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CONVERGENCE THEOREMS FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. The aim of this paper to study a Noor-type iteration process with errors for approximating common fixed point of a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings in the framework of convex metric spaces. We give a necessary and sufficient condition for strong convergence of said iteration scheme involving a finite family of above said mappings and also establish a strong convergence theorem by using condition (A). The results presented in this paper extend, improve and unify some existing results in the previous work.

1. Introduction and Preliminaries

Throughout this paper, we assume that E is a metric space, $F(T_i) = \{x \in E : T_i x = x\}$ be the set of all fixed points of the mappings T_i $(i = 1, 2, \dots, N)$, $D(T)$ be the domain of T and $\mathbb{N}$ is the set of all positive integers. The set of common fixed points of T_i $(i = 1, 2, \dots, N)$ denoted by F, that is, $F = \cap_{i=1}^{N} F(T_i)$.

Definition 1.1. (See [1]) Let $T: D(T) \subset E \to E$ be a mapping.

(1) The mapping T is said to be L-Lipschitzian if there exists a constant $L > 0$ such that

$$d(Tx, Ty) \leq Ld(x, y), \quad \forall\, x,\, y \in D(T). \tag{1.1}$$

(2) The mapping T is said to be nonexpansive if

$$d(Tx, Ty) \leq d(x, y), \quad \forall\, x,\, y \in D(T). \tag{1.2}$$

(3) The mapping T is said to be quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p), \quad \forall\, x \in D(T),\ \forall\, p \in F(T). \tag{1.3}$$

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(4) The mapping T is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, T^n y) \quad \leq \quad k_n d(x, y), \qquad \forall\, x,\, y \in D(T),\ \forall\, n \in \mathbb{N}. \tag{1.4}$$

(5) The mapping T is said to be asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d(T^n x, p) \quad \leq \quad k_n d(x, p), \qquad \forall\, x \in D(T),\ \forall\, p \in F(T),\ \forall\, n \in \mathbb{N}. \tag{1.5}$$

(6) The mapping T is said to be asymptotically nonexpansive type, if

$$\limsup_{n\to\infty} \left\{ \sup_{x,\, y\, \in\, D(T)} \Big(d(T^n x, T^n y) - d(x, y) \Big) \right\} \quad \leq \quad 0. \tag{1.6}$$

(7) The mapping T is said to be asymptotically quasi-nonexpansive type, if $F(T) \neq \emptyset$ and

$$\limsup_{n\to\infty} \left\{ \sup_{x \in D(T),\, p\, \in\, F(T)} \Big(d(T^n x, p) - d(x, p) \Big) \right\} \quad \leq \quad 0. \tag{1.7}$$

Remark 1.2. It is easy to see that if $F(T)$ is nonempty, then nonexpansive mapping, quasi-nonexpansive mapping, asymptotically nonexpansive mapping, asymptotically quasi-nonexpansive mapping and asymptotically nonexpansive type mapping all are the special cases of asymptotically quasi-nonexpansive type mappings.

In recent years, the problem concerning convergence of iterative sequences (and sequences with errors) for asymptotically nonexpansive mappings or asymptotically quasi-nonexpansive mappings converging to some fixed points in Hilbert spaces or Banach spaces have been considered by many authors.

In 1973, Petryshyn and Williamson [13] obtained a necessary and sufficient condition for Picard iterative sequences and Mann iterative sequences to converge to a fixed point for quasi-nonexpansive mappings. In 1994, Tan and Xu [16] also proved some convergence theorems of Ishikawa iterative sequences satisfies Opial's condition or has a Frechet differential norm. In 1997, Ghosh and Debnath [5] extended the result of Petryshyn and Williamson [13] and gave a necessary and sufficient condition for Ishikawa iterative sequences to converge to a fixed point of quasi-nonexpansive mappings. Also in 2001 and 2002, Liu [10, 11, 12] obtained some necessary and sufficient conditions for Ishikawa iterative sequences or Ishikawa iterative sequences with errors to converge to a fixed point for asymptotically quasi-nonexpansive mappings.

In 2004, Chang et al. [1] extended and improved the result of Liu [12] in convex metric space. Further in the same year, Kim et al. [8] gave the necessary and sufficient conditions for asymptotically quasi-nonexpansive mappings in convex metric spaces which generalized and improved some previous known results.

Very recently, Tian and Yang [18] gave some necessary and sufficient conditions for a new Noor-type iterative sequences with errors to approximate a common fixed point for a finite family of uniformly quasi-Lipschitzian mappings in convex metric spaces.

The purpose of this paper is to give some necessary and sufficient conditions for Noor-type iteration process with errors to approximate a common fixed point for a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings in convex metric spaces. The results presented in this paper generalize, improve and unify some main results of [1]-[3], [5]-[8], [10]-[17], [19] and [21].

Let T be a given self mapping of a nonempty convex subset C of an arbitrary normed space. The sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\ y_n &= a_n x_n + b_n T z_n + c_n v_n, \\ z_n &= d_n x_n + e_n T x_n + f_n w_n, \end{aligned} \tag{1.8}$$

is called the Noor-type iterative procedure with errors [2], where $\alpha_n, \beta_n, \gamma_n, a_n, b_n, c_n, d_n, e_n$ and f_n are appropriate sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1$, $n \geq 0$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C. If $d_n = 1 (e_n = f_n = 0)$, $n \geq 0$, then (1.8) reduces to the Ishikawa iterative procedure with errors [20] defined as follows: $x_0 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0, \\ y_n &= a_n x_n + b_n T x_n + c_n v_n. \end{aligned} \tag{1.9}$$

If $a_n = 1 (b_n = c_n = 0)$, then (1.9) reduces to the following Mann type iterative procedure with errors [20]: $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0. \tag{1.10}$$

For the sake of convenience, we first recall some definitions and notations.

Definition 1.3. (See [1]) Let (E, d) be a metric space and $I = [0, 1]$. A mapping $W \colon E^3 \times I^3 \to E$ is said to be a convex structure on E if it satisfies the following condition:

$$d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z),$$

for any $u, x, y, z \in E$ and for any $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$.

If (E, d) is a metric space with a convex structure W, then (E, d) is called a *convex metric space* and denotes it by (E, d, W). Let (E, d) be a convex metric space, a nonempty subset C of E is said to be convex if

$$W(x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C, \quad \forall (x, y, z, \lambda_1, \lambda_2, \lambda_3) \in C^3 \times I^3.$$

Remark 1.4. It is easy to prove that every linear normed space is a convex metric space with a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in E$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. But there exist some convex metric spaces which can not be embedded into any linear normed spaces (see, Takahashi [15]).

Definition 1.5. Let (E, d, W) be a convex metric space and $T_i \colon E \to E$ be a finite family of asymptotically quasi-nonexpansive type mappings with $i = 1, 2, \dots, N$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$ and $\{f_n\}$ be nine sequences in $[0, 1]$ with

$$\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad n = 0, 1, 2, \dots. \tag{1.11}$$

For a given $x_0 \in E$, define a sequence $\{x_n\}$ as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(f(x_n), T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(f(x_n), T_n^n x_n, w_n; d_n, e_n, f_n), \end{aligned} \tag{1.12}$$

where $T_n^n = T_{n(mod\ N)}^n$, $f \colon E \to E$ is a Lipschitz continuous mapping with a Lipschitz constant $\xi > 0$ and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ are any given three sequences in E. Then $\{x_n\}$ is called the Noor-type iterative sequence with errors for a finite family of asymptotically quasi-nonexpansive type mappings $\{T_i\}_{i=1}^N$. If $f = I$ (the identity mapping on E) in (1.12), then the sequence $\{x_n\}$ defined by (1.12) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(x_n, T_n^n z_n, v_n; a_n, b_n, c_n), \\ z_n &= W(x_n, T_n^n x_n, w_n; d_n, e_n, f_n), \end{aligned} \tag{1.13}$$

If $d_n = 1 (e_n = f_n = 0)$ for all $n \geq 0$ in (1.12), then $z_n = x_n$ for all $n \geq 0$ and the sequence $\{x_n\}$ defined by (1.12) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(f(x_n), T_n^n x_n, v_n; a_n, b_n, c_n). \end{aligned} \tag{1.14}$$

If $f = I$ and $d_n = 1 (e_n = f_n = 0)$ for all $n \geq 0$, then the sequence $\{x_n\}$ defined by (1.12) can be written as follows:

$$\begin{aligned} x_{n+1} &= W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0, \\ y_n &= W(x_n, T_n^n x_n, v_n; a_n, b_n, c_n), \end{aligned} \tag{1.15}$$

which is the Ishikawa type iterative sequence with errors considered in [17]. Further, if $f = I$ and $d_n = a_n = 1 (e_n = f_n = b_n = c_n = 0)$ for all $n \geq 0$, then $z_n = y_n = x_n$ for all $n \geq 0$ and (1.12) reduces to the following Mann type iterative sequence with errors [17]:

$$x_{n+1} = W(x_n, T_n^n y_n, u_n; \alpha_n, \beta_n, \gamma_n), \quad n \geq 0. \tag{1.16}$$

Recall that a family $\{T_i : i \in \mathcal{N} = 1, 2, \ldots, N\}$ of N asymptotically quasi-nonexpansive type self mappings of C with $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ is said to satisfy condition (A) ([4]) if there exists a nondecreasing function $f\colon [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$ holds for at least one $T \in \{T_i : i \in \mathcal{N}\}$.

In the sequel, we shall need the following lemmas.

Lemma 1.6. *(See [11]) Let $\{p_n\}$, $\{q_n\}$, $\{r_n\}$ be three nonnegative sequences of real numbers satisfying the following conditions:*

$$p_{n+1} \leq (1+q_n)p_n + r_n, \quad n \geq 0, \quad \sum_{n=0}^{\infty} q_n < \infty, \quad \sum_{n=0}^{\infty} r_n < \infty. \tag{1.17}$$

Then

(1) $\lim_{n\to\infty} p_n$ exists.

(2) In addition, if $\liminf_{n\to\infty} p_n = 0$, then $\lim_{n\to\infty} p_n = 0$.

Lemma 1.7. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings for $i = 1, 2, \ldots, N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $f\colon C \to C$ be a contractive mapping with a contractive constant $\xi \in (0,1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.12) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be three bounded sequences in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$ be sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$.

Then the following conclusions hold:

(1) for all $p \in F$ and $n \geq 0$,

$$d(x_{n+1}, p) \leq (1 + 3\beta_n)d(x_n, p) + 3K\sigma_n + M\sigma_n, \tag{1.18}$$

where $\sigma_n = \beta_n + \gamma_n$ for all $n \geq 0$ and

$$M = \sup_{p\in F,\ n\geq 0} \Big\{d(u_n,p) + d(v_n,p) + d(w_n,p) + 2d(f(p),p)\Big\},$$

(2) there exists a constant $M_1 > 0$ such that

$$d(x_{n+m}, p) \leq M_1 d(x_n, p) + 3KM_1 \sum_{k=n}^{n+m-1} \sigma_k + MM_1 \sum_{k=n}^{n+m-1} \sigma_k, \ \forall p \in F, \tag{1.19}$$

for all $n, m \geq 0$.

Proof. (1) Let $p \in F$. It follows from (1.7) that

$$\limsup_{n\to\infty}\left\{\sup_{x\in E,\ p\in F}\Big(d(T^n x,p)-d(x,p)\Big)\right\}\le 0.$$

This implies that for any given $K>0$, there exists a positive integer n_0 such that for $n\ge n_0$ we have

$$\sup_{x\in E,\ p\in F}\Big(d(T^n x,p)-d(x,p)\Big)<K. \tag{1.20}$$

Since $\{x_n\},\{y_n\},\{z_n\}\subset E$, we have

$$\begin{aligned}
d(T_n^n x_n,p)-d(x_n,p)&<K, \quad \forall p\in F, \quad \forall n\ge n_0\\
d(T_n^n y_n,p)-d(y_n,p)&<K, \quad \forall p\in F, \quad \forall n\ge n_0\\
d(T_n^n z_n,p)-d(z_n,p)&<K, \quad \forall p\in F, \quad \forall n\ge n_0.
\end{aligned} \tag{1.21}$$

Thus for each $n\ge 0$ and for any $p\in F$, using (1.12), and (1.21), we have

$$\begin{aligned}
d(x_{n+1},p) &= d(W(x_n,T_n^n y_n,u_n;\alpha_n,\beta_n,\gamma_n),p)\\
&\le \alpha_n d(x_n,p)+\beta_n d(T_n^n y_n,p)+\gamma_n d(u_n,p)\\
&\le \alpha_n d(x_n,p)+\beta_n[d(y_n,p)+K]+\gamma_n d(u_n,p)\\
&\le \alpha_n d(x_n,p)+\beta_n d(y_n,p)+\beta_n K+\gamma_n d(u_n,p),
\end{aligned} \tag{1.22}$$

and

$$\begin{aligned}
d(y_n,p) &= d(W(f(x_n),T_n^n z_n,v_n;a_n,b_n,c_n),p)\\
&\le a_n d(f(x_n),p)+b_n d(T_n^n z_n,p)+c_n d(v_n,p)\\
&\le a_n d(f(x_n),f(p))+a_n d(f(p),p)\\
&\quad +b_n[d(z_n,p)+K]+c_n d(v_n,p)\\
&\le a_n\xi d(x_n,p)+a_n d(f(p),p)+b_n d(z_n,p)\\
&\quad +b_n K+c_n d(v_n,p),
\end{aligned} \tag{1.23}$$

and

$$\begin{aligned}
d(z_n,p) &= d(W(f(x_n),T_n^n x_n,w_n;d_n,e_n,f_n),p)\\
&\le d_n d(f(x_n),p)+e_n d(T_n^n x_n,p)+f_n d(w_n,p)\\
&\le d_n d(f(x_n),f(p))+d_n d(f(p),p)\\
&\quad +e_n[d(x_n,p)+K]+f_n d(w_n,p)\\
&\le d_n\xi d(x_n,p)+d_n d(f(p),p)+e_n d(x_n,p)\\
&\quad +e_n K+f_n d(w_n,p)\\
&\le (d_n\xi+e_n)d(x_n,p)+d_n d(f(p),p)\\
&\quad +e_n K+f_n d(w_n,p).
\end{aligned} \tag{1.24}$$

Substituting (1.23) into (1.22) and simplifying it, we have

$$\begin{aligned}
d(x_{n+1},p) &\leq \alpha_n d(x_n,p)+\beta_n\Big[a_n\xi d(x_n,p)+a_n d(f(p),p)\\
&\quad +b_n d(z_n,p)+b_n K+c_n d(v_n,p)\Big]+\beta_n K+\gamma_n d(u_n,p)\\
&\leq (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(f(p),p)+b_n\beta_n K\\
&\quad +b_n\beta_n d(z_n,p)+c_n\beta_n d(v_n,p)+\beta_n K+\gamma_n d(u_n,p)\\
&= (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(f(p),p)+(1+b_n)\beta_n K\\
&\quad +b_n\beta_n d(z_n,p)+c_n\beta_n d(v_n,p)+\gamma_n d(u_n,p)\\
&\leq (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(f(p),p)+2\beta_n K\\
&\quad +b_n\beta_n d(z_n,p)+c_n\beta_n d(v_n,p)+\gamma_n d(u_n,p). \qquad (1.25)
\end{aligned}$$

Substituting (1.24) into (1.25) and simplifying it, we have

$$\begin{aligned}
d(x_{n+1},p) &\leq (\alpha_n+a_n\beta_n\xi)d(x_n,p)+a_n\beta_n d(f(p),p)+2\beta_n K\\
&\quad +b_n\beta_n\Big[(d_n\xi+e_n)d(x_n,p)+d_n d(f(p),p)+e_n K\\
&\quad +f_n d(w_n,p)\Big]+c_n\beta_n d(v_n,p)+\gamma_n d(u_n,p)\\
&\leq \Big[\alpha_n+a_n\beta_n\xi+b_n\beta_n(d_n\xi+e_n)\Big]d(x_n,p)\\
&\quad +\beta_n(a_n+b_n d_n)d(f(p),p)+\beta_n K(2+b_n e_n)\\
&\quad +b_n\beta_n f_n d(w_n,p)+c_n\beta_n d(v_n,p)+\gamma_n d(u_n,p)\\
&\leq \Big[\alpha_n+\beta_n(a_n\xi+b_n d_n\xi+b_n e_n)\Big]d(x_n,p)\\
&\quad +2\beta_n d(f(p),p)+3\beta_n K+\beta_n d(w_n,p)\\
&\quad +\beta_n d(v_n,p)+\gamma_n d(u_n,p)\\
&\leq (1+3\beta_n)d(x_n,p)+2\beta_n d(f(p),p)\\
&\quad +2\gamma_n d(f(p),p)+3\beta_n K+3\gamma_n K+\beta_n d(w_n,p)\\
&\quad +\gamma_n d(w_n,p)+\beta_n d(v_n,p)+\gamma_n d(v_n,p)\\
&\quad +\beta_n d(u_n,p)+\gamma_n d(u_n,p)\\
&= (1+3\beta_n)d(x_n,p)+3K(\beta_n+\gamma_n)+2(\beta_n+\gamma_n)d(f(p),p)\\
&\quad +(\beta_n+\gamma_n)\Big[d(u_n,p)+d(v_n,p)+d(w_n,p)\Big]\\
&= (1+3\beta_n)d(x_n,p)+3K(\beta_n+\gamma_n)+(\beta_n+\gamma_n)\Big[d(u_n,p)\\
&\quad +d(v_n,p)+d(w_n,p)+2d(f(p),p)\Big]\\
&= (1+3\beta_n)d(x_n,p)+3K\sigma_n+M\sigma_n, \quad \forall n\geq 0,\ p\in F, \qquad (1.26)
\end{aligned}$$

where

$$M=\sup_{p\in F\ \ n\geq 0}\Big\{d(u_n,p)+d(v_n,p)+d(w_n,p)+2d(f(p),p)\Big\}, \quad \sigma_n=\beta_n+\gamma_n.$$

This completes the proof of part (1).

(2) Since $1+x \leq e^x$ for all $x \geq 0$, it follows from (1.26) that, for $n, m \geq 0$ and $p \in F$, we have

$$
\begin{aligned}
d(x_{n+m}, p) &\leq (1+3\beta_{n+m-1})d(x_{n+m-1}, p) + 3K\sigma_{n+m-1} + M\sigma_{n+m-1} \\
&\leq e^{3\beta_{n+m-1}} d(x_{n+m-1}, p) + 3K\sigma_{n+m-1} + M\sigma_{n+m-1} \\
&\leq e^{3\beta_{n+m-1}} \Big[e^{3\beta_{n+m-2}} d(x_{n+m-2}, p) + 3K\sigma_{n+m-2} + M\sigma_{n+m-2} \Big] \\
&\quad +3K\sigma_{n+m-1} + M\sigma_{n+m-1} \\
&\leq e^{3(\beta_{n+m-1}+\beta_{n+m-2})} d(x_{n+m-2}, p) + 3K\Big[e^{3\beta_{n+m-1}} \sigma_{n+m-2} \\
&\quad + \sigma_{n+m-1} \Big] + M\Big[e^{3\beta_{n+m-1}} \sigma_{n+m-2} + \sigma_{n+m-1} \Big] \\
&\leq \dots \\
&\leq \dots \\
&\leq M_1 d(x_n, p) + 3KM_1 \sum_{k=n}^{n+m-1} \sigma_k + MM_1 \sum_{k=n}^{n+m-1} \sigma_k, \\
&= M_1 d(x_n, p) + (3K+M)M_1 \sum_{k=n}^{n+m-1} \sigma_k, \qquad (1.27)
\end{aligned}
$$

where

$$M_1 = e^{3\sum_{k=0}^{\infty} \beta_k}.$$

This completes the proof of part (2). □

2. **Main Results**

Theorem 2.1. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i\colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings for $i = 1, 2, \dots, N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $f\colon C \to C$ be a contractive mapping with a contractive constant $\xi \in (0,1)$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.12) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be three bounded sequences in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$ be nine sequences in $[0,1]$ satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty} (\beta_n + \gamma_n) < \infty$.

Then the sequence $\{x_n\}$ converges to a common fixed point p in F if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$, *where* $d(x, F) = \inf_{p\in F} d(x, p)$.

Proof. The necessity is obvious. Now, we prove the sufficiency. In fact, from Lemma 1.7, we have

$$d(x_{n+1}, F) \leq (1+3\beta_n)d(x_n, F) + (3K+M)\sigma_n, \quad \forall n \geq 0, \qquad (2.1)$$

where $\sigma_n = \beta_n + \gamma_n$. By conditions (i) and (ii), we know that

$$\sum_{n=0}^{\infty} \sigma_n < \infty, \qquad \sum_{n=0}^{\infty} \beta_n < \infty. \qquad (2.2)$$

It follows from Lemma 1.6 that $\lim_{n\to\infty} d(x_n, F)$ exists. Since $\liminf_{n\to\infty} d(x_n, F) = 0$, we have

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{2.3}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, for any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ (where n_0 is the positive integer appeared in Lemma 1.7) such that for any $n \geq n_1$, we have

$$d(x_n, F) < \frac{\varepsilon}{8M_1}, \quad \sum_{n=n_1}^{\infty} \sigma_n < \frac{\varepsilon}{12(K+M)M_1}, \quad \forall n \geq 0. \tag{2.4}$$

From (2.4), there exists $p_1 \in F$ and positive integer $n_2 \geq n_1$ such that

$$d(x_{n_2}, p_1) < \frac{\varepsilon}{4M_1}. \tag{2.5}$$

Thus Lemma 1.7(2) implies that, for any positive integer n, m with $n \geq n_2$, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p_1) + d(x_n, p_1) \\ &\leq M_1 d(x_{n_2}, p_1) + 3(K+M)M_1 \sum_{k=n_2}^{n+m-1} \sigma_k \\ &\quad + M_1 d(x_{n_2}, p_1) + 3(K+M)M_1 \sum_{k=n_2}^{n+m-1} \sigma_k \\ &\leq 2M_1 d(x_{n_2}, p_1) + 6(K+M)M_1 \sum_{k=n_2}^{n+m-1} \sigma_k \\ &< 2M_1 . \frac{\varepsilon}{4M_1} + 6(K+M)M_1 . \frac{\varepsilon}{12(K+M)M_1} \\ &< \varepsilon. \end{aligned} \tag{2.6}$$

This shows that $\{x_n\}$ is a Cauchy sequence in a nonempty closed convex subset C of a complete convex metric space E. Without loss of generality, we can assume that $\lim_{n\to\infty} x_n = q \in E$. Now we will prove that $q \in F$. Since $x_n \to q$ and $d(x_n, F) \to 0$ as $n \to \infty$, for any given $\varepsilon_1 > 0$, there exists a positive integer $n_2 \geq n_1 \geq n_0$ such that for $n \geq n_2$, we have

$$d(x_n, q) < \varepsilon_1, \quad d(x_n, F) < \varepsilon_1. \tag{2.7}$$

Again from (2.7), there exists $q_1 \in F$ and positive integer $n_3 \geq n_2$ such that

$$d(x_{n_3}, q_1) < 2\varepsilon_1. \tag{2.8}$$

Moreover, it follows from (1.20) that for any $n \geq n_3$, we have

$$d(T^n q, q_1) - d(q, q_1) < K. \tag{2.9}$$

Thus for any $i = 1, 2, \dots, N$, from (2.7) - (2.9) and for any $n \geq n_3$, we have

$$\begin{aligned} d(T_i^n q, q) &\leq d(T_i^n q, q_1) + d(q_1, q) \\ &\leq d(q, q_1) + K + d(q_1, q) \\ &= K + 2d(q, q_1) \\ &\leq K + 2[d(q, x_{n_3}) + d(x_{n_3}, q_1)] \\ &< K + 2(\varepsilon_1 + 2\varepsilon_1) = K + 6\varepsilon_1 = \varepsilon', \end{aligned} \tag{2.10}$$

where $\varepsilon' = K + 6\varepsilon_1$, since $K > 0$ and $\varepsilon_1 > 0$, it follows that $\varepsilon' > 0$. By the arbitrariness of $\varepsilon' > 0$, we know that $T_i^n q = q$ for all $i = 1, 2, \dots, N$.

Again since for any $n \geq n_3$, we have

$$\begin{aligned} d(T_i^n q, T_i q) &\leq d(T_i^n q, q_1) + d(T_i q, q_1) \\ &\leq d(q, q_1) + K + d(T_i q, q_1) \\ &\leq d(q, q_1) + K + L d(q, q_1) \\ &= (1 + L) d(q, q_1) + K \\ &\leq (1 + L)[d(q, x_{n_3}) + d(x_{n_3}, q_1)] + K \\ &< (1 + L)[\varepsilon_1 + 2\varepsilon_1] + K \\ &= 3(1 + L)\varepsilon_1 + K = \varepsilon'', \end{aligned} \tag{2.11}$$

where $\varepsilon'' = 3(1 + L)\varepsilon_1 + K$, since $K > 0$ and $\varepsilon_1 > 0$, it follows that $\varepsilon'' > 0$. By the arbitrariness of $\varepsilon'' > 0$, we know that $T_i^n q = T_i q$ for all $i = 1, 2, \dots, N$. From the uniqueness of limit, we have $q = T_i q$ for all $i = 1, 2, \dots, N$, that is, $q \in F$. This shows that q is a common fixed point of the mappings $\{T_i\}_{i=1}^N$. This completes the proof. □

Taking $f = I$ in Theorem 2.1, then we have the following result.

Theorem 2.2. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i \colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings for $i = 1, 2, \dots, N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the iterative sequence with errors defined by (1.13) and $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ be three bounded sequences in C. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$ be nine sequences in $[0, 1]$ satisfying the conditions (i) and (ii) of Theorem 2.1. Then the sequence $\{x_n\}$ converges to a common fixed point p in F if and only if*

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2.12}$$

where $d(x, F) = \inf_{p \in F} d(x, p)$.

Taking $d_n = 1 (e_n = f_n = 0)$ for all $n \geq 0$ in Theorem 2.1, then we have the following result.

Theorem 2.3. *Let (E, d, W) be a complete convex metric space and C be a nonempty closed convex subset of E. Let $T_i \colon C \to C$ be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings for $i = 1, 2, \dots, N$ such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$ and $f \colon C \to C$ be a contractive mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$ be the iterative sequence with errors defined* *by (1.14) and* $\{u_n\}$, $\{v_n\}$ *be two bounded sequences in* C. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ *be six sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = 1$, *for all* $n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$.

Then the sequence $\{x_n\}$ *converges to a common fixed point* p *in* F *if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0, \tag{2.13}$$

where $d(x, F) = \inf_{p\in F} d(x, p)$.

As an application of Theorem 2.1, we establish another strong convergence result as follows.

Theorem 2.4. *Let* (E, d, W) *be a complete convex metric space and* C *be a nonempty closed convex subset of* E. *Let* $T_i \colon C \to C$ *be a finite family of uniformly L-Lipschitzian asymptotically quasi-nonexpansive type mappings for* $i = 1, 2, \ldots, N$ *such that* $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$ *and* $f \colon C \to C$ *be a contractive mapping with a contractive constant* $\xi \in (0,1)$. *Let* $\{x_n\}$ *be the iterative sequence with errors defined by (1.12) and* $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ *be three bounded sequences in* C. *Let* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{d_n\}$, $\{e_n\}$, $\{f_n\}$ *be nine sequences in* $[0,1]$ *satisfying the following conditions:*

(i) $\alpha_n + \beta_n + \gamma_n = a_n + b_n + c_n = d_n + e_n + f_n = 1, \quad \forall n \geq 0$;

(ii) $\sum_{n=0}^{\infty}(\beta_n + \gamma_n) < \infty$.

Assume that $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ *for* $l = 1, 2, \ldots, N$. *If* $\{T_i : i \in \mathcal{N}\}$ *satisfies condition* (A), *then the sequence* $\{x_n\}$ *converges strongly to a point in* F.

Proof. As in the proof of Theorem 2.1, we have that $\lim_{n\to\infty} d(x_n, F)$ exists. Again by hypothesis of the theorem $\lim_{n\to\infty} d(x_n, T_l x_n) = 0$ for $l = 1, 2, \ldots, N$. So condition (A) guarantees that $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a non-decreasing function and $f(0) = 0$, it follows that $\lim_{n\to\infty} d(x_n, F) = 0$. Therefore, Theorem 2.1 implies that $\{x_n\}$ converges strongly to a point in F. This completes the proof. □

Remark 2.5. Theorems 2.1 - 2.3 generalize, improve and unify some corresponding result in [1]-[3], [5]-[8], [10]-[17], [19] and [21].

Remark 2.6. Our results also extend the corresponding results of [18] to the case of more general class of uniformly quasi-Lipschitzian mappings considered in this paper.

Example 2.7. *Let* $E = [-\pi, \pi]$ *and let* T *be defined by*

$$Tx = x\, cos x$$

for each $x \in E$. *Clearly* $F(T) = \{0\}$. T *is a quasi-nonexpansive mapping since if* $x \in E$ *and* $z = 0$, *then*

$$d(Tx, z) = d(Tx, 0) = |x||cos\, x| \leq |x| = |x - z| = d(x, z),$$

and T is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. Hence by remark 1.1, T is asymptotically quasi-nonexpansive type mapping. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{\pi}{2}$ and $y = \pi$, then

$$d(Tx, Ty) = \Big|\frac{\pi}{2}\, cos\, \frac{\pi}{2} - \pi\, cos\, \pi\Big| = \pi,$$

whereas

$$d(x, y) = \Big|\frac{\pi}{2} - \pi\Big| = \frac{\pi}{2}.$$

Example 2.8. *Let $E = \mathbb{R}$ and let T be defined by*

$$T(x) = \begin{cases} \frac{x}{2}\, cos\, \frac{1}{x}, & if\ x \neq 0, \\ 0, & if\ x = 0. \end{cases}$$

If $x \neq 0$ and $Tx = x$, then $x = \frac{x}{2}\, cos\, \frac{1}{x}$. Thus $2 = cos\, \frac{1}{x}$. This is impossible. T is a quasi-nonexpansive mapping since if $x \in E$ and $z = 0$, then

$$d(Tx, z) = d(Tx, 0) = |\frac{x}{2}||cos\, \frac{1}{x}| \leq \frac{|x|}{2} < |x| = |x - z| = d(x, z),$$

and T is asymptotically quasi-nonexpansive mapping with constant sequence $\{k_n\} = \{1\}$. Hence by remark 1.1, T is asymptotically quasi-nonexpansive type mapping. But it is not a nonexpansive mapping and hence asymptotically nonexpansive mapping. In fact, if we take $x = \frac{2}{3\pi}$ and $y = \frac{1}{\pi}$, then

$$d(Tx, Ty) = \Big|\frac{1}{3\pi}\, cos\, \frac{3\pi}{2} - \frac{1}{2\pi}\, cos\, \pi\Big| = \frac{1}{2\pi},$$

whereas

$$d(x, y) = \Big|\frac{2}{3\pi} - \frac{1}{\pi}\Big| = \frac{1}{3\pi}.$$

3. Conclusion

If $F(T)$ is nonempty, then asymptotically nonexpansive mapping and asymptotically quasi-nonexpansive mappings are asymptotically quasi-nonexpansive type mappings by Remark 1.2, thus our results are good improvement and extension of some previous work from the existing literature (see, e.g., [1]-[3], [5]-[8], [10]-[21] and many others).

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