $\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_{\mathbb{T}}\right)$ and taking limit as $t \to \infty$, we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right)}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_{\mathbb{T}}\right)}\\ &\qquad \leq \frac{1}{\varepsilon^p}\lim_{t\to\infty}\frac{1}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_{\mathbb{T}}\right)}\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}} \left|f\left(s\right)-L\right|^p \Delta s = 0,\end{aligned}$$

which yields that $s_{\mathbb{T}}^{\theta,m}\text{-}\lim_{t\to\infty}\left(f\left(t\right)\right)=L$.

(ii) Suppose $f$ is bounded and $(\theta,m)$-uniform lacunary statistically convergent to $L$ on $\mathbb{T}$. Then, there exists a positive number $M$ such that $\left|f\left(s\right)\right| \leq M$ for all $s \in \mathbb{T}$, and also

$$\lim_{t\to\infty}\frac{\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right)}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_{\mathbb{T}}\right)} = 0, \tag{2.3}$$

where $\Omega\left(t,\theta,m\right)$ as defined before. Since

$$\begin{aligned}\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}} \left|f\left(s\right)-L\right|^p \Delta s &= \int\limits_{\Omega(t,\theta,m)} \left|f\left(s\right)-L\right|^p \Delta s\\ &\quad + \int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}/\Omega(t,\theta,m)} \left|f\left(s\right)-L\right|^p \Delta s\\ &\leq \left(M+\left|L\right|\right)^p \int\limits_{\Omega(t,\theta,m)} \Delta s + \varepsilon^p \int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}} \Delta s\\ &= \left(M+\left|L\right|\right)^p \mu_\Delta\left(\Omega\left(t,\theta,m\right)\right)\\ &\quad +\varepsilon^p \mu_\Delta\left(\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}\right),\end{aligned}$$

we obtain

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_{\mathbb{T}}\right)}\int\limits_{\left(k_{t-2t_0+m+1},k_{t-t_0+m+1}\right]_{\mathbb{T}}} \left|f\left(s\right)-L\right|^p \Delta s\\ &\qquad \leq \left(M+\left|L\right|\right)^p \lim_{t\to\infty}\frac{\mu_\Delta\left(\Omega\left(t,\theta,m\right)\right)}{\mu_\Delta\left(\left(k_{t-2t_0+m},k_{t-t_0+m}\right]_{\mathbb{T}}\right)} + \varepsilon^p. \end{aligned}\tag{2.4}$$

Since $\varepsilon$ is an arbitrary, the proof follows from (2.3) and (2.4). □

**Theorem 2.3.** *Let $\theta = \left\{k_{t-t_0+m+1}\right\}$ be a lacunary sequence for $t \in \mathbb{T}$. Then*

(i) $\left[W_{\theta p}^m\right]_{\mathbb{T}} \subset \left[W_p^m\right]_{\mathbb{T}}$ *if* $\limsup_t \left(\dfrac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \infty$,

(ii) $\left[W_p^m\right]_{\mathbb{T}} \subset \left[W_{\theta p}^m\right]_{\mathbb{T}}$ *if* $\liminf_t \left(\dfrac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) > 1$,

(iii) $\left[W_p^m\right]_{\mathbb{T}} = \left[W_{\theta p}^m\right]_{\mathbb{T}}$ *if* $1 < \liminf_t \left(\dfrac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \limsup_t \left(\dfrac{k_{t-t_0+m+1}}{k_{t-2t_0+m+1}}\right) < \infty$.

*Proof.* We can prove by using similar techniques to Theorem 2.2, Theorem 2.3 and Theorem 2.4 of [31] in case of $p = 1$. □

## 3. Conclusion

In this study, we defined the concept of $(\theta, m)$-uniform lacunary density, $(\theta, m)$-uniform lacunary statistical convergence and $(\theta, m)$-uniform strongly $p$-lacunary summability on $\mathbb{T}$. We emphasize that the results that we obtained are more general than classical results mentioned in the theory of $m$-uniform statistical convergence. For example, Definition 1.6 is a generalization of the Definition 1.4 which is given by Nuray [30] to the lacunary type sequences. We firstly defined $(\theta, m)$-uniform lacunary statistical convergence in classical case to define it on $\mathbb{T}$. Then, we generalized this definition into $\mathbb{T}$. Furthermore, we defined $m$-uniform strongly $p$-summable functions and $m$-uniform statistical convergence on $\mathbb{T}$ by considering curicial results of Turan and Duman [38].

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