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GENERALIZED STABILITIES OF EULER-LAGRANGE-JENSEN (a, b)-SEXTIC FUNCTIONAL EQUATIONS IN QUASI-β-NORMED SPACES

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Abstract. The aim of this paper is to investigate generalized Ulam-Hyers stabilities of the following Euler-Lagrange-Jensen-(a, b)-sextic functional equation

$$\begin{aligned} f(ax+by)+f(bx+ay)+(a-b)^6\left[f\left(\frac{ax-by}{a-b}\right)+f\left(\frac{bx-ay}{b-a}\right)\right] \\ = 64(ab)^2\left(a^2+b^2\right)\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)\right] \\ + 2\left(a^2-b^2\right)\left(a^4-b^4\right)[f(x)+f(y)] \end{aligned}$$

where $a \neq b$, such that $k \in \mathbb{R}$; $k = a+b \neq 0, \pm 1$ and $\lambda = 1+(a-b)^6-2\left(a^6+b^6\right)-62(ab)^2\left(a^2+b^2\right) \neq 0$, in quasi-$\beta$-normed spaces by using fixed point method. In particular, we prove generalized stabilities involving the sum of powers of norms, product of powers of norms and the mixed product-sum of powers of norms of the above functional equation in quasi-β-normed spaces by using fixed point method. A counter-example for a singular case is also indicated.

1. Introduction

The classical theory of stability of functional equations was instigated by the question of Ulam [42] in the year 1940. In the subsequent year 1941, Hyers [16] was the foremost mathematician to establish the pioneering result connected with the stability of functional equations. The result obtained by Hyers is called as Hyers-Ulam stability of functional equation. Later in the year 1950, Aoki [4] made a further simplification to the theorem of Hyers. In the year 1978, Th.M. Rassias [41] took a broad view in the Hyers result by taking the upper bound as a sum of powers of norms. The result obtained by Th.M. Rassias is recognized as Hyers-Ulam-Rassias stability of functional equation. John M. Rassias ([29], [30], [31]) provided a further generalization of the result of Hyers by using weaker conditions controlled by a product of different powers of norms. The result proved by John M. Rassias is termed as Ulam-Gavruta-Rassias stability of functional equation. Further, in the year 1994, Gavruta [14] provided a generalization of Th.M. Rassias theorem by replacing a general control function as an upper bound. The stability result ascertained by Gavruta is celebrated as generalized Ulam-Hyers stability of functional equation. In the year 2008, Ravi et al. [39] investigated the stability of the following quadratic functional equation

$$q(\ell x+y)+q(\ell x-y)=2q(x+y)+2q(x-y)+2\left(\ell^2-2\right)q(x)-2q(y)$$

for any arbitrary but fixed real constant ℓ with $\ell \neq 0$; $\ell \neq \pm 1$; $\ell \neq \pm\sqrt{2}$ using mixed product-sum of powers of norms. This stability result acquired by Ravi et al. is known as J.M. Rassias stability involving mixed product-sum of powers of norms.

Several stability results have recently been obtained for various functional equations and functional inequalities, also for mappings with more general domains and ranges (see [6], [7], [8], [11], [13], [22], [23], [24], [40]). Many research monographs are also available on functional equations, one can see ([1], [2], [3], [10], [17], [20], [21]).

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In 1996, Isac and Th.M. Rassias [18] were the first to provide applications of the stability theory of functional equations for the proof of new fixed point theorems with applications. The stability problems of several various functional equations have been extensively investigated by a number of authors using fixed point methods (see [5], [26], [43], [45]).

John M. Rassias [32] introduced Euler-Lagrange type quadratic functional equation of the form

$$f(ax+by)+f(bx-ay)=\left(a^2+b^2\right)\left(f(x)+f(y)\right) \tag{1.1}$$

motivated from the following pertinent algebraic equation

$$|ax+by|^2+|bx-ay|^2=\left(a^2+b^2\right)\left(|x|^2+|y|^2\right). \tag{1.2}$$

The solution of the functional equation (1.1) is called an Euler-Lagrange quadratic type mapping. In addition, John M. Rassias ([32], [33], [34], [35], [36]) generalized the standard quadratic equation to the following quadratic equation

$$\begin{aligned} m_1m_2\left|a_1x_1+a_2x_2\right|^2+\left|m_2a_2x_1-m_1a_1x_2\right|^2 \\ =\left(m_1\left|a_1\right|^2+m_2\left|a_2\right|^2\right)\left(m_2\left|x_1\right|^2+m_1\left|x_2\right|^2\right). \end{aligned}$$

He introduced and investigated the general pertinent Euler-Lagrange quadratic mappings. These Euler-Lagrange mappings are named Euler-Lagrange-Rassias mappings, and the corresponding Euler-Lagrange equations are called Euler-Lagrange-Rassias equations (see [15], [25], [27], [28]). These notions provide a cornerstone in analysis, because of their particular interest in probability theory and stochastic analysis in marrying these fields of research to functional equations via the pioneering introduction of the Euler-Lagrange-Rassias quadratic weighted means and fundamental mean equations ([15], [34], [35]).

John M. Rassias [38] introduced the cubic functional equation, as follows:

$$f(x+2y)-3f(x+y)+3f(x)-f(x-y)=6f(y). \tag{1.3}$$

This inspiring cubic functional equation was the transition from the following famous Euler-Lagrange-Rassias quadratic functional equation:

$$f(x+y)-2f(x)+f(x-y)=2f(y)$$

to the cubic functional equations.

John M. Rassias [37] introduced also the following quartic functional equation:

$$f(x+2y)+f(x-2y)+6f(x)=4f(x+y)+4f(x-y)+24f(y). \tag{1.4}$$

It is easy to see that $f(x)=x^4$ is a solution of equation (1.4). For this reason, the equation (1.4) is called a quartic functional equation. The general solution of (1.4) is determined without assuming any regularity conditions on the unknown function (refer [9]). Since the solution of equation (1.4) is even, we can rewrite (1.4) as

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y). \tag{1.5}$$

In 2010, Xu et al. [44] achieved the general solution and proved the stability of the quintic functional equation

$$f(x+3y)-5f(x+2y)+10f(x+y)-10f(x)+5f(x-y)-f(x-2y)=120f(y) \tag{1.6}$$

and the sextic functional equation

$$\begin{aligned} f(x+3y)-6f(x+2y)+15f(x+y)-20f(x) \\ +15f(x-y)-6f(x+2y)+f(x-3y)=720f(y) \end{aligned} \tag{1.7}$$

in quasi-β-normed spaces using fixed point method.

In this paper, the first author of this paper introduces a new Euler-Lagrange-Jensen $(a, b; k = a+b)$-sextic functional equation

$$\begin{aligned} f(ax+by)+f(bx+ay)+(a-b)^6\left[f\left(\frac{ax-by}{a-b}\right)+f\left(\frac{bx-ay}{b-a}\right)\right] \\ = 64(ab)^2\left(a^2+b^2\right)\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)\right] \\ +2\left(a^2-b^2\right)\left(a^4-b^4\right)[f(x)+f(y)] \end{aligned} \tag{1.8}$$

where $a \neq b$, such that $k \in \mathbb{R}$; $k = a+b \neq 0, \pm 1$ and $\lambda = 1+(a-b)^6-2\left(a^6+b^6\right)-62(ab)^2\left(a^2+b^2\right) \neq 0$. Then we investigate the generalized Ulam-Hyers stability of the equation (1.8) in quasi-β-normed spaces using fixed point method. We extend the stability results involving sum of powers of norms, product of powers of norms and mixed product-sum of powers of norms of the above functional equation. We also provide a counter-example to show that the functional equation (1.8) is not stable for singular case. It is easy to see that the function $f(x) = kx^6$ is a solution of the equation (1.8). Hence we say that it is a sextic functional equation.

2. Preliminaries

In this section, we recall some fundamental notions in association with quasi-β-normed spaces and m-additive symmetric mappings.

Let β be a fixed real number with $0 < \beta \leq 1$ and let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$.

Definition 2.1. *Let $\mathcal{X}$ be a linear space over $\mathbb{K}$. A quasi-β-norm $\|\cdot\|$ is a real-valued function on $\mathcal{X}$ satisfying the following conditions:*

(i) $\|a\| \geq 0$ *for all* $a \in \mathcal{X}$ *and* $\|a\| = 0$ *if and only if* $a = 0$.
(ii) $\|\eta a\| = |\eta|^\beta \cdot \|a\|$ *for all* $\eta \in \mathbb{K}$ *and all* $a \in \mathcal{X}$.
(iii) *There is a constant* $K \geq 1$ *such that*

$$\|a+b\| \leq K(\|a\|+\|b\|) \quad \text{for all } a, b \in \mathcal{X}.$$

The pair $(\mathcal{X}, \|\cdot\|)$ is called quasi-β-normed space if $\|\cdot\|$ is a quasi-β-norm on $\mathcal{X}$. The smallest possible K is called the modulus of concavity of $\|\cdot\|$.

Definition 2.2. *A complete quasi-β-normed space is called a quasi-β-Banach space.*

Definition 2.3. *A quasi-β-norm $\|\cdot\|$ is called a (β, p)-norm $(0 < p < 1)$ if*

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x, y \in \mathcal{X}$. In this case, a quasi-β-Banach space is called a (β, p)-Banach space.

3. Generalized Ulam-Hyers stability of equation (1.8)

Throughout this section, we assume that $\mathcal{X}$ is a linear space and $\mathcal{Y}$ is a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_\mathcal{Y}$. Let K be the modulus of concavity of $\|\cdot\|_\mathcal{Y}$. For notational convenience, we define the difference operator for a given mapping $f : \mathcal{X} \to \mathcal{Y}$ as

$$\begin{aligned} D_s f(x,y) = f(ax+by)+f(bx+ay)+(a-b)^6\left[f\left(\frac{ax-by}{a-b}\right)+f\left(\frac{bx-ay}{b-a}\right)\right] \\ = 64(ab)^2\left(a^2+b^2\right)\left[f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)\right] \\ +2\left(a^2-b^2\right)\left(a^4-b^4\right)[f(x)+f(y)] \end{aligned}$$

for all $x, y \in \mathcal{X}$.

Lemma 3.1. (see [44]). *Let $j \in \{-1, 1\}$ be fixed, $m, b \in \mathbb{N}$ with $b \geq 2$ and $\Phi : \mathcal{X} \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\Phi\left(b^j x\right) \leq b^{jm\beta} L\Phi(x)$ for all $x \in \mathcal{X}$. Let $g : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|g(bx) - b^m g(x)\|_\mathcal{Y} \leq \Phi(x) \tag{3.1}$$

for all $x \in \mathcal{X}$, then there exists a uniquely determined mapping $G : \mathcal{X} \to \mathcal{Y}$ such that $G(bx) = b^m G(x)$ and

$$\|g(x) - G(x)\|_{\mathcal{Y}} \leq \frac{1}{b^{m\beta}\left|1 - L^j\right|}\Phi(x) \tag{3.2}$$

for all $x \in \mathcal{X}$.

Theorem 3.1. *Let $i \in \{-1, 1\}$ be fixed. Let $\phi : \mathcal{X} \times \mathcal{X} \to [0, \infty)$ be a function such that there exists an $L < 1$ with $\phi\left(k^i x, k^i y\right) \leq k^{6i\beta} L \phi(x, y)$ for all $x, y \in \mathcal{X}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|D_s f(x, y)\|_{\mathcal{Y}} \leq \phi(x, y) \tag{3.3}$$

for all $x, y \in \mathcal{X}$. Then there exists a unique sextic mapping $S : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - S(x)\|_{\mathcal{Y}} \leq \frac{1}{k^{6\beta}\left|1 - L^i\right|}\Psi(x) \tag{3.4}$$

for all $x \in \mathcal{X}$, where

$$\Psi(x) = \frac{K}{2^\beta}\left[\phi(x, x) + \frac{32^\beta (ab)^{2\beta}\left(a^2 + b^2\right)^\beta}{\lambda^\beta}\phi(0, 0)\right].$$

Proof. Plugging (x, y) into $(0, 0)$ in (3.3), we obtain

$$\|f(0)\|_{\mathcal{Y}} \leq \frac{1}{2^\beta \lambda^\beta}\phi(0, 0). \tag{3.5}$$

Switching (x, y) to (x, x) in (3.3), one finds

$$\left\|f(kx) - k^6 f(x) - 32(ab)^2\left(a^2 + b^2\right) f(0)\right\|_{\mathcal{Y}} \leq \frac{1}{2^\beta}\phi(x, x) \tag{3.6}$$

for all $x \in \mathcal{X}$. Using (3.5) and (3.6), we arrive

$$\left\|f(kx) - k^6 f(x)\right\|_{\mathcal{Y}} \leq \Psi(x) \tag{3.7}$$

for all $x \in \mathcal{X}$. By Lemma 3.1, there exists a unique mapping $S : \mathcal{X} \to \mathcal{Y}$ such that $S(kx) = k^6 S(x)$ and

$$\|f(x) - S(x)\|_Y \leq \frac{1}{k^{6\beta}\left|1 - L^i\right|}\Psi(x)$$

for all $x \in \mathcal{X}$. It remains to show that S is a sextic map. By (3.3), we have

$$\begin{aligned}
\left\|\frac{1}{k^{6in}} D_s f\left(k^{in} x, k^{in} y\right)\right\|_{\mathcal{Y}} &\leq k^{-6in\beta}\phi\left(k^{in} x, k^{in} y\right) \\
&\leq k^{-6in\beta}\left(k^{6i\beta} L\right)^n \phi(x, y) \\
&= L^n \phi(x, y)
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and $n \in \mathbb{N}$. So $\|D_s S(x, y)\|_{\mathcal{Y}} = 0$ for all $x, y \in \mathcal{X}$. Thus the mapping $S : \mathcal{X} \to \mathcal{Y}$ is sextic, which completes the proof of theorem. ☐

Corollary 3.1. *Let $\mathcal{X}$ be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_{\mathcal{X}}$, and let $\mathcal{Y}$ be a (β, p)-Banach space with (β, p)-norm $\|\cdot\|_{\mathcal{Y}}$. Let k_1, p be positive numbers with $p \neq \frac{6\beta}{\alpha}$ and $f : \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying*

$$\|D_s f(x, y)\|_{\mathcal{Y}} \leq k_1\left(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p\right)$$

for all $x, y \in \mathcal{X}$. Then there exists a unique sextic mapping $S : \mathcal{X} \to \mathcal{Y}$ such that

$$\|f(x) - S(x)\|_{\mathcal{Y}} \leq \begin{cases} \frac{k_1 K}{2^\beta\left(k^{6\beta} - k^{p\alpha}\right)}\|x\|_{\mathcal{X}}^p, & p \in \left(0, \frac{6\beta}{\alpha}\right) \\ \frac{k^{p\alpha} k_1 K}{k^{6\beta} 2^\beta\left(k^{p\alpha} - k^{6\beta}\right)}\|x\|_{\mathcal{X}}^p, & p \in \left(\frac{6\beta}{\alpha}, \infty\right) \end{cases}$$

for all $x \in \mathcal{X}$.

Proof. The proof is obtained by taking $\phi(x, y) = k_1\left(\|x\|_{\mathcal{X}}^p + \|y\|_{\mathcal{X}}^p\right)$, for all $x, y \in \mathcal{X}$ and $L = \frac{k^{p\alpha}}{k^{6\beta}}$ in Theorem 3.1. ☐

Corollary 3.2. *Let $\mathcal{X}$ be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_{\mathcal{X}}$, and let $\mathcal{Y}$ be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_{\mathcal{Y}}$. Let k_2,p,q be positive numbers with $\rho=p+q\neq\frac{6\beta}{\alpha}$ and $f:\mathcal{X}\to\mathcal{Y}$ be a mapping satisfying*

$$\|D_s f(x,y)\|_Y \leq k_2\,\|x\|_{\mathcal{X}}^p\,\|y\|_{\mathcal{X}}^q$$

for all $x,y\in X$. Then there exists a unique sextic mapping $S:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(x)-S(x)\|_{\mathcal{Y}} \leq \begin{cases} \frac{k_2 K}{2^{\beta}(k^{6\beta}-k^{\rho\alpha})}\,\|x\|_{\mathcal{X}}^{\rho}, & \rho\in\left(0,\frac{6\beta}{\alpha}\right)\\ \frac{k^{\rho\alpha}k_2 K}{k^{6\beta}2^{\beta}(k^{\rho\alpha}-k^{6\beta})}\,\|x\|_{\mathcal{X}}^{\rho}, & \rho\in\left(\frac{6\beta}{\alpha},\infty\right)\end{cases}$$

for all $x\in\mathcal{X}$.

Proof. Letting $\phi(x,y)=k_2\,\|x\|_{\mathcal{X}}^p\,\|y\|_{\mathcal{X}}^q$, for all $x,y\in\mathcal{X}$ and $L=\frac{k^{\rho\alpha}}{k^{6\beta}}$ in Theorem 3.1, we obtain the required results. □

Corollary 3.3. *Let $\mathcal{X}$ be a quasi-α-normed space with quasi-α-norm $\|\cdot\|_{\mathcal{X}}$, and let $\mathcal{Y}$ be a (β,p)-Banach space with (β,p)-norm $\|\cdot\|_{\mathcal{Y}}$. Let k_3,r be positive numbers $r\neq\frac{3\beta}{\alpha}$ and $f:\mathcal{X}\to\mathcal{Y}$ be a mapping satisfying*

$$\|D_s f(x,y)\|_Y \leq k_3\left[\|x\|_{\mathcal{X}}^r\,\|y\|_{\mathcal{X}}^r+\left(\|x\|_{\mathcal{X}}^{2r}+\|y\|_{\mathcal{X}}^{2r}\right)\right]$$

for all $x,y\in\mathcal{X}$. Then there exists a unique sextic mapping $S:\mathcal{X}\to\mathcal{Y}$ such that

$$\|f(x)-S(x)\|_{Y} \leq \begin{cases} \frac{3k_3 K}{2^{\beta}(k^{6\beta}-k^{2r\alpha})}\,\|x\|_{\mathcal{X}}^{2r}, & r\in\left(0,\frac{3\beta}{\alpha}\right)\\ \frac{3k^{2r\alpha}k_3 K}{k^{6\beta}2^{\beta}(k^{2r\alpha}-k^{6\beta})}\,\|x\|_{\mathcal{X}}^{2r}, & r\in\left(\frac{3\beta}{\alpha},\infty\right)\end{cases}$$

for all $x\in\mathcal{X}$.

Proof. By taking $\varphi(x,y)=k_3\left[\|x\|_{\mathcal{X}}^r\,\|y\|_{\mathcal{X}}^r+\left(\|x\|_{\mathcal{X}}^{2r}+\|y\|_{\mathcal{X}}^{2r}\right)\right]$, for all $x,y\in\mathcal{X}$ and $L=\frac{k^{2r\alpha}}{k^{6\beta}}$ in Theorem 3.1, we arrive at the desired results. □

4. Counter-example

In this section, using the idea of the well-known counter-example provided by Z. Gajda [12], we illustrate a counter-example that the functional equation (1.8) is not stable for $p=\frac{6\beta}{\alpha}$ in Corollary 3.1.

We consider the function

$$\varphi(x)=\begin{cases} x^6, & \text{for } |x|<1\\ 1, & \text{for } |x|\geq 1.\end{cases} \tag{4.1}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x)=\sum_{n=0}^{\infty}2^{-6n}\varphi(2^n x) \tag{4.2}$$

for all $x\in\mathbb{R}$. The function f serves as a counter-example for the fact that the functional equation (1.8) is not stable for $p=\frac{6\beta}{\alpha}$ in Corollary 3.1 in the following theorem.

Theorem 4.1. *If the function f defined in (4.2) satisfies the functional inequality*

$$|D_s f(x,y)| \leq \frac{64^3\delta}{63}\left(|x|^6+|y|^6\right) \tag{4.3}$$

where $\delta=2\left[1+(a-b)^6-2\left(a^6+b^6\right)-62(ab)^2\left(a^2+b^2\right)\right]>0$, for all $x,y\in\mathbb{R}$, then there do not exist a sextic mapping $S:\mathbb{R}\to\mathbb{R}$ and a constant $\epsilon>0$ such that

$$|f(x)-S(x)|\leq\epsilon\,|x|^6, \qquad \textit{for all } x\in\mathbb{R}.$$

Proof. First, we are going to show that f satisfies (4.3).

$$|f(x)|=\left|\sum_{n=0}^{\infty}2^{-6n}\varphi(2^n x)\right|\leq\sum_{n=0}^{\infty}\frac{1}{2^{6n}}=\frac{64}{63}.$$

Therefore, we see that f is bounded by $\frac{64}{63}$ on $\mathbb{R}$. If $|x|^6+|y|^6=0$ or $|x|^6+|y|^6 \geq \frac{1}{64}$, then

$$|D_s f(x,y)| \leq \frac{64\delta}{63} \leq \frac{64^2\delta}{63}\left(|x|^6+|y|^6\right).$$

Now, suppose that $0<|x|^6+|y|^6<\frac{1}{64}$. Then there exists a non-negative integer k such that

$$\frac{1}{64^{k+1}} \leq |x|^6+|y|^6 < \frac{1}{64^k}. \tag{4.4}$$

Hence $64^k|x|^6<1$, $64^k|y|^6<1$ and $2^n(ax+by)$, $2^n(bx+ay)$, $2^n\left(\frac{ax-by}{a-b}\right)$, $2^n\left(\frac{bx-ay}{b-a}\right)$, $2^n\left(\frac{x+y}{2}\right)$, $2^n\left(\frac{x-y}{2}\right)$, 2^nx, $2^ny \in (-1,1)$ for all $n=0,1,2,\ldots,k-1$. Hence for $n=0,1,2,\ldots,k-1$,

$$\begin{aligned}
&\varphi\left(2^n(ax+by)\right)+\varphi\left(2^n(bx+ay)\right)\\
&\quad+(a-b)^6\left[\varphi\left(2^n\left(\frac{ax-by}{a-b}\right)\right)+\varphi\left(2^n\left(\frac{bx-ay}{b-a}\right)\right)\right]\\
&\qquad-64(ab)^2\left(a^2+b^2\right)\left[\varphi\left(2^n\left(\frac{x+y}{2}\right)\right)+\varphi\left(2^n\left(\frac{x-y}{2}\right)\right)\right]\\
&\qquad\quad-2\left(a^2-b^2\right)\left(a^4-b^4\right)\left[\varphi\left(2^nx\right)+\varphi\left(2^ny\right)\right]=0.
\end{aligned}\tag{4.5}$$

From the definition of f and the inequality (4.4), we obtain that

$$\begin{aligned}
&|D_s f(x,y)|\\
&=\Bigg|\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^n(ax+by)\right)+\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^n(bx+ay)\right)\\
&\quad+(a-b)^6\left[\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^n\left(\frac{ax-by}{a-b}\right)\right)+\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^n\left(\frac{bx-ay}{b-a}\right)\right)\right]\\
&\quad-64(ab)^2\left(a^2+b^2\right)\left[\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^n\left(\frac{x+y}{2}\right)\right)+\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^n\left(\frac{x-y}{2}\right)\right)\right]\\
&\quad-2\left(a^2-b^2\right)\left(a^4-b^4\right)\left[\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^nx\right)+\sum_{n=0}^{\infty}2^{-6n}\varphi\left(2^ny\right)\right]\Bigg|\\
&\leq\sum_{n=0}^{\infty}2^{-6n}\Bigg|\varphi\left(2^n(ax+by)\right)+\varphi\left(2^n(bx+ay)\right)\\
&\qquad+(a-b)^6\left[\varphi\left(2^n\left(\frac{ax-by}{a-b}\right)\right)+\varphi\left(2^n\left(\frac{bx-ay}{b-a}\right)\right)\right]\\
&\qquad\quad-64(ab)^2\left(a^2+b^2\right)\left[\varphi\left(2^n\left(\frac{x+y}{2}\right)\right)+\varphi\left(2^n\left(\frac{x-y}{2}\right)\right)\right]\\
&\qquad\qquad-2\left(a^2-b^2\right)\left(a^4-b^4\right)\left[\varphi\left(2^nx\right)+\varphi\left(2^ny\right)\right]\Bigg|\\
&\leq\sum_{n=0}^{\infty}2^{-6n}\delta=\frac{2^{6(1-k)}\delta}{63}\leq\frac{64^3\delta}{63}\left(|x|^6+|y|^6\right).
\end{aligned}\tag{4.6}$$

Therefore, f satisfies (4.3) for all $x,y\in\mathbb{R}$. Now, we claim that the functional equation (1.8) is not stable for $p=\frac{6\beta}{\alpha}$ in Corollary 3.1. Suppose on the contrary that there exists a sexticc mapping $S:\mathbb{R}\to\mathbb{R}$ and a constant $\epsilon>0$ such that

$$|f(x)-S(x)|\leq\epsilon|x|^6, \quad \text{for all } x\in\mathbb{R}.$$

Then there exists a constant $c\in\mathbb{R}$ such that $S(x)=cx^6$ for all rational numbers x (see [19]). So we obtain that

$$|f(x)|\leq(\epsilon+|c|)|x|^6 \tag{4.7}$$

for all $x \in \mathbb{Q}$. Let $m \in \mathbb{N}$ with $m+1 > \epsilon + |c|$. If x is a rational number in $(0, 2^{-m})$, then $2^n x \in (0,1)$ for all $n = 0, 1, 2, \ldots, m$, and for this x, we get

$$f(x) = \sum_{n=0}^{\infty} 2^{-6n} \varphi(2^n x) \geq \sum_{n=0}^{m} 2^{-6n} \left(2^n x\right)^6 = (m+1)x^6 > (\epsilon + |c|)x^6 \tag{4.8}$$

which contradicts (4.7). Hence the functional equation (1.8) is not stable for $p = \frac{6\beta}{\alpha}$ in Corollary 3.1. $\square$

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