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QUADRUPLE FIXED POINT OF MULTIVALUED NONLINEAR CONTRACTION MAPPINGS

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Abstract. The notion of Quadruple fixed point is introduced by Karapinar E. [6]. Samet and Vetro [12] established some coupled fixed point theorems for multivalued non linear contraction mapping in partially ordered metric spaces. In this paper, we obtain existence of quadrupled fixed point of multivalued non linear contraction mappings in framework work of partially ordered metric spaces. Also, we give an example.

1. Introduction and Preliminary

Let (X,d) be a metric space. We denote by $CB(X)$ the collection of non- empty closed bounded subsets of X. For $A,B \in CB(X)$ and $x \in X$, suppose that

$$\begin{aligned} D(x,A) &= \inf_{a\in A} d(x,a) \\ H(A,B) &= \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}. \end{aligned}$$

Such mapping H is called a Housdorff metric on $CB(X)$ induced by d.

Definition 1. *An element $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to CB(X)$ iff $x \in Tx$.*

In 1969, Nadlar [8] extended the famous Banach contraction principle from single valued mapping to multivalued mapping and proved the following fixed point theorem for the multivalued contraction which state as follows,

Theorem 2. *Let (X,d) be a complete metric space and let T be a mapping from X into $CB(X)$. Assume that there exists $c \in [0,1)$ such that $H(Tx,Ty) \le cd(x,y)$ for all $x,y \in X$. Then T has a fixed point.*

The existence of fixed points for various multi valued contraction mappings has been studied by many authors under different conditions. In 1989, Mizoguchi and Takahashi [7] proved the following interesting fixed point theorem for a weak contraction.

Theorem 3. *Let (X,d) be a complete metric space and let T be a mapping form X into $CB(X)$. Assume that there exists $c \in [0,1)$ such that $H(Tx,Ty) \le \alpha(d(x,y))d(x,y)$ for all $x,y \in X$, where α is a function from $[0,\infty)$ into $[0,1)$,*

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satisfying the condition $\limsup_{s\to t^+}\alpha(s)<1$ *for all* $t\in[0,\infty)$. *Then* T *has a fixed point.*

Several authors studies the problem of existence of fixed point of multivalued mappings satisfying different contractive conditions (see e.g., [1, 2, 3, 4, 5, 7, 10, 11]). The theory of multivalued mapping has application in control theory, convex optimization, differential equations, and economics.

Existence of fixed points in ordered metric spaces has been initiated in 2004 by Ran and Reurings [10] further studied by Nieto and Rodriguez - Lopez [9]. Samet and Vetro [12] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for $N=4$ (Quadruple case) i.e., Let $(X,\preceq)$ be partially ordered set and (X,d) be a complete metric space. We consider the following partial order on the product space $X^4=X\times X\times X\times X$:

$$(u,v,r,t)\preceq(x,y,z,w)\ \ iff\ \ x\preceq u,\ \ y\preceq\ \ v,\ \ z\preceq r,\ \ t\preceq w, \tag{1.1}$$

where $(u,v,r,t),(x,y,z,w)\in X^4$.

Regarding this partial order Karapinar [6] give the following definitions,

Definition 4. *Let* $(X,\preceq)$ *be partially ordered set and* $F:X^4\to X$. *We say that* F *has the mixed monotone property if* $F(x,y,z,w)$ *is monotone non decreasing in* x *and* z *and it is monotone non increasing in* y *and* w, *that is, for any* $x,y,z,w\in X$

$$\begin{aligned}
x_1,x_2\in X,\ \ x_1 &\preceq x_2\Longrightarrow F(x_1,y,z,w)\preceq F(x_2,y,z,w)\\
y_1,y_2\in X,\ \ y_1 &\preceq y_2\Longrightarrow F(x,y_2,z,w)\preceq F(x,y_1,z,w)\\
z_1,z_2\in X,\ \ z_1 &\preceq z_2\Longrightarrow F(x,y,z_1,w)\preceq F(x,y,z_2,w)\\
w_1,w_2\in X,\ \ w_1 &\preceq w_2\Longrightarrow F(x,y,z,w_2)\preceq F(x,y,z,w_1).
\end{aligned} \tag{1.2}$$

Definition 5. *An element* $(x,y,z,w)\in X^4$ *is called a quadruple fixed point of* $F:X^4\to X$ *if*

$$\begin{aligned}
F(x,y,z,w) &= x,\ \ F(y,z,w,x)=y,\\
F(z,w,x,y) &= z,\ \ F(w,x,y,z)=w.
\end{aligned} \tag{1.3}$$

For a metric space (X,d) the function $\rho:X^4\to[0,\infty)$, *given by*

$$\rho((x,y,z,w),(u,v,r,t))=d(x,u)+d(y,v)+d(z,r)+d(w,t) \tag{1.4}$$

forms a metric space on X^4, *that is,* (X^4,ρ) *is a metric induced by (X,d).*

2. Quadruple Fixed Point Result for Multivalued Mappings

First we introduced the following concepts.

Definition 6. *An element* $(x,y,z,w)\in X^4$ *is called a Quadruple fixed point of* $F:X^4\to CL(X)$ *if*

$$\begin{aligned}
x &\in F(x,y,z,w),\ \ y\in F(y,z,w,x),\\
z &\in F(z,w,x,y),\ \ w\in F(w,x,y,z)
\end{aligned} \tag{2.1}$$

Definition 7. *A mapping* $f:X^4\to\mathbf{R}$ *is called lower semi continuous if, for the sequences* $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$ *in* X *and* $(x,y,z,w)\in X^4$, *one has*

$$\begin{aligned} \lim_{n\to\infty}(\{x_n\},\{y_n\},\{z_n\},\{w_n\}) &= (x,y,z,w) \Longrightarrow f(x,y,z,w) \\ &\preceq \lim_{n\to\infty}\inf\{\{x_n\},\{y_n\},\{z_n\},\{w_n\}\} \end{aligned} \tag{2.2}$$

Let (X,d) be a metric space endowed with the partial order $\preceq$ and $T: X \to X$. Define the set $\Psi \subset X^4$ by,

$$\Psi = \{(x,y,z,w) \in X^4 : T(x) \preceq T(y) \preceq T(z) \preceq T(w)\} \tag{2.3}$$

Definition 8. *A mapping $F: X^4 \to X$ is said to have a Ψ- property if,*

$$(x,y,z,w) \in \Psi \Longrightarrow F(x,y,z,w) \times F(y,z,w,x) \times F(z,w,x,y) \times F(w,x,y,z) \subset \Psi. \tag{2.4}$$

We give some examples to illustrate Definition 8.

Example 9. *Let $X = \mathbf{R}$ be endowed with the usual order $\preceq$ and $T: X \to X$. Define $F: X^4 \to CL(X)$ by,*

$$F(x,y,z,w) = \{x\} \tag{2.5}$$

Obviously F has the Ψ- property.

Example 10. *Let $X = \mathbf{R}^+$ be endowed with the usual order $\leq$ and $T: X \to X$ be defined by $Tx = exp(x)$. Define $F: X^4 \to CL(X)$ by,*

$$F(x,y,z,w) = \{x+w\} \quad \forall x,y,z,w \in \mathbf{R}^+ \tag{2.6}$$

We have $\Psi = \{(x,y,z,w) \in X^4, exp(x) \preceq exp(y) \preceq exp(z) \preceq exp(w)\}$. Moreover, F has the $\Psi-$ property.

Now, we prove the following theorem.

Theorem 11. *Let (X,d) be a complete metric space endowed with a partial order $\preceq$ and $\Psi \neq \phi$ that is there exists $(x_0,y_0,z_0,w_0) \in Psi$. Suppose that $F: X^4 \to CL(X)$ has a $\Psi-$ property such that $f: X^4 \to [0,\infty)$ given by for all $x,y,z,w \in X$,*

$$\begin{aligned} f(x,y,z,w) &= D(x,F(x,y,z,w)) + D(y,F(y,z,w,y)) \\ &+ D(z,F(z,w,x,y)) + D(w,F(w,x,y,z)) \end{aligned} \tag{2.7}$$

is lower semi continuous and there exists a function $\phi: [0,\infty) \to [M,1)$, $0 < M < 1$ satisfying

$$\lim_{r\to s^+}\sup \phi(r) < 1 \ \text{ for each } \ s \in [0,\infty) \tag{2.8}$$

if for any $(x,y,z,w) \in \Psi$ there exist $u \in F(x,y,z,w), v \in F(y,z,w,x), r \in F(z,w,x,y), t \in F(w,x,y,z)$ with

$$\sqrt{\phi(f(x,y,z,w))}[d(x,u)+d(y,v)+d(z,r)+d(w,t)] \quad \leq \quad f(x,y,z,w) \tag{2.9}$$

such that

$$f(u,v,r,t) \quad \leq \quad \phi(f(x,y,z,w))[d(x,u)+d(y,v)+d(z,r)+d(w,t)] \tag{2.10}$$

then F has a quadruple fixed point.

Proof. By our assumption, $\phi(f(x,y,z,w)) < 1$ for each $(x,y,z,w) \in X^4$. Hence , for any $(x,y,z,w) \in X^4$, there exist $u \in F(x,y,z,w), v \in F(y,z,w,x), r \in F(z,w,x,y), t \in F(w,x,y,z)$ satisfying

$$\begin{aligned}
\sqrt{\phi(f(x,y,z,w))}d(x,u) &\preceq D(x,F(x,y,z,w)) \\
\sqrt{\phi(f(x,y,z,w))}d(y,v) &\preceq D(y,F(y,z,w,x)) \\
\sqrt{\phi(f(x,y,z,w))}d(z,r) &\preceq D(z,F(z,w,x,y)) \\
\sqrt{\phi(f(x,y,z,w))}d(w,t) &\preceq D(w,F(w,x,y,z)). \qquad (2.11)
\end{aligned}$$

Let (x_0,y_0,z_0,w_0) be an arbitrary point in Ψ. From (2.8) and (2.9) we can choose $x_1 \in F(x_0,y_0,z_0,w_0), y_1 \in F(y_0,z_0,w_0,x_0), z_1 \in F(z_0,w_0,x_0,y_0), w_1 \in F(w_0,x_0,y_0,z_0)$ satisfying

$$\sqrt{\phi(f(x_0,y_0,z_0,w_0))}[d(x_0,x_1)+d(y_0,y_1)+d(z_0,z_1)+d(w_0,w_1)] \preceq f(x_0,y_0,z_0,w_0) \qquad (2.12)$$

such that

$$f(x_1,y_1,z_1,w_1) \preceq \phi(f(x_0,y_0,z_0,w_0))[d(x_0,x_1)+d(y_0,y_1)+d(z_0,z_1)+d(w_0,w_1)] \qquad (2.13)$$

By 2.12 and 2.13, we obtain

$$\begin{aligned}
f(x_1,y_1,z_1,w_1) &\preceq \phi(f(x_0,y_0,z_0,w_0))[d(x_0,x_1)+d(y_0,y_1)+d(z_0,z_1)+d(w_0,w_1)] \\
&\preceq \sqrt{\phi(f(x_0,y_0,z_0,w_0))}\,(\phi(f(x_0,y_0,z_0,w_0)) \\
&\quad [d(x_0,x_1)+d(y_0,y_1)+d(z_0,z_1)+d(w_0,w_1)]) \\
f(x_1,y_1,z_1,w_1) &\preceq \sqrt{\phi(f(x_0,y_0,z_0,w_0))}f(x_0,y_0,z_0,w_0).
\end{aligned}$$

Since F has a $\Psi-$ property and $(x_0,y_0,z_0,w_0) \in \Psi$, so we have

$$F(x_0,y_0,z_0,w_0) \times F(y_0,z_0,w_0,x_0) \times F(z_0,w_0,x_0,y_0) \times F(w_0,x_0,y_0,z_0) \subset \Psi \qquad (2.14)$$

which implies that $(x_1,y_1,z_1,w_1) \in \Psi$.

Again by 2.9 and 2.10 we can choose ,

$x_2 \in F(x_1,y_1,z_1,w_1), y_2 \in F(y_1,z_1,w_1,x_1), z_2 \in F(z_1,w_1,x_1,y_1), w_2 \in F(w_1,x_1,y_1,z_1)$ satisfying

$$\sqrt{\phi(f(x_1,y_1,z_1,w_1))}[d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)] \preceq f(x_1,y_1,z_1,w_1) \qquad (2.15)$$

such that

$f(x_1,y_1,z_1,w_1) \preceq \phi(f(x_1,y_1,z_1,w_1))[d(x_1,x_2)+d(y_1,y_2)+d(z_1,z_2)+d(w_1,w_2)]$ (2.16)

Thus we have

$$f(x_1,y_1,z_1,w_1) \preceq \sqrt{\phi(f(x_1,y_1,z_1,w_1))}f(x_1,y_1,z_1,w_1) \tag{2.17}$$

which implies that $(x_2,y_2,z_2,w_2) \in \Psi$.

Continuing this process, we can choose sequences $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$, in X such that for each $n \in N$ with $(x_n,y_n,z_n,w_n) \in \Psi$.

Now $x_{n+1} \in F(x_n,y_n,z_n,w_n)$, $y_{n+1} \in F(y_n,z_n,w_n,x_n)$, $z_{n+1} \in F(z_n,w_n,x_n,y_n)$, $w_{n+1} \in F(w_n,x_n,y_n,z_n)$ satisfying

$$\begin{aligned}\sqrt{\phi(f(x_n,y_n,z_n,w_n))}[d(x_n,x_{n+1}) \quad &+ \\ d(y_n,y_{n+1})+d(z_n,z_{n+1}) \quad &+ \quad d(w_n,w_{n+1})] \preceq f(x_n,y_n,z_n,w_n)\end{aligned} \tag{2.18}$$

such that

$$\begin{aligned}f(x_{n+1},y_{n+1},z_{n+1},w_{n+1}) \quad &\preceq \quad \phi(f(x_n,y_n,z_n,w_n)[d(x_n,x_{n+1}) \\ &\quad +d(y_n,y_{n+1})+d(z_n,z_{n+1})+d(w_n,w_{n+1})].\end{aligned} \tag{2.19}$$

Hence, we obtain

$$f(x_{n+1},y_{n+1},z_{n+1},w_{n+1}) \preceq \sqrt{\phi(f(x_n,y_n,z_n,w_n)}f(x_n,y_n,z_n,w_n) \tag{2.20}$$

with

$$(x_{n+1},y_{n+1},z_{n+1},w_{n+1}) \in \Psi. \tag{2.21}$$

We claim that $f(x_n,y_n,z_n,w_n) \to 0$ as $n \to \infty$.
If $f(x_n,y_n,z_n,w_n) = 0$ for some $n \in N$, then

$$\begin{gathered} D(x_n, F(x_n,y_n,z_n,w_n)) = 0 \text{ implies that} \\ x_n \in \overline{F(x_n,y_n,z_n,w_n)} = F(x_n,y_n,z_n,w_n). \end{gathered}$$

Analogously,

$$\begin{gathered} D(y_n, F(y_n,z_n,w_n,x_n)) = 0 \text{ implies that} \\ y_n \in \overline{F(y_n,z_n,w_n,x_n)} = F(y_n,z_n,w_n,x_n)\ , \end{gathered}$$

$$\begin{gathered} D(z_n, F(z_n,w_n,x_n,y_n)) = 0 \text{ implies that} \\ z_n \in \overline{F(z_n,w_n,x_n,y_n)} = F(z_n,w_n,x_n,y_n)\ , \end{gathered}$$

$$D(w_n, F(w_n, x_n, y_n, z_n)) = 0 \text{ implies that}$$
$$w_n \in \overline{F(w_n, x_n, y_n, z_n)} = F(w_n, x_n, y_n, z_n).$$

Hence (x_n, y_n, z_n, w_n) become a quadruple fixed point of F for such n and the result follows. Suppose that $f(x_n, y_n, z_n, w_n) > 0$ for all $n \in N$.
Using 2.20 and $\phi(t) < 1$, we conclude that $\{F(x_n, y_n, z_n, w_n)\}$ is decreasing sequence of positive real numbers. Thus, there exists a $\delta \geq 0$ such that

$$\lim_{n\to\infty} f(x_n, y_n, z_n, w_n) \quad = \quad \delta \tag{2.22}$$

We will show that $\delta = 0$. Assume on contrary that $\delta > 0$. Let $n \to \infty$ in 2.20 and by assumption 2.8 we obtain

$$\delta \quad \preceq \quad \lim_{f(x_n,y_n,z_n,w_n)\to\delta^+} \sup \sqrt{\phi(f(x_n, y_n, z_n, w_n))\delta} < \delta, \tag{2.23}$$

a contradiction, Hence

$$\lim_{n\to\infty} f(x_n, y_n, z_n, w_n) \quad = \quad 0^+ \tag{2.24}$$

Now, we prove that sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ in X are Cauchy sequences in (X, d). Assume that

$$\alpha \quad = \quad \lim f(x_n, y_n, z_n, w_n) \to 0^+ \sup \sqrt{\phi(f(x_n, y_n, z_n, w_n))}. \tag{2.25}$$

By 2.8 we conclude that $\alpha < 1$. Let k be a real number such that $\alpha < k < 1$. Thus there exists $n_0 \in N$ such that

$$\sqrt{\phi(f(x_n, y_n, z_n, w_n))} \preceq k \quad for \;\; each \;\; n \quad \geq \quad n_0. \tag{2.26}$$

Using 2.20 we obtain

$$f(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) \preceq kf(x_n, y_n, z_n, w_n) \quad for \;\; each \;\; n \quad \geq \quad n_0. \tag{2.27}$$

By mathematical induction,

$$f(x_{n+1}, y_{n+1}, z_{n+1}, w_{n+1}) \preceq k^{n+1-n_0} f(x_{n_0}, y_{n_0}, z_{n_0}, w_{n_0}) \quad for \;\; each \;\; n \quad \geq \quad n_0. \tag{2.28}$$

Since $\phi(t) \geq M < 0$ for all $t \geq 0$ so 2.18 and 2.28 gives that

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1}) + d(z_n, z_{n+1}) + d(w_n, w_{n+1})] \preceq \frac{k^{n-n_0}}{\sqrt{M}}(x_{n_0}, y_{n_0}, z_{n_0}, w_{n_0}) \tag{2.29}$$

$for \;\; each \;\; n \geq n_0$, which yields that the sequences $\{x_n\}, \{y_n\}, \{z_n\}, \{w_n\}$ in X are Cauchy sequences in (X, d). Since X is complete then there exists $(a, b, c, d) \in X^4$ such that

$$\begin{aligned} \lim_{n\to\infty} x_n &= a, \lim_{n\to\infty} y_n = b, \\ \lim_{n\to\infty} z_n &= c, \lim_{n\to\infty} w_n = d. \end{aligned} \tag{2.30}$$

Finally we show that $(a, b, c, d) \in X^4$ is quadruple fixed point of F. As f is lower semi continuous 2.24 implies that

$$\begin{aligned} 0 &\preceq f(a, b, c, d) \\ &= D(a, F(a, b, c, d)) + D(b, F(b, c, d, a)) \\ &+ D(c, F(c, d, a, b)) + D(d, F(d, a, b, c)) \\ &\preceq \lim_{n\to\infty} \inf f(x_n, y_n, z_n, w_n) = \delta. \end{aligned} \tag{2.31}$$

Hence,

$$D(a, F(a, b, c, d)) = D(b, F(b, c, d, a)) = 0$$

$$D(c, F(c, d, a, b)) = D(d, F(d, a, b, c)) = 0$$

gives that (a, b, c, d) is a quadruple fixed point of F.

□

Theorem 12. *Let (X, d) be a complete metric space endowed with a partial order $\preceq$ and $\Psi \neq \phi$ that is there exists $(x_0, y_0, z_0, w_0) \in Psi$. Suppose that $F : X^4 \to CL(X)$ has a $\Psi-$ property such that $f : X^4 \to [0, \infty)$ given by*

$$\begin{aligned} f(x, y, z, w) &= D(x, F(x, y, z, w)) + D(y, F(y, z, w, y)) \\ &+ D(z, F(z, w, x, y)) + D(w, F(w, x, y, z)) \end{aligned} \tag{2.32}$$

for all $x, y, z, w \in X$ and f is lower semi continuous and there exists a function $\phi : [0, \infty) \to [M, 1)$, $0 < M < 1$, satisfying

$$\lim_{r\to s^+} \sup \phi(r) < 1 \quad \textit{for each } s \in [0, \infty) \tag{2.33}$$

if for any $(x, y, z, w) \in \Psi$ there exist $u \in F(x, y, z, w), v \in F(y, z, w, x), r \in F(z, w, x, y), t \in F(w, x, y, z)$ with

$$\begin{aligned} \sqrt{\phi(\Delta)}\Delta &\preceq D(x, F(x, y, z, w)) + D(y, F(y, z, w, y)) \\ &+ D(z, F(z, w, x, y)) + D(w, F(w, x, y, z)) \end{aligned} \tag{2.34}$$

such that

$$\begin{aligned} D(u, F(u, v, r, t)) + D(v, F(v, r, t, u)) + D(r, F(r, t, u, v)) &+ D(t, F(t, u, v, r))) \\ &\preceq \phi(\Delta)\Delta \end{aligned} \tag{2.35}$$

where $\Delta = \Delta((x, y, z, w), (u, v, r, t)) = [d(x, u) + d(y, v) + d(z, r) + d(w, t)]$ then F has a quadruple fixed point.

Proof. By replacing $\phi(f(x,y,z,w))$ with $[d(x,u)+d(y,v)+d(z,r)+d(w,t)]$ in the proof of Theorem 11 we obtain sequences $\{x_n\},\{y_n\},\{z_n\},\{w_n\}$, in X such that for each $n \in N$ with, $(x_n,y_n,z_n,w_n) \in \Psi$

$$\begin{aligned} x_{n+1} &\in F(x_n,y_n,z_n,w_n), \quad y_{n+1} \in F(y_n,z_n,w_n,x_n) \\ z_{n+1} &\in F(z_n,w_n,x_n,y_n), \quad w_{n+1} \in F(w_n,x_n,y_n,z_n) \end{aligned} \tag{2.36}$$

such that

$$\begin{aligned} \sqrt{\phi(\Delta_n)}\Delta_n &\preceq D(x_n,F(x_n,y_n,z_n,w_n))+D(y_n,F(y_n,z_n,w_n,y_n)) \\ &+ D(z_n,F(z_n,w_n,x_n,y_n))+D(w_n,F(w_n,x_n,y_n,z_n)) \end{aligned} \tag{2.37}$$

$$\begin{aligned} D(x_{n+1},F(x_{n+1},y_{n+1},z_{n+1},w_{n+1})) &+ D(y_{n+1},F(y_{n+1},z_{n+1},w_{n+1},y_{n+1})) \\ &+ D(z_{n+1},F(z_{n+1},w_{n+1},x_{n+1},y_{n+1})) \\ &+ D(w_{n+1},F(w_{n+1},x_{n+1},y_{n+1},z_{n+1})) \\ &\preceq \sqrt{\phi(\Delta_n)}(D(x_n,F(x_n,y_n,z_n,w_n)) \\ &+ D(y_n,F(y_n,z_n,w_n,y_n)) \\ &+ D(z_n,F(z_n,w_n,x_n,y_n)) \\ &+ D(w_n,F(w_n,x_n,y_n,z_n))). \end{aligned} \tag{2.38}$$

where

$$\Delta_n = \Delta((x_n,y_n,z_n,w_n)(x_{n+1},y_{n+1},z_{n+1},w_{n+1})) \tag{2.39}$$
$$= d(x_n,x_{n+1})+d(y_n,y_{n+1})+d(z_n,z_{n+1})+d(w_n,w_{n+1}). \tag{2.40}$$

Again following arguments similar to those given in proof of Theorem 11 we deduce that

$$\begin{aligned} D(x_n,F(x_n,y_n,z_n,w_n)) &+ D(y_n,F(y_n,z_n,w_n,y_n)) \\ &+ D(z_n,F(z_n,w_n,x_n,y_n))+D(w_n,F(w_n,x_n,y_n,z_n)). \end{aligned} \tag{2.41}$$

is a decreasing sequence of real numbers. Thus, there exists a $\delta > 0$ such that

$$\begin{aligned} &\lim_{n\to\infty}(D(x_n,F(x_n,y_n,z_n,w_n))+D(y_n,F(y_n,z_n,w_n,y_n)) \\ &+ D(z_n,F(z_n,w_n,x_n,y_n))+D(w_n,F(w_n,x_n,y_n,z_n)))=\delta. \end{aligned} \tag{2.42}$$

Now we need to proof that $\{\Delta_n\}$ admits a subsequence converging to certain η^+ for some $\eta \geq 0$. Since $\phi(t) \leq M > 0$, using 2.37 we obtain

$$\begin{aligned} \delta_n &\preceq \frac{1}{\sqrt{a}}(D(x_n,F(x_n,y_n,z_n,w_n))+D(y_n,F(y_n,z_n,w_n,y_n)) \\ &+ D(z_n,F(z_n,w_n,x_n,y_n))+D(w_n,F(w_n,x_n,y_n,z_n))) \end{aligned} \tag{2.43}$$

from 2.42 and 2.43 it is clear that the sequence

$$\begin{aligned}(D(x_n, F(x_n, y_n, z_n, w_n)) &+ D(y_n, F(y_n, z_n, w_n, y_n)) \\ &+ D(z_n, F(z_n, w_n, x_n, y_n)) + D(w_n, F(w_n, x_n, y_n, z_n)))\end{aligned} \tag{2.44}$$

is bounded. Therefore, there is some $\theta \geq 0$ such that

$$\lim_{n\to\infty} \inf \Delta_n = \theta \tag{2.45}$$

from 2.36 we have

$$x_{n+1} \in F(x_n, y_n, z_n, w_n), y_{n+1} \in F(y_n, z_n, w_n, x_n),$$

$$z_{n+1} \in F(z_n, w_n, x_n, y_n), w_{n+1} \in F(w_n, x_n, y_n, z_n),$$

$$\begin{aligned}\Delta_n &\succeq D(x_n, F(x_n, y_n, z_n, w_n)) + D(y_n, F(y_n, z_n, w_n, y_n)) \\ &+ D(z_n, F(z_n, w_n, x_n, y_n)) + D(w_n, F(w_n, x_n, y_n, z_n)).\end{aligned} \tag{2.46}$$

By comparing 2.42 to 2.45 we get that $\theta \geq \delta$. Now, we shall show that $\theta = \delta$. If $\delta = 0$, by 2.42 and 2.43 we get $\theta = lim \quad inf_{n\to\infty^+} \Delta_n = 0$ and consequently $\theta = \delta = 0$. Suppose that $\delta > 0$. Assume on contrary that $\theta > \delta$. From 2.42 and 2.45 there is a positive integer n_0 such that

$$\begin{aligned}D(x_n, F(x_n, y_n, z_n, w_n)) &+ D(y_n, F(y_n, z_n, w_n, y_n)) \\ &+ D(z_n, F(z_n, w_n, x_n, y_n)) + D(w_n, F(w_n, x_n, y_n, z_n)) \\ &\preceq \delta + \frac{\theta - \delta}{4}\end{aligned} \tag{2.47}$$

$$\delta - \frac{\theta - \delta}{4} \preceq \Delta_n \tag{2.48}$$

for all $n \geq n_0$. We combine 2.37, 2.47 and 2.48 to obtain

$$\begin{aligned}\sqrt{\phi((\Delta_n)}\left(\delta - \frac{\theta - \delta}{4}\right) &\preceq \sqrt{\phi((\Delta_n)}\Delta_n \\ &\preceq D(x_n, F(x_n, y_n, z_n, w_n)) + D(y_n, F(y_n, z_n, w_n, y_n)) \\ &+ D(z_n, F(z_n, w_n, x_n, y_n)) + D(w_n, F(w_n, x_n, y_n, z_n)) \\ &\preceq \delta + \frac{\theta - \delta}{4}\end{aligned} \tag{2.49}$$

for all $n \geq n_0$. It follows that

$$\sqrt{\phi((\Delta_n)} \preceq \frac{\theta + 3\delta}{3\theta + \delta} \quad \forall n \geq n_0. \tag{2.50}$$

By 2.38 and 2.50 we have

$$
\begin{aligned}
D(x_{n+1},F(x_{n+1},y_{n+1},z_{n+1},w_{n+1})) \quad &+ \quad D(y_{n+1},F(y_{n+1},z_{n+1},w_{n+1},y_{n+1}))\\
&+ \quad D(z_{n+1},F(z_{n+1},w_{n+1},x_{n+1},y_{n+1}))\\
&+ \quad D(w_{n+1},F(w_{n+1},x_{n+1},y_{n+1},z_{n+1}))\\
&\leq \quad hD(x_n,F(x_n,y_n,z_n,w_n))\\
&+ \quad D(y_n,F(y_n,z_n,w_n,y_n))\\
&+ \quad D(z_n,F(z_n,w_n,x_n,y_n))\\
&+ \quad D(w_n,F(w_n,x_n,y_n,z_n)).
\end{aligned}
\tag{2.51}
$$

where $h = \frac{\theta+3\delta}{3\theta+\delta}$. Since $\theta > \delta > 0$, therefore $h < 1$, so proceeding by induction and combining the above inequalities, it follows that

$$
\begin{aligned}
\delta \quad &\preceq \quad D(x_{n_0+k_0},F(x_{n_0+k_0},y_{n_0+k_0},z_{n_0+k_0},w_{n_0+k_0}))\\
&+ \quad D(y_{n_0+k_0},F(y_{n_0+k_0},z_{n_0+k_0},w_{n_0+k_0},y_{n_0+k_0}))\\
&+ \quad D(z_{n_0+k_0},F(z_{n_0+k_0},w_{n_0+k_0},x_{n_0+k_0},y_{n_0+k_0}))\\
&+ \quad D(w_{n_0+k_0},F(w_{n_0+k_0},x_{n_0+k_0},y_{n_0+k_0},z_{n_0+k_0}))\\
&\preceq \quad h^{k_0}[D(x_{n_0},F(x_{n_0},y_{n_0},z_{n_0},w_{n_0}))\\
&+ \quad D(y_{n_0},F(y_{n_0},z_{n_0},w_{n_0},y_{n_0}))\\
&+ \quad D(z_{n_0},F(z_{n_0},w_{n_0},x_{n_0},y_{n_0}))\\
&+ \quad D(w_{n_0},F(w_{n_0},x_{n_0},y_{n_0},z_{n_0}))]\delta.
\end{aligned}
\tag{2.52}
$$

for a positive integer k_0. Then, we obtain a contradiction, so we must have $\theta = \delta$. Now, we shall show that $\theta = 0$. Since

$$
\begin{aligned}
\theta = \delta \quad &\preceq \quad D(x_n,F(x_n,y_n,z_n,w_n)) + D(y_n,F(y_n,z_n,w_n,y_n))\\
&+ \quad D(z_n,F(z_n,w_n,x_n,y_n)) + D(w_n,F(w_n,x_n,y_n,z_n))\Delta_n
\end{aligned}
\tag{2.53}
$$

then we rewrite 2.45 as

$$
\lim_{n\to\infty^+} \inf \Delta_n = \theta^+. \tag{2.54}
$$

Hence, there exists a subsequence $\{\Delta_{n_k}\}$ of $\{\Delta_n\}$ such that $lim_{k\to\infty^+} inf \Delta_{n_k} = \theta^+$.

By 2.33 we have

$$
\lim_{\Delta_{n_k}\to\infty^+} \sup \sqrt{\phi(\Delta_{n_k})} < 1. \tag{2.55}
$$

From 2.38 we obtain

$$\begin{aligned}
D(x_{n_k+1}, F(x_{n_k+1}, y_{n_k+1}, z_{n_k+1}, w_{n_k+1})) &+ D(y_{n_k+1}, F(y_{n_k+1}, z_{n_k+1}, w_{n_k+1}, y_{n_k+1})) \\
+D(z_{n_k+1}, F(z_{n_k+1}, w_{n_k+1}, x_{n_k+1}, y_{n_k+1})) &+ D(w_{n_k+1}, F(w_{n_k+1}, x_{n_k+1}, y_{n_k+1}, z_{n_k+1})) \\
&\preceq \sqrt{\phi(\Delta_{n_k})}[D(x_{n_k}, F(x_{n_k}, y_{n_k}, z_{n_k}, w_{n_k})) \\
&+ D(y_{n_k}, F(y_{n_k}, z_{n_k}, w_{n_k}, y_{n_k})) \\
&+ D(z_{n_k}, F(z_{n_k}, w_{n_k}, x_{n_k}, y_{n_k})) \\
&+ D(w_{n_k}, F(w_{n_k}, x_{n_k}, y_{n_k}, z_{n_k}))].
\end{aligned} \tag{2.56}$$

Taking the limit as $k \to \infty$ and using 2.42 we have

$$\begin{aligned}
\delta &= \lim_{k\to\infty^+} \{\sup[D(x_{n_k+1}, F(x_{n_k+1}, y_{n_k+1}, z_{n_k+1}, w_{n_k+1})) \\
&+ D(y_{n_k+1}, F(y_{n_k+1}, z_{n_k+1}, w_{n_k+1}, y_{n_k+1})) \\
&+ D(z_{n_k+1}, F(z_{n_k+1}, w_{n_k+1}, x_{n_k+1}, y_{n_k+1})) \\
&+ D(w_{n_k+1}, F(w_{n_k+1}, x_{n_k+1}, y_{n_k+1}, z_{n_k+1}))]\} \\
&\preceq \lim_{k\to\infty^+} \sup \left[\sqrt{\phi(\Delta_{n_k})}\right]
\end{aligned} \tag{2.57}$$

$$\begin{aligned}
\lim_{k\to\infty^+} \{\sup[D(x_{n_k}, F(x_{n_k}, y_{n_k}, z_{n_k}, w_{n_k})) &+ D(y_{n_k}, F(y_{n_k}, z_{n_k}, w_{n_k}, y_{n_k})) \\
&+ D(z_{n_k}, F(z_{n_k}, w_{n_k}, x_{n_k}, y_{n_k})) \\
&+ D(w_{n_k}, F(w_{n_k}, x_{n_k}, y_{n_k}, z_{n_k}))] \\
&\preceq \left(\lim_{k\to\infty^+} \sup \sqrt{\phi(\Delta_{n_k})}\right) \delta.
\end{aligned} \tag{2.58}$$

Assume that $\delta > 0$, then from 2.57 we get that

$$1 \preceq \lim_{k\to\infty^+} \sup \sqrt{\phi(\Delta_{n_k})} \tag{2.59}$$

a contradiction with respect to 2.55 so $\delta = 0$.
Now, from 2.38 and 2.42 we have

$$\alpha = \lim_{\Delta_n \to 0} \sup \sqrt{\phi(\Delta_n)} < 1 \tag{2.60}$$

The rest of the proof is similar to the proof of the Theorem 11 so it is omitted. $\square$

We improve and corrected the example of Samet and Vetro [12].

3. Examples

Example 13. *Let $X = [0, 2]$, and let $d : X \times X \to [0, \infty)$ be the usual metric. Suppose that $T(x) = M$ for all $x \in [0, 2]$ where M is a constant in [0,2], and $F : X^4 \to CL(X)$ is defined for all $x, y, z, w \in X$ as follows*

$$F(x, y, z, w) = \begin{cases} \frac{x^2}{4} & if \;\; x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 2\right] \\ \{\frac{15}{96}, \frac{1}{5}\} & if \;\; x = \frac{15}{32} \end{cases}$$

Oviously, F has the Ψ- property. Set $\phi : [0,\infty) \to [0,\infty)$

$$\phi(s) = \begin{cases} \frac{11}{12}s & if \ \ s \in \left[0, \frac{2}{3}\right] \\ \frac{10}{16} & if \ \ s \in \left(\frac{2}{3}, \infty\right) \end{cases}$$

Consider a function

$$f(x,y,z,w) = \begin{cases} A & if \ \ x,y,z,w \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 2\right] \\ B & if \ \ x,y,z \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 2\right] \ \ with \ \ w = \frac{15}{32} \\ C & if \ \ x,y \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 2\right] \ \ with \ \ z = w = \frac{15}{32} \\ D & if \ \ x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{5}{32}, 2\right] \ \ with \ \ y = z = w = \frac{15}{32} \\ E & if \ \ x = y = z = w = \frac{15}{32} \end{cases}$$

where, $A = x+y+z+w-\frac{1}{4}(x^2+y^2+z^2+w^2)$, $B = x+y+z-\frac{1}{4}(x^2+y^2+z^2)+\frac{43}{160}$, $C = x+y-\frac{1}{4}(x^2+y^2)+\frac{86}{160}$, $D = x-\frac{1}{4}(x^2)+\frac{129}{160}$, $E = \frac{172}{160}$

which is lower semicontinuous. Thus for all $x,y,z,w \in X$ with $x,y,z,w \neq \frac{5}{32}$, there exists $u \in F(x,y,z,w) = \frac{x^2}{4}$, $v \in F(y,z,w,x) = \frac{y^2}{4}$, $r \in F(z,w,x,y) = \frac{z^2}{4}$, $t \in F(w,x,y,z) = \frac{w^2}{4}$ such that

$$D(u,F(u,v,r,t)) + D(v,F(v,r,t,u)) + D(r,F(r,t,u,v)) + D(t,F(t,u,v,r))$$

$$\begin{aligned}
&= \frac{x^2}{4} - \frac{x^4}{16} + \frac{y^2}{4} - \frac{y^4}{16} + \frac{z^2}{4} - \frac{z^4}{16} + \frac{w^2}{4} - \frac{w^4}{16} \\
&= \frac{1}{4}\left[\left(x+\frac{x^2}{4}\right)\left(x-\frac{x^2}{4}\right) + \left(y+\frac{y^2}{4}\right)\left(y-\frac{y^2}{4}\right) + \left(z+\frac{z^2}{4}\right)\left(z-\frac{z^2}{4}\right)\right. \\
&\quad \left. + \left(w+\frac{w^2}{4}\right)\left(w-\frac{w^2}{4}\right)\right] \\
&\leq \frac{1}{4}\left[\left(x+\frac{x^2}{4}\right)d(x,u) + \left(y+\frac{y^2}{4}\right)d(y,v) + \left(z+\frac{z^2}{4}\right)d(z,r) + \left(w+\frac{w^2}{4}\right)d(w,t)\right] \\
&\preceq \frac{1}{4}max\{\left(x+\frac{x^2}{4}\right), \left(y+\frac{y^2}{4}\right), \left(z+\frac{z^2}{4}\right), \left(w+\frac{w^2}{4}\right)\} \\
&\quad d(x,u)+d(y,v)+d(z,r)+d(w,t) \\
&\preceq \frac{10}{12}max\{\left(x-\frac{x^2}{4}\right), \left(y-\frac{y^2}{4}\right), \left(z-\frac{z^2}{4}\right), \left(w-\frac{w^2}{4}\right)\} \\
&\quad d(x,u)+d(y,v)+d(z,r)+d(w,t)
\end{aligned}$$

$$\preceq \phi(d(x,u)+d(y,v)+d(z,r)+d(w,t))[d(x,u)+d(y,v)+d(z,r)+d(w,t)]$$

Hence for all $x,y,z,w \in X$ with $x,y,z,w \neq \frac{15}{32}$, the conditions 2.9 and 2.10 are satisfied. Analogously, one can easy show that conditions 2.9 and 2.10 are satisfied for the cases $x,y,z \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 2\right]$ with $w = \frac{15}{32}$ and $x \in \left[0, \frac{15}{32}\right) \cup \left(\frac{15}{32}, 2\right]$ with $y = z = w = \frac{15}{32}$. For the last case, that is $x = y = z = w = \frac{15}{36}$, we assume that $u = v = r = t = \frac{15}{96}$, it follows that,

$$[d(x,u)+d(y,v)+d(z,r)+d(w,t)] = \frac{5}{4} > \frac{2}{3}$$

As a consequence, we conclude that all the conditions of Theorem 2.7 are satisfied and F admits a quadruple fixed point i.e. $(0, 0, 0, 0)$.

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