International Journal of Analysis and Applications
ISSN 2291-8639
Volume 14, Number 2 (2017), 203-208
http://www.etamaths.com

INEQUALITIES FOR THE MODIFIED k-BESSEL FUNCTION

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Abstract. The article considers the generalized k-Bessel functions and represents it as Wright functions. Then we study the monotonicity properties of the ratio of two different orders k- Bessel functions, and the ratio of the k-Bessel and the k-Bessel functions. The log-convexity with respect to the order of the k-Bessel also given. An investigation regarding the monotonicity of the ratio of the k-Bessel and k-confluent hypergeometric functions are discussed.

1. Introduction

One of the generalization of the classical gamma function Γ studied in [4] is defined by the limit formula

$$\Gamma_k(x) := \lim_{n\to\infty} \frac{n!\, k^n (n^k)^{\frac{x}{k}-1}}{(x)_{n,k}}, \quad k>0, \tag{1.1}$$

where $(x)_{n,k} := x(x+k)(x+2k)\ldots(x+(n-1)k)$ is called k-Pochhammer symbol. The above $k-$gamma function also have an integral representation as

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-\frac{t^k}{k}} dt, \quad \Re(x)>0. \tag{1.2}$$

Properties of the k-gamma functions have been studies by many researchers [6, 8–11]. Following properties are required in sequel:

(i) $\Gamma_k(x+k) = x\Gamma_k(x)$
(ii) $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$
(iii) $\Gamma_k(k) = 1$
(iv) $\Gamma_k(x+nk) = \Gamma_k(x)(x)_{n,k}$

Motivated with the above generalization of the k-gamma functions, Romero et. al. [1] introduced the $k-$Bessel function of the first kind defined by the series

$$J_{k,\nu}^{\gamma,\lambda}(x) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,\,k}}{\Gamma_k(\lambda n+\upsilon+1)} \frac{(-1)^n (x/2)^n}{(n!)^2}, \tag{1.3}$$

where $k\in\mathbb{R}^+$; $\alpha,\lambda,\gamma,\upsilon\in C$; $\Re(\lambda)>0$ and $\Re(\upsilon)>0$. They also established two recurrence relations for $J_{k,\nu}^{\gamma,\lambda}$.

In this article, we are considering the following function:

$$I_{k,\nu}^{\gamma,\lambda}(x) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,\,k}}{\Gamma_k(\lambda n+\upsilon+1)} \frac{(x/2)^n}{(n!)^2}, \tag{1.4}$$

Since

$$\lim_{k,\lambda,\gamma\to 1} I_{k,\nu}^{\gamma,\lambda}(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+\upsilon+1)} \frac{(x/2)^n}{n!} = \left(\frac{2}{x}\right)^{\frac{\nu}{2}} I_\nu(\sqrt{2x}),$$

Received 18th March, 2017; accepted 25th May, 2017; published 3rd July, 2017.
2010 *Mathematics Subject Classification.* 33C10, 26D07.
Key words and phrases. generalized k-Bessel functions; monotonicity; log-convexity; Turán type inequality.

the classical modified Bessel functions of first kind. In this sense, we can call $I^{\gamma,\lambda}_{k,\nu}$ as the modified k-Bessel functions of first kind. In fact, we can express both $J^{\gamma,\lambda}_{k,\nu}$ and $I^{\gamma,\lambda}_{k,\nu}$ together in

$$\mathtt{W}^{\gamma,\lambda}_{k,\nu,c}(x) := \sum_{n=0}^{\infty} \frac{(\gamma)_{n,\,k}}{\Gamma_k(\lambda n+\nu+1)} \frac{(-c)^n (x/2)^n}{(n!)^2}, \quad c \in \mathbb{R}. \tag{1.5}$$

We can termed $\mathtt{W}^{\gamma,\lambda}_{k,\nu}$ as the generalized k-Bessel function.

First we study the representation formulas for $\mathtt{W}^{\gamma,\lambda}_{k,\nu}$ in term of the classical Wright functions. Then we will study about the monotonicity and log-convexity properties of $I^{\gamma,\lambda}_{k,\nu}$.

2. Representation formula for the generalized k-Bessel function

The generalized hypergeometric function ${}_pF_q(a_1,\dots,a_p;c_1,\dots,c_q;x)$, is given by the power series

$${}_pF_q(a_1,\dots,a_p;c_1,\dots,c_q;z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(c_1)_k \cdots (c_q)_k (1)_k} z^k, \qquad |z|<1, \tag{2.1}$$

where the c_i can not be zero or a negative integer. Here p or q or both are allowed to be zero. The series (2.1) is absolutely convergent for all finite z if $p \le q$ and for $|z|<1$ if $p=q+1$. When $p>q+1$, then the series diverge for $z \neq 0$ and the series does not terminate.

The generalized Wright hypergeometric function ${}_p\psi_q(z)$ is given by the series

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{array}{c} (a_i,\alpha_i)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{array} \middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)} \frac{z^k}{k!}, \tag{2.2}$$

where $a_i, b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R}$ $(i=1,2,\dots,p; j=1,2,\dots,q)$. The asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ were studied in [13,14] and under the condition

$$\sum_{j=1}^{q}\beta_j - \sum_{i=1}^{p}\alpha_i > -1 \tag{2.3}$$

in literature [18,19]. The more properties of the Wright function are investigated in [14–16].

Now we will give the representation of the generalized k-Bessel functions in terms of the Wright and generalized hypergeometric functions.

Proposition 2.1. *Let, $k \in \mathbb{R}$ and $\lambda, \gamma, \nu \in \mathbb{C}$ such that $\Re(\lambda)>0, \Re(\nu)>0$. Then*

$$\mathtt{W}^{\gamma,\lambda}_{k,\nu,c}(x) = \frac{1}{k^{\frac{\nu+k+1}{k}}\Gamma\left(\frac{\gamma}{k}\right)} {}_1\psi_2\left[\begin{array}{c} \left(\frac{\gamma}{k},1\right) \\ \left(\frac{\nu+1}{k},\frac{\gamma}{k}\right) \quad (1,1) \end{array} \middle| -\frac{cx}{2k^{\frac{\lambda}{k}-1}}\right]$$

Proof. Using the relations $\Gamma_k(x) = k^{\frac{x}{k}-1}\Gamma\left(\frac{x}{k}\right)$ and $\Gamma_k(x+nk) = \Gamma_k(x)(x)_{n,k}$, the generalized k-Bessel functions defined in (1.5) can be rewrite as

$$\begin{aligned}
\mathtt{W}^{\gamma,\lambda}_{k,\nu,c}(x) &= \sum_{n=0}^{\infty} \frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\lambda n+\nu+1)\Gamma_k(\gamma)} \frac{(-c)^n}{(n!)^2}\left(\frac{x}{2}\right)^n && (2.4)\\
&= \frac{1}{k^{\frac{\nu+k+1}{k}}\Gamma\left(\frac{\gamma}{k}\right)} \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma}{k}+n\right)}{\Gamma\left(\frac{\lambda}{k}n+\frac{\nu+1}{k}\right)\Gamma\left(\frac{\gamma}{k}\right)} \frac{(-c)^n}{\Gamma(n+1)\Gamma(n+1)} \left(\frac{x}{2k^{\frac{\lambda}{k}-1}}\right)^n && (2.5)\\
&= \frac{1}{k^{\frac{\nu+k+1}{k}}\Gamma\left(\frac{\gamma}{k}\right)} {}_1\psi_2\left[\begin{array}{c} \left(\frac{\gamma}{k},1\right) \\ \left(\frac{\nu+1}{k},\frac{\gamma}{k}\right) \quad (1,1) \end{array} \middle| -\frac{cx}{2k^{\frac{\lambda}{k}-1}}\right] && (2.6)
\end{aligned}$$

Hence the result follows. □

3. Monotonicity and log-convexity properties

This section discuss the monotonicity and log-convexity properties for the modified k-Bessel functions $\mathtt{W}^{\gamma,\lambda}_{k,\nu,-1}(x) = \mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)$.

Following lemma due to Biernacki and Krzyż [7] will be required.

Lemma 3.1. *[7] Consider the power series $f(x)=\sum_{k=0}^{\infty}a_kx^k$ and $g(x)=\sum_{k=0}^{\infty}b_kx^k$, where $a_k\in\mathbb{R}$ and $b_k>0$ for all k. Further suppose that both series converge on $|x|<r$. If the sequence $\{a_k/b_k\}_{k\geq 0}$ is increasing (or decreasing), then the function $x\mapsto f(x)/g(x)$ is also increasing (or decreasing) on $(0,r)$.*

The above lemma still holds when both f and g are even, or both are odd functions.

Theorem 3.1. *The following results holds true for the modified k-Bessel functions.*

1. *For $\mu\geq\nu>-1$, the function $x\mapsto \mathtt{I}^{\gamma,\lambda}_{k,\mu}(x)/\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)$ is increasing on $(0,\infty)$ for some fixed $k>0$.*
2. *If $k\geq\lambda\geq m>0$, the function $x\mapsto \mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)/\mathtt{I}^{\gamma,\lambda}_{m,\nu}(x)$ is increasing on $(0,\infty)$ for some fixed $\nu>-1$ and $\gamma\geq\nu+1$.*
3. *The function $\nu\mapsto\mathcal{I}^{\gamma,\lambda}_{k,\nu}(x)$ is log-convex on $(0,\infty)$ for some fixed $k,\gamma>0$ and $x>0$. Here, $\mathcal{I}^{\gamma,\lambda}_{k,\nu}(x):=\Gamma_k(\nu+1)\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)$.*
4. *Suppose that $\lambda\geq k>0$ and $\nu>-1$. Then*
   - (a) *The function $x\mapsto\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)/\Phi_k(a,c;x)$ is decreasing on $(0,\infty)$ for $a\geq c>0$ and $0<\gamma\leq\nu+1$. Here, $\Phi_k(a;c;x)$ is the k-confluent hypergeometric functions.*
   - (b) *The function $x\mapsto\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)/\Phi_k(\gamma;\lambda;x/2)$ is decreasing on $(0,1)$ for $\gamma>0$ and $0<k\leq\lambda\leq\nu+1$.*
   - (c) *The function $x\mapsto\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)/\Phi_k(\gamma;\lambda;x/2)$ is decreasing on $[1,\infty)$ for $\gamma>0$ and $0<k\leq\min\{\lambda,\nu+1\}$.*

Proof. (**1**) Form (1.4) it follows that

$$\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)=\sum_{n=0}^{\infty}a_n(\nu)x^n\quad\text{and}\quad\mathtt{I}^{\gamma,\lambda}_{k,\nu}(x)=\sum_{n=0}^{\infty}a_n(\mu)x^n,$$

where

$$a_n(\nu)=\frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+1)(n!)^2 2^n}\quad\text{and}\quad a_n(\mu)=\frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\mu+1)(n!)^2 2^n}$$

Consider the function

$$f(t):=\frac{\Gamma_k(\lambda t+\mu+1)}{\Gamma_k(\lambda t+\nu+1)}.$$

Then the logarithmic differentiation yields

$$\frac{f'(t)}{f(t)}=\lambda(\Psi_k(\lambda t+\mu+1)-\Psi_k(\lambda t+\nu+1)).$$

Here, $\Psi_k=\Gamma_k'/\Gamma_k$ is the k-digamma functions studied in [5] and defined by

$$\Psi_k(t)=\frac{\log(k)-\gamma_1}{k}-\frac{1}{t}+\sum_{n=1}^{\infty}\frac{t}{nk(nk+t)}\tag{3.1}$$

where γ_1 is the Euler-Mascheronis constant.

A calculation yields

$$\Psi_k'(t)=\sum_{n=0}^{\infty}\frac{1}{(nk+t)^2},\quad k>0\quad\text{and}\quad t>0.\tag{3.2}$$

Clearly, Ψ_k is increasing on $(0,\infty)$ and hence $f'(t)>0$ for all $t\geq 0$ if $\mu\geq\nu>-1$. This, in particular, implies that the sequence $\{d_n\}_{n\geq 0}=\{a_n(\nu)/a_n(\mu)\}_{n\geq 0}$ is increasing and hence the conclusion follows from Lemma 3.1.

(2). This result also follows from Lemma 3.1 if the sequence $\{d_n\}_{n\geq 0}=\{a_n^k(\nu)/a_n^m(\mu)\}_{n\geq 0}$ is increasing for $k\geq m>0$. Here,

$$a_n^k(\nu)=\frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+1)(n!)^2}\quad\text{and}\quad a_n^m(\nu)=\frac{(\gamma)_{n,m}}{\Gamma_m(\lambda n+\nu+1)(n!)^2},$$

which together with the identity $\Gamma_k(x+nk)=\Gamma_k(x)(x)_{n,k}$ gives

$$d_n=\frac{(\gamma)_{n,k}}{(\gamma)_{n,m}}\frac{\Gamma_m(\lambda n+\nu+1)}{\Gamma_k(\lambda n+\nu+1)}$$
$$=\frac{\Gamma_k(\gamma+nk)\,\Gamma_m(\lambda n+\nu+1)}{\Gamma_k(\gamma+nm)\,\Gamma_k(\lambda n+\nu+1)}.$$

Now to show that $\{d_n\}$ is increase, consider the function

$$f(y):=\frac{\Gamma_k(\gamma+yk)\,\Gamma_m(\lambda y+\nu+1)}{\Gamma_k(\gamma+ym)\,\Gamma_k(\lambda y+\nu+1)}$$

The logarithmic differentiation of f yields

$$\frac{f'(y)}{f(y)}=k\Psi_k(\gamma+yk)+\lambda\Psi_m(\lambda y+\nu+1)-m\Psi_m(\gamma+ym)-\lambda\Psi_k(\lambda y+\nu+1) \tag{3.3}$$

If $\gamma\geq\nu+1$ and $k\geq\lambda\geq m$, then (3.3) can be rewrite as

$$\frac{f'(y)}{f(y)}\geq\lambda\big(\Psi_k(\nu+1+yk)-\Psi_k(\lambda y+\nu+1)\big)+m\big(\Psi_m(\lambda y+\nu+1)-\Psi_m(\nu+1+ym)\big)\geq 0. \tag{3.4}$$

This conclude that f, and consequently the sequence $\{d_n\}_{n\geq0}$, is increasing. Finally the result follows from the Lemma 3.1.

(3). It is known that sum of the log-convex functions is log-convex. Thus, to prove the result it is enough to show that

$$\nu\mapsto a_n^k(\nu):=\frac{(\gamma)_{n,k}\,\Gamma_k(\nu+1)}{\Gamma_k(\lambda n+\nu+1)\,(n!)^2}$$

is log-convex.

A logarithmic differentiation of $a_n(\nu)$ with respect to ν yields

$$\frac{\partial}{\partial\nu}\log\left(a_n^k(\nu)\right)=\Psi_k(\nu+1)-\Psi_k(\lambda n+\nu+1).$$

This along with (3.2) gives

$$\frac{\partial^2}{\partial\nu^2}\log\left(a_n^k(\nu)\right)=\Psi_k'(\nu+1)-\Psi_k'(\lambda n+\nu+1)$$
$$=\sum_{r=0}^{\infty}\frac{1}{(rk+\nu+1)^2}-\sum_{r=0}^{\infty}\frac{1}{(rk+\lambda n+\nu+1)^2}$$
$$=\sum_{r=0}^{\infty}\frac{\lambda n(2rk+\lambda n+2\nu+2)}{(rk+\nu+1)^2(rk+\lambda n+\nu+1)^2}>0,$$

for all $n\geq0$, $k>0$ and $\nu>-1$. Thus, $\nu\mapsto a_n^k(\nu)$ is log-convex and hence the conclusion.

(4). Denote $\Phi_k(a,c;x)=\sum_{n=0}^{\infty}c_{n,k}(a,c)x^n$ and $\mathtt{I}_{k,\nu}^{\gamma,\lambda}(x)=\sum_{n=0}^{\infty}a_n(\nu)x^n$, where

$$a_n(\nu)=\frac{(\gamma)_{n,k}}{\Gamma_k(\lambda n+\nu+1)(n!)^2 2^n}\quad\text{and}\quad d_{n,k}(a,c)=\frac{(a)_{n,k}}{(c)_{n,k}\,n!}$$

with $v>-1$ and $a,c,\lambda,\gamma,k>0$. To apply Lemma 3.1, consider the sequence $\{w_n\}_{n\geq0}$ defined by

$$w_n=\frac{a_n(\nu)}{d_{n,k}(a,c)}=\frac{\Gamma_k(\gamma+nk)}{2^n\Gamma_k(\gamma)\,\Gamma_k(\lambda n+\alpha+1)\,(n!)^2}.\frac{\Gamma_k(a)\,\Gamma_k(c+nk)\,n!}{\Gamma_k(a+nk)\,\Gamma_k(c)}$$
$$=\frac{\Gamma_k(a)}{\Gamma_k(\gamma)\,\Gamma_k(c)}\rho_k(n)$$

where

$$\rho_k(x)=\frac{\Gamma_k(\gamma+xk)\,\Gamma_k(c+xk)}{\Gamma_k(\lambda x+\nu+1)\,\Gamma_k(a+xk)\,2^x\Gamma(x+1)}.$$

In view of the increasing properties of Ψ_k on $(0,\infty)$, and

$$\frac{\rho'(x)}{\rho(x)} = k\psi_k(\gamma+xk) + k\psi_k(c+xk) - \lambda\psi_k(\lambda x+\alpha+1) - k\psi_k(a+xk),$$

it follows that for $a \geq c > 0$, $\lambda \geq k$ and $\nu+1 \geq \gamma$, the function ρ is decreasing on $(0,\infty)$ and thus the sequence $\{w_n\}_{n\geq 0}$ also decreasing. Finally the conclusion for (a) follows from the Lemma 3.1.

In the case (b) and (c), the sequence $\{w_n\}$ reduces to

$$w_n = \frac{a_n(\nu)}{d_{n,k}(\gamma,\lambda)} = \frac{\rho_k(n)}{\Gamma_k(\lambda)}$$

where

$$\rho_k(x) = \frac{\Gamma_k(\lambda+xk)}{\Gamma_k(\nu+1+\lambda x)\Gamma(x+1)}.$$

Now as in the proof of part (a)

$$\frac{\rho_k'(x)}{\rho_k(x)} = k\Psi_k(\lambda+xk) - \lambda\Psi_k(\nu+1+xk) - \Psi(x+1) > 0,$$

if $\nu+1+\lambda x \geq \lambda+xk$. Now for $x \in (0,1)$, this inequality holds if $0 < k \leq \lambda \leq \nu+1$, while for $x \geq 1$, it is required that $k \leq \min\{\lambda, \nu+1\}$. $\square$

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