International Journal of Analysis and Applications
ISSN 2291-8639
Volume 14, Number 2 (2017), 175-179
http://www.etamaths.com

FACTORS FOR ABSOLUTE WEIGHTED ARITHMETIC MEAN SUMMABILITY OF INFINITE SERIES

HÜSEYİN BOR*

Abstract. In this paper, we proved a general theorem dealing with absolute weighted arithmetic mean summability factors of infinite series under weaker conditions. We have also obtained some known results.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by u_n^α the nth Cesàro mean of order α, with $\alpha > -1$, of the sequence (s_n), that is (see [4])

$$u_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1.1}$$

where

$$A_n^\alpha = \frac{(\alpha+1)(\alpha+2)....(\alpha+n)}{n!} = O(n^\alpha), \quad A_{-n}^\alpha = 0 \quad \texttt{for} \quad n > 0. \tag{1.2}$$

A series $\sum a_n$ is said to be summable $\mid C, \alpha \mid_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} \mid u_n^\alpha - u_{n-1}^\alpha \mid^k < \infty. \tag{1.3}$$

If we take α=1, then we obtain $\mid C, 1 \mid_k$ summability. Let (p_n) be a sequence of positive numbers such that $P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1)$. The sequence-to-sequence transformation

$$w_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

defines the sequence (w_n) of the weighted arithmetic mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [1])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} \mid w_n - w_{n-1} \mid^k < \infty. \tag{1.5}$$

If we take $p_n = 1$ for all values of n, then we obtain $\mid C, 1 \mid_k$ summability. Also if we take $k = 1$, then we obtain $\mid \bar{N}, p_n \mid$ summability (see [11]). For any sequence (λ_n) we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

2. Known Result

The following theorem is known dealing with $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series.

2010 *Mathematics Subject Classification.* 26D15, 40D15, 40F05, 40G99.
Key words and phrases. weighted arithmetic mean; absolute summability; summability factors; infinite series; Hölder inequality; Minkowski inequality.

Theorem 2.1. [2] *Let (X_n) be a positive non-decreasing sequence and suppose that there exists sequences (β_n) and (λ_n) such that*

$$| \Delta\lambda_n |\leq \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{2.3}$$

$$| \lambda_n | X_n = O(1). \tag{2.4}$$

If

$$\sum_{n=1}^{m} \frac{| s_n |^k}{n} = O(X_m) \quad as \quad m \to \infty, \tag{2.5}$$

and (p_n) is a sequence such that

$$P_n = O(np_n), \tag{2.6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{2.7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $| \bar{N}, p_n |_k$, $k \geq 1$.

Remark 2.1. *It should be noted that, under the conditions on the sequence (λ_n) we have that (λ_n) is bounded and $\Delta\lambda_n = O(1/n)$ [2].*

3. Main Result

The aim of this paper is to prove Theorem 2.1 under weaker conditions. Now, we shall prove the following theorem.

Theorem 3.1. *Let (X_n) be a positive non-decreasing sequence. If the sequences (X_n), (β_n), (λ_n), and (p_n) satisfy the conditions (2.1)-(2.4), (2.6)-(2.7), and*

$$\sum_{n=1}^{m} \frac{| s_n |^k}{nX_n^{k-1}} = O(X_m) \quad as \quad m \to \infty, \tag{3.1}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n\lambda_n}{np_n}$ is summable $| \bar{N}, p_n |_k$, $k \geq 1$.

Remark 3.1. *It should be noted that condition (3.1) is the same as condition (2.5) when k=1. When $k > 1$, condition (3.1) is weaker than condition (2.5) but the converse is not true. As in [10], we can show that if (2.5) is satisfied, then we get*

$$\sum_{n=1}^{m} \frac{| s_n |^k}{nX_n^{k-1}} = O(\frac{1}{X_1^{k-1}}) \sum_{n=1}^{m} \frac{| s_n |^k}{n} = O(X_m) \quad as \quad m \to \infty.$$

To show that the converse is false when $k > 1$, as in [3], the following example is sufficient. We can take $X_n = n^\delta$, $0 < \delta < 1$, and then construct a sequence (u_n) such that

$$u_n = \frac{|s_n|^k}{n{X_n}^{k-1}} = X_n - X_{n-1},$$

hence

$$\sum_{n=1}^{m} \frac{|s_n|^k}{n{X_n}^{k-1}} = X_m = m^\delta,$$

and so

$$\begin{aligned} \sum_{n=1}^{m} \frac{|s_n|^k}{n} &= \sum_{n=1}^{m} (X_n - X_{n-1}) X_n^{k-1} = \sum_{n=1}^{m} (n^\delta - (n-1)^\delta) n^{\delta(k-1)} \\ &\geq \delta \sum_{n=1}^{m} n^{\delta-1} n^{\delta(k-1)} = \delta \sum_{n=1}^{m} n^{\delta k-1} \sim \frac{m^{\delta k}}{k} \quad as \quad m \to \infty. \end{aligned}$$

It follows that

$$\frac{1}{X_m}\sum_{n=1}^{m}\frac{|s_n|^k}{n}\to\infty \quad as \quad m\to\infty$$

provided $k>1$. This shows that (2.5) implies (3.1) but not conversely.
We require the following lemmas for the proof of Theorem 3.1.

Lemma 3.1. [7] *Under the conditions on (X_n), (β_n) and (λ_n) as as expressed in the statement of the theorem, we have the following;*

$$nX_n\beta_n = O(1), \tag{3.2}$$

$$\sum_{n=1}^{\infty}\beta_n X_n<\infty. \tag{3.3}$$

Lemma 3.2. [9] *If the conditions (2.6) and (2.7) are satisfied, then $\Delta\left(\frac{P_n}{np_n}\right)=O\left(\frac{1}{n}\right)$.*

4. Proof of Theorem 3.1

Proof. Let (T_n) be the sequence of $(\bar{N},p_n)$ mean of the series $\sum_{n=1}^{\infty}\frac{a_nP_n\lambda_n}{np_n}$. Then, by definition, we have

$$T_n=\frac{1}{P_n}\sum_{v=1}^{n}p_v\sum_{r=1}^{v}\frac{a_rP_r\lambda_r}{rp_r}=\frac{1}{P_n}\sum_{v=1}^{n}(P_n-P_{v-1})\frac{a_vP_v\lambda_v}{vp_v}.$$

Then we get that

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}P_va_v\lambda_v}{vp_v},\quad n\geq 1,\quad (P_{-1}=0).$$

By using Abel's transformation, we have that

$$T_n-T_{n-1}=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}s_v\Delta\left(\frac{P_{v-1}P_v\lambda_v}{vp_v}\right)+\frac{\lambda_ns_n}{n}$$

$$=\frac{s_n\lambda_n}{n}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}s_v\frac{P_{v+1}P_v\Delta\lambda_v}{(v+1)p_{v+1}}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vs_v\lambda_v\Delta\left(\frac{P_v}{vp_v}\right)-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}s_vP_v\lambda_v\frac{1}{v}$$

$$=T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}.$$

To complete the proof of the Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,r}\mid^k<\infty,\ \texttt{for}\ r=1,2,3,4. \tag{4.1}$$

Applying Abel's transformation, we have that

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\mid T_{n,1}\mid^k=\sum_{n=1}^{m}\left(\frac{P_n}{np_n}\right)^{k-1}\mid\lambda_n\mid^{k-1}\mid\lambda_n\mid\frac{\mid s_n\mid^k}{n}=O(1)\sum_{n=1}^{m}\frac{\mid s_n\mid^k}{n}\left(\frac{1}{X_n}\right)^{k-1}\mid\lambda_n\mid$$

$$=O(1)\sum_{n=1}^{m-1}\Delta\mid\lambda_n\mid\sum_{v=1}^{n}\frac{\mid s_v\mid^k}{vX_v{}^{k-1}}+O(1)\mid\lambda_m\mid\sum_{n=1}^{m}\frac{\mid s_n\mid^k}{nX_n{}^{k-1}}=O(1)\sum_{n=1}^{m-1}\mid\Delta\lambda_n\mid X_n+O(1)\mid\lambda_m\mid X_m$$

$$=O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)\mid\lambda_m\mid X_m=O(1),\quad as\quad m\to\infty,$$

by the hypotheses of Theorem 3.1 and Lemma 3.1. Now, by using (2.6) and applying Hölder's inequality, we obtain that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,2} \mid^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k} \mid \sum_{v=1}^{n-1} P_v s_v \Delta\lambda_v \mid^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}\frac{P_v}{p_v} \mid s_v \mid p_v \mid \Delta\lambda_v \mid\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v{\beta_v}^k \times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^k \mid s_v \mid^k p_v{\beta_v}^k \sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{k-1} {\beta_v}^{k-1}\beta_v \mid s_v \mid^k = O(1)\sum_{v=1}^{m}(v\beta_v)^{k-1}\beta_v \mid s_v \mid^k$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1}\beta_v \mid s_v \mid^k = O(1)\sum_{v=1}^{m} v\beta_v\frac{\mid s_v \mid^k}{v{X_v}^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{\mid s_r \mid^k}{r{X_r}^{k-1}} + O(1)m\beta_m\sum_{v=1}^{m}\frac{\mid s_v \mid^k}{v{X_v}^{k-1}} = O(1)\sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1)m\beta_mX_m$$

$$= O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1)m\beta_mX_m$$

$$= O(1)\sum_{v=1}^{m-1} v \mid \Delta\beta_v \mid X_v + O(1)\sum_{v=1}^{m-1} X_v\beta_v + O(1)m\beta_mX_m = O(1),$$

as $m \to \infty$, by the hypotheses of the Theorem 3.1 and Lemma 3.1. Again, as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,3} \mid^k = \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} P_v s_v \lambda_v \Delta\left(\frac{P_v}{vp_v}\right) \mid^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1} P_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k = O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k}\left\{\sum_{v=1}^{n-1}\left(\frac{P_v}{p_v}\right) p_v \mid s_v \mid\mid \lambda_v \mid \frac{1}{v}\right\}^k$$

$$= O(1)\sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k p_v \mid s_v \mid^k\mid \lambda_v \mid^k \times \left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right\}^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k \mid s_v \mid^k p_v \mid \lambda_v \mid^k \sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}} = O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k p_v \mid s_v \mid^k\mid \lambda_v \mid^k \frac{1}{P_v}.\frac{v}{v}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1} \mid \lambda_v \mid^{k-1}\mid \lambda_v \mid \frac{\mid s_v \mid^k}{v} = O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1} \mid \lambda_v \mid \frac{\mid s_v \mid^k}{v} = O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \frac{\mid s_v \mid^k}{v{X_v}^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1} X_v\beta_v + O(1)X_m \mid \lambda_m \mid = O(1), \quad as \quad m \to \infty,$$

by the hypotheses of the Theorem 3.1, Lemma 3.1 and Lemma 3.2. Finally, using Hölder's inequality, as in $T_{n,3}$, we have get

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1} \mid T_{n,4} \mid^k = \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v\frac{P_v}{v}\lambda_v \mid^k$$

$$= \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}^k} \mid \sum_{v=1}^{n-1} s_v\frac{P_v}{vp_v}p_v\lambda_v \mid^k \leq \sum_{n=2}^{m+1}\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1} \mid s_v \mid^k \left(\frac{P_v}{vp_v}\right)^k p_v \mid \lambda_v \mid^k \times \left(\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\right)^{k-1}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k \mid s_v \mid^k p_v \mid \lambda_v \mid^k \frac{1}{P_v}.\frac{v}{v} = O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^{k-1} \mid \lambda_v \mid^{k-1} \mid \lambda_v \mid \frac{\mid s_v \mid^k}{v}$$

$$= O(1)\sum_{v=1}^{m}\left(\frac{1}{X_v}\right)^{k-1} \mid \lambda_v \mid \frac{\mid s_v \mid^k}{v} = O(1)\sum_{v=1}^{m} \mid \lambda_v \mid \frac{\mid s_v \mid^k}{v{X_v}^{k-1}}$$

$$= O(1)\sum_{v=1}^{m-1} X_v \beta_v + O(1) X_m \mid \lambda_m \mid = O(1), \quad as \quad m \to \infty.$$

This completes the proof of Theorem 3.1. □

5. Conclusions

It should be noted that if we take $p_n = 1$ for all n, then we obtain a known result of Mishra and Srivastava dealing with $\mid C, 1 \mid_k$ summability factors of infinite series (see [8]). Also, if we set $k = 1$, then we have a known result of Mishra and Srivastava concerning the $\mid \bar{N}, p_n \mid$ summability factors of infinite series (see [9]).

References

[1] H. Bor, On two summability methods, Math. Proc. Camb. Philos Soc. 97 (1985), 147-149.
[2] H. Bor, A note on $\mid \bar{N}, p_n \mid_k$ summability factors of infinite series, Indian J. Pure Appl. Math. 18 (1987), 330-336.
[3] H. Bor, Quasi-monotone and almost increasing sequences and their new applications, Abstr. Appl. Anal. 2012, Art. ID 793548, 6 pp.
[4] E. Cesàro, Sur la multiplication des séries, Bull. Sci. Math. 14 (1890), 114-120.
[5] T. M. Flett, On an extension of absolute summability and some theorems of Littlewood and Paley, Proc. London Math. Soc., 7 (1957), 113-141.
[6] G. H. Hardy, Divergent Series, Clarendon Press, Oxford, (1949).
[7] K. N. Mishra, On the absolute Nörlund summability factors of infinite series, Indian J. Pure Appl. Math. 14 (1983), 40-43.
[8] K. N. Mishra and R. S. L. Srivastava, On the absolute Cesàro summability factors of infinite series, Portugal. Math. 42 (1983/84), 53-61.
[9] K. N. Mishra and R. S. L. Srivastava, On $\mid \bar{N}, p_n \mid$ summability factors of infinite series, Indian J. Pure Appl. Math. 15 (1984), 651-656.
[10] W. T. Sulaiman, A note on $|A|_k$ summability factors of infinite series, Appl. Math. Comput. 216 (2010), 2645-2648.
[11] G. Sunouchi, Notes on Fourier analysis. XVIII. Absolute summability of series with constant terms, Tôhoku Math. J. (2), 1 (1949), 57-65.

P. O. Box 121, TR-06502 Bahçelievler, Ankara, Turkey

*Corresponding author: hbor33@gmail.com