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INTEGRAL REPRESENTATIONS OF SEMI-INNER PRODUCTS IN FUNCTION SPACES

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Abstract. Various spaces of measurable functions are usually endowed with semi-inner products expressed in terms of positive measures. Trying to give answers to the inverse problem, we present integral representations for some semi-inner products on function spaces of measurable functions, obtained either directly or by adapting and extending techniques from the theory of moment problems.

1. Introduction

Let $(\Omega, \mathfrak{S})$ be a *measurable space*, that is, Ω is an arbitrary (nonempty) set and $\mathfrak{S}$ is a σ-algabra of subsets of Ω. Giving a positive measure μ on $\mathfrak{S}$, we denote by $\mathcal{L}^p(\Omega, \mu)\,(p = 1, 2)$ the set of those $\mathfrak{S}$-measurable functions $f : \Omega \mapsto \mathbb{C}$ such that $|f|^p$ is μ-integrable. (Note that we do not identify the functions μ-equal almost everywhere.)

Let $\mathcal{S}$ be a vector space consisting of $\mathfrak{S}$-measurable complex-valued functions on Ω, invariant under complex conjugation. Assume that $\mathcal{S}$ is endowed with a semi-inner product (in the sense of [7], page 96), denoted by $\langle *, *\rangle_0$, having real values when applied to real-valued functions.

A natural question is to find a positive measure μ on Ω such that $\mathcal{S} \subset \mathcal{L}^2(\Omega, \mu)$ and

$$\langle f, g\rangle_0 = \int_\Omega f(\omega)\overline{g(\omega)}d\mu(\omega), \quad f, g \in \mathcal{S}. \tag{1.1}$$

One possible approach to a solution of this problem is to connect it with a moment problem of a general type, that is, not necessarily directly related to spaces of polynomials (see for instance [19]). Of course, when (1.1) is fulfilled, the space $\mathcal{S}^{(2)}$, spanned by all products fg with $f, g \in \mathcal{S}$, lies in $\mathcal{L}^1(\Omega, \mu)$. As it is reasonable and useful to look for a probability measure μ, we should suppose that $1 \in \mathcal{S}$ and $\langle 1, 1\rangle_0 = 1$. Setting $\Lambda_0(f) = \langle f, 1\rangle_0$, $f \in \mathcal{S}$, formula (1.1) leads to a linear extension of Λ_0 to the space $\mathcal{S}^{(2)}$. An arbitrary element in $\mathcal{S}^{(2)}$ may have various representations as a sum of products of two functions from $\mathcal{S}$ but we clearly have

$$\sum_{j\in J} |f_j|^2 = \sum_{k\in K} |g_k|^2 \Longrightarrow \sum_{j\in J} \|f_j\|_0^2 = \sum_{k\in K} \|g_k\|_0^2 \tag{1.2}$$

in the presence of (1.1) for all $f_j, g_k \in \mathcal{S}$, $j \in J, k \in K$, J, K finite, where $\| * \|_0$ is the semi-norm asociated to the inner-product $\langle *, *\rangle_0$. Condition (1.2) becomes a necessary one when trying to extend Λ_0 from $\mathcal{S}$ to $\mathcal{S}^{(2)}$, in order to look for a unknown measure μ satisfying (1.1) (see Section 3).

Thanks to an argument originating in [23], in many situations of interest we may restrict ourselves to the case when the space $\mathcal{S}$ is finite dimensional (see Theorem 3). The finite dimensionality of the space $\mathcal{S}$ leads to the possibility to replace an existing measure μ by another one consisting of a finite number of atoms, via an argument going back to [25] (see also [4]). Hence, under appropriate conditions (see Theorem 4), there exist a finite subset $\{\omega_1, \ldots, \omega_d\} \subset \Omega$, and positive numbers $\lambda_1, \ldots, \lambda_d$ with $\lambda_1 + \cdots + \lambda_d = 1$, such that

$$\langle f, g\rangle_0 = \sum_{j=1}^{d} \lambda_j f(\omega_j)g(\omega_j), \ f, g \in \mathcal{S}.$$

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As already hinted above, when trying the represent semi-inner products under the form (1.1), we either obtain some direct statements (as in Theorem 1 and Theorem 6) or adapt some methods and results coming from the theory moment problems (as in Theorem 8 and Theorem 9). Because the classical moment problems are well reflected in the literature, we shall mainly mention some more recent contributions, as for instance [5], [9], [10], [12], [14], [17], [20], [29], [31], etc.

Let us briefly describe the contents of this work. The next chapter, devided into five subsections, contains the basic definitions, some examples, and some proofs as well. We introduce a general concept of function space, consisting of measurable complex-valued functions, endowed with a compatible semi-inner product (Definition 1), which extends the concept of unital square positive functional (see [21, 29–31]). Such a pair will be called a quasi-Hilbert function space (Definition 2), and will always be associated with a canonical Hilbert space (see Remark 1(2)). The formal general question, whose various particular cases are discussed in this work, is stated as Problem 2. A first abstract partial solution to Problem 2 is provided by Theorem 1, when the cardinal of the set of continuous point evaluations equals the dimension of the associated Hilbert space.

Although the framework of general function spaces leads to some interesting results (as for instance Theorem 1), to recapture as many as possible results known for the spaces of polynomials we restrict ourselves to spaces of functions having a finite numbers of generators (see Subsection 2.3). A useful consequence of such a hypothesis is the possibility of approaching Problem 2, using only finite dimensional function spaces (see Theorem 3, extending a result which goes back to [23]).

The concept of idempotent element with respect to a given semi-inner product (see Definition 4) extends the homonymous one, defined in [30]. This concept plays a central role in our development: two of our main results (Theorem 8 and Theorem 9), and other results as well, depend on the existence of orthogonal bases consisting of idempotent elements, with special properties.

Many of the results of this work are stated in terms of semi-inner products. Nevertheless, in some cases (see Theorem 9) we need the slightly stronger concept of unital square positive functional (see Subsection 2.1). The connection between these two concepts is given by Proposition 1.

A strong relation between Problem 2 and an interpolation-type problem is presented in Theorem 6, which uses an extreme situation, that is, when the cardinal of the representing measure is supposed to be equal to the dimension of the associated Hilbert space. In fact, such a hypothesis is present in most of our results. A relaxation of this assumption shows the necessity of working with projections of idempotent elements rather than with idempotents, as shown by Theorem 7.

In the function spaces having a finite number of generators, the existence of representing measures is characterized by the existence of a orthogonal bases consisting of idempotents, satisfying a certain "multiplicativity condition" given by (6). The necessary condition (6) is essential for the proof of Theorem 8, which is one of the main results of this work.

The concept of dimensional stability (see Definition 7) goes back the concept of *flatness*, introduced in [9] in the context of spaces of polynomials (see also [29]). This property is used to prove Theorem 9, which is another main result of this paper. We note that the proof of Theorem 9 is an application of Theorem 8, which makes it shorter and more transparent than those of its predecessors from [9] or [29].

An example, originating in [14] and [16], treated in our spirit, concludes this work.

2. Preliminaries

2.1. **Function Spaces and Compatible Semi-Inner Products.** Let $(\Omega, \mathfrak{S})$ be a measurable space, and let also $\mathcal{M}_{\mathfrak{S}}(\Omega)$ be the algebra of all complex-valued $\mathfrak{S}$-measurable functions on Ω (that is, $f^{-1}(B) \in \mathfrak{S}$ for each $f \in \mathcal{M}_{\mathfrak{S}}(\Omega)$ and all Borel sets $B \subset \mathbb{C}$).

Among some other reasons, the choice of this framework (see also [19]) is related to the use, in the proof of Theorem 4, of an abstract version of Tchakaloff's theorem giving a quadrature formula, due in the actual form to C. Bayer and J. Teichmann (see [4]). Nevertheless, except for such particular but important situations, we may restrict ouselves to the case when Ω is a Hausdorff space and $\mathfrak{S}$ is the σ-algabra of all Borel subsets of Ω and, when working with finite atomic measures, even to the case when Ω is an arbitrary (nonempty) set and $\mathfrak{S}$ is family of all subsets of Ω.

For convenience, and following [29–31], a vector subspace $\mathcal{S} \subset \mathcal{M}_{\mathfrak{S}}(\Omega)$ such that $1 \in \mathcal{S}$ and if $f \in \mathcal{S}$, then $\bar{f} \in \mathcal{S}$, is said to be a *function space* (we specify "*on* $(\Omega, \mathfrak{S})$" or simply "*on* Ω", whenever necessary).

Fixing a function space $\mathcal{S}$, let $\mathcal{S}^{(2)}$ be the vector space spanned by all products of the form fg with $f, g \in \mathcal{S}$, which is itself a function space. We have $\mathcal{S} \subset \mathcal{S}^{(2)}$, and $\mathcal{S} = \mathcal{S}^{(2)}$ when $\mathcal{S}$ is an algebra.

For any vector subspace $\mathcal{T} \subset \mathcal{S}$ invariant under complex conjugation, the symbol $\mathcal{RT}$ will designate the "real part" of $\mathcal{T}$, that is $\{f \in \mathcal{T}; f = \bar{f}\}$.

Functions spaces on a Hausdorff (topological) space consisting of Borel measurable functions were considered in [31].

Important examples of function spaces are associated with the space $\mathcal{P}^n$ of all polynomials in $n \geq 1$ real variables, usually denoted by $t_1, \ldots, t_n$, with complex coefficients. For every integer $m \geq 0$, let $\mathcal{P}^n_m$ be the subspace of $\mathcal{P}^n$ consisting of all polynomials p with $\deg(p) \leq m$, where $\deg(p)$ is the total degree of p. Both $\mathcal{P}^n_m$ and $\mathcal{P}^n$ are function spaces (of continuous functions) on $\mathbb{R}^n$.

We occasionally use the notation $\mathcal{P}_m$ instead of $\mathcal{P}^n_m$ and $\mathcal{P}$ instead of $\mathcal{P}^n$ when the number n is given. Note that $\mathcal{P}^{(2)}_m = \mathcal{P}_{2m}$ and $\mathcal{P}^{(2)} = \mathcal{P}$, the latter being an algebra.

Let $\mathcal{S}$ be a function space, endowed with a semi-inner product denoted by $\langle *, * \rangle_0$, whose associated semi-norm is denoted by $\| * \|_0$. Note that we have the *Cauchy-Schwarz inequality*, that is,

$$|\langle f, g \rangle_0| \leq \|f\|_0 \|g\|_0, \; f, g \in \mathcal{S}. \tag{2.1}$$

Set

$$\mathcal{I} := \{f \in \mathcal{S}; \|f\|_0 = 0\} = \{f \in \mathcal{S}, \langle f, g \rangle_0 = 0 \;\forall g \in \mathcal{S}\}, \tag{2.2}$$

which is a vector subspace of $\mathcal{S}$, via the Cauchy-Schwarz inequality. The semi-inner product $\langle *, * \rangle_0$ induces on the quotient space $\mathcal{H}^0 := \mathcal{S}/\mathcal{I}$ an inner product given by

$$\langle f + \mathcal{I}, g + \mathcal{I} \rangle = \langle f, g \rangle_0, \; f, g \in \mathcal{S}, \tag{2.3}$$

because the right hand side of this equality depends only on the equivalent classes $f + \mathcal{I}, g + \mathcal{I}$, again by the Cauchy-Schwarz inequality. The norm corresponding to (2.3) will be denoted by $\| * \|$. For the sake of simplicity, the equivalence class $f + \mathcal{I}$ will be often denoted by $\hat{f}$ for every $f \in \mathcal{S}$.

Definition 1. Let $\mathcal{S}$ be a function space. A semi-inner product $\langle *, * \rangle_0$ of $\mathcal{S}$ is said to be *compatible* with (the structure of) $\mathcal{S}$ if

$$\|1\|_0 = 1, \quad \langle \bar{f}, \bar{g} \rangle_0 = \overline{\langle f, g \rangle_0} \quad \forall f, g \in \mathcal{S}. \tag{2.4}$$

Similarly, the associated seminorm $\| * \|_0$ is said to be *compatible* with (the structure of) $\mathcal{S}$.

Example 1. An arbitrary semi-inner product on a function space is not necessarily compatible. Here is an example.

Let $\Omega := \{z = x + iy \in \mathbb{C}; 0 < x, y < 1\}$. We consider a subspace of the space $\mathcal{L}^2(\Omega)$ of square integrable functions with respect to the planar Lebesgue measure on Ω, defined as follows:

$$\mathcal{S} := \{f \in C^1(\Omega); f, \partial_z f, \partial_{\bar{z}} f \in \mathcal{L}^2(\Omega)\},$$

where $\partial_z = 2^{-1}(\partial/\partial x - i\partial/\partial y), \partial_{\bar{z}} = 2^{-1}(\partial/\partial x + i\partial/\partial y)$.

If $f, \partial_z f, \partial_{\bar{z}} f \in \mathcal{L}^2(\Omega)$, and so $\bar{f}, \overline{\partial_z f}, \overline{\partial_{\bar{z}} f} \in \mathcal{L}^2(\Omega)$, we also have $\partial_z \bar{f}, \partial_{\bar{z}} \bar{f} \in \mathcal{L}^2(\Omega)$, as one can easily see, showing that $\mathcal{S}$ is a function space on Ω.

Let us define

$$\langle f, g \rangle_0 := \int_0^1 \int_0^1 (f + i\partial_z f + \partial_{\bar{z}} f)\overline{(g + i\partial_z g + \partial_{\bar{z}} g)} dxdy, \; f, g \in \mathcal{L}^2(\Omega),$$

which is a semi-inner product on $\mathcal{S}$. In particular, if $f(x, y) = y$ and $g(x, y) = 1$, we obtain $\langle f, g \rangle_0 = 1 + 2^{-1} i$, showing that this semi-inner product is not compatible with the structure of $\mathcal{S}$.

Remark 1. Let $\mathcal{S}$ be a function space, endowed with a compatible semi-inner product $\langle *, * \rangle_0$.

(1) Because $\langle *, * \rangle_0$ is compatible with $\mathcal{S}$, its restriction to $\mathcal{RS} \times \mathcal{RS}$ is a real semi-inner product. In addition, the sum of spaces $\mathcal{RS} + i\mathcal{RS}$ is direct, equals $\mathcal{S}$, and

$$\|f + ig\|_0^2 = \|f\|_0^2 + \|g\|_0^2, \; f, g \in \mathcal{RS}. \tag{2.5}$$

We also have $\mathcal{I}=\mathcal{RI}+i\mathcal{RI}$, where the sum of the spaces is direct.

It is easily seen that the $\mathbb{R}$-linear map $\mathcal{RS}/\mathcal{RI} \ni f+\mathcal{RI} \mapsto f+\mathcal{I} \in \mathcal{H}^0$ is injective, because $\mathcal{RS} \cap \mathcal{I} = \mathcal{RI}$, allowing us to identify the space $\mathcal{RS}/\mathcal{RI}$ with a real vector subspace of $\mathcal{H}^0$, denoted by $\mathcal{RH}^0$. In fact we have the direct sum $\mathcal{H}^0 = \mathcal{RH}^0 + i\mathcal{RH}^0$. Setting

$$\langle f+\mathcal{RI}, g+\mathcal{RI}\rangle = \langle f,g\rangle_0,\ \ f,g \in \mathcal{RS},$$

and therefore

$$\langle f+\mathcal{RI}, g+\mathcal{RI}\rangle = \langle f+\mathcal{I}, g+\mathcal{I}\rangle,\ \ f,g \in \mathcal{RS},$$

we infer that the map $\mathcal{RH}^0 \ni f+\mathcal{RI} \mapsto f+\mathcal{I} \in \mathcal{H}^0$ is actually an isometry, and

$$\|\phi+i\psi\|^2 = \|\phi\|^2 + \|\psi\|^2,\ \phi,\psi \in \mathcal{RH}^0. \tag{2.6}$$

(2) We may complete the space $\mathcal{H}^0$ with respect to the inner product $\langle *,*\rangle$, to get a Hilbert space. Note that a sequence $(\sigma_k = \phi_k + i\psi_k)_{k\ge1}$ in $\mathcal{H}^0$, with $\phi_k, \psi_k \in \mathcal{RH}^0$ for all $k \ge 1$ is a Cauchy sequence if and only if both $(\phi_k)_{k\ge1}, (\psi_k)_{k\ge1}$ are Cauchy sequences. Consequently, denoting by $\mathcal{H}$ (resp. $\mathcal{RH}$) the completion of $\mathcal{H}^0$ (resp. $\mathcal{RH}^0$), we must have the direct sum $\mathcal{H} = \mathcal{RH} + i\mathcal{RH}$, and equality (2.6) is still valid for $\phi,\psi \in \mathcal{RH}$. This decomposition allows us to represent any element $\phi \in \mathcal{H}$ as a sum $\phi = \phi_1 + i\phi_2$, with $\phi_1,\phi_2 \in \mathcal{RH}$. In addition, the "complex conjugate" of ϕ is given by $\bar\phi = \phi_1 - i\phi_2$. We also note that $\|1+\mathcal{I}\| = 1$, as well as the identity

$$\overline{\langle \phi,\psi\rangle} = \langle \bar\phi, \bar\psi\rangle,\ \ \phi,\psi \in \mathcal{H}_\Lambda,$$

which is a consequence of (2.4).

Given a function space $\mathcal{S}$ endowed with a compatible semi-inner product $\langle *,*\rangle_0$, the Hilbert space $\mathcal{H}$ obtained as above will be referred to as the *Hilbert space associated to* $(\mathcal{S}, \langle *,*\rangle_0)$. When $\mathcal{H}^0$ is finite dimensional, in particular when $\mathcal{S}$ is finite dimensional, we have $\mathcal{H} = \mathcal{H}^0$, and if $\mathcal{S}$ is actually a Hilbert space, obviously $\mathcal{H} = \mathcal{S}$.

(3) The spaces $\mathcal{H}^0$ and $\mathcal{H}$ have induced inner products with a property similar to (2.4). A natural question is related to the possibility to organize them as function spaces. Nevertheless, this operation is not always possible. To explain this assertion, let us assume that $\mathcal{S}$ is a function space on $(\Omega, \mathfrak{S})$, and that

$$\Omega_0 := \cap_{f\in\mathcal{I}} \mathcal{Z}(f) \in \mathfrak{S}. \tag{2.7}$$

where $\mathcal{Z}(f) := \{\omega \in \Omega; f(\omega) = 0\}$.

Condition (2.7) is automatically fulfilled if Ω is a Hausdorff space and $\mathcal{S}$ consists of continuous functions or if the space $\mathcal{I}$ has an at most countable algebraic basis. If $\langle *,*\rangle_0$ is actually an inner product, which is equivalent to $\mathcal{I} = \{0\}$, we have again $\Omega_0 = \Omega \in \mathfrak{S}$.

Condition (2.7) allows us to regard the elements of $\mathcal{H}^0$ as functions on Ω_0. Indeed, if $\hat f = f + \mathcal{I}$ is an arbitrary element of $\mathcal{H}^0$, we put $\hat f(\omega) = f(\omega)$ for all $\omega \in \Omega_0$, which is correctly defined. In addition, $\hat f^{-1}(B) = (f|\Omega_0)^{-1}(B)$ for all Borel sets $B \subset \mathbb{C}$. In fact, $\mathcal{S}_0 := \{f|\Omega_0; f \in \mathcal{S}\}$ is a function space on $(\Omega_0, \mathfrak{S}_0)$, where $\mathfrak{S}_0 = \{A \cap \Omega_0; A \in \mathfrak{S}\}$. However, the map $\mathcal{H}^0 \ni \hat f \mapsto f|\Omega_0 \in \mathcal{S}_0$ is not injective, so $\hat f(\omega) = 0$ for all $\omega \in \Omega_0$ does not necessarily imply $\hat f(\omega) = 0$ (see Example 2).

Of course, when $\mathcal{I} = \{f \in \mathcal{S}; f|\Omega_0 = 0\}$, then $\mathcal{H}^0$ may be regarded as a function space on Ω_0.

(4) When a function space $\mathcal{S}$ and a compatible inner product $\langle *,*\rangle_0$ are given, we use the notation $\mathcal{I}, \mathcal{H}^0, \mathcal{H}$, and so on, in the sense from this remark, if not otherwise specified.

The construction in Remark 1 has an important particular case, to be discussed in the following.

Let $\mathcal{S}$ be a function space and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a linear map with the following properties:

(1) $\Lambda(\bar f) = \overline{\Lambda(f)}$ for all $f \in \mathcal{S}^{(2)}$;
(2) $\Lambda(|f|^2) \ge 0$ for all $f \in \mathcal{S}$;
(3) $\Lambda(1) = 1$.

Adapting some terminology from [21] to our context (see also [29–31]), a linear map Λ with the properties (1)-(3) is said to be a *unital square positive functional*, briefly a *uspf*.

A simple example of a uspf is given by a probability measure μ and a functions space $\mathcal{S}$ on $(\Omega, \mathfrak{S})$, consisting of square μ-integrable functions. Then the map $\mathcal{S}^{(2)} \ni f \mapsto \int_\Omega f d\mu \in \mathbb{C}$ is a uspf, as one can easily see.

When the linear map $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, we may define a semi-inner product by the equality

$$\langle f, g\rangle_0 = \Lambda(f\bar{g}),\ \ f, g \in \mathcal{S},$$

which is easily seen to be compatible with $\mathcal{S}$. Then, as in (2.2), the set

$$\mathcal{I}_\Lambda := \{f \in \mathcal{S}; \Lambda(|f|^2) = 0\} = \{f \in \mathcal{S}; \Lambda(fg) = 0\ \forall g \in \mathcal{S}\}$$

is a vector subspace of $\mathcal{S}$. Moreover, the quotient $\mathcal{H}^0_\Lambda := \mathcal{S}/\mathcal{I}_\Lambda$ is an inner product space, with the inner product given by

$$\langle \hat{f}, \hat{g}\rangle = \Lambda(f\bar{g}), \tag{2.8}$$

where $\hat{f} = f + \mathcal{I}_\Lambda$ is the equivalence class of $f \in \mathcal{S}$ modulo $\mathcal{I}_\Lambda$.

The Hilbert space obtained by completion of the inner product space $\mathcal{H}^0_\Lambda$ will be denoted by $\mathcal{H}_\Lambda$. Of course, if $\mathcal{S}$ is finite dimensional, we have $\mathcal{H}_\Lambda = \mathcal{S}/\mathcal{I}_\Lambda$.

When the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is given, we shall use the notation $\mathcal{I}_\Lambda$, $\mathcal{H}_\Lambda$, $\hat{f}$, with the meaning from above, if not otherwise specified.

Problem 1. The *(abstract) moment problem* for a given uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, where $\mathcal{S}$ is a fixed function space on $(\Omega, \mathfrak{S})$, means to find necessary and sufficient conditions insuring the existence of a probability measure μ, defined on $\mathfrak{S}$, such that $\mathcal{S} \subset \mathcal{L}^2(\Omega, \mu)$ and $\Lambda(f) = \int_\Omega f d\mu$, $f \in \mathcal{S}^{(2)}$. When such a measure μ exists, it is said to be a *representing measure of* Λ *(with support) in* Ω.

In the classical moment problem on spaces of polynomials, the role of the uspf Λ is played by the so-called Riesz functional (see for instance [17]).

In some special cases, a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ may have an *atomic representing measure in* Ω, which (in this text) means that there exists a finite subset $\Omega_\Lambda = \{\omega_1, \ldots, \omega_d\} \subset \Omega$ consisting of distinct points, and positive numbers $\lambda_1, \ldots, \lambda_d$, with $\lambda_1 + \cdots + \lambda_d = 1$, such that $\Lambda(f) = \sum_{j=1}^d \lambda_j f(\omega_j)$.

When we want to specify the number points $\{\omega_1, \ldots, \omega_d\}$, the corresponding atomic measure will be sometimes called a *d-atomic representing measure* (of Λ in Ω). Of course, in this case we can write $\Lambda(f) = \int_\Omega f(\omega)d\mu(\omega)$, where μ is a probability measure defined on a σ-algebra $\mathfrak{S}$ containing Ω_Λ and its subsets, such that $\mu(\{\omega_j\}) = \lambda_j\ j = 1, \ldots, d$. In particular, we may take as $\mathfrak{S}$ the family of all subsets of Ω.

When $\mathcal{S}$ is finite dimensional and the uspf Λ on $\mathcal{S}^{(2)}$ has an arbitrary representing measure, then one expects that this measure may be replaced by an atomic one. Such a property, going back to Tchakaloff (see Corollary 2 in [25]), will be also discussed in this work (see Theorem 4).

The concept of representing measure can be also defined for a compatible semi-inner product $\langle *, *\rangle_0$ of a function space $\mathcal{S}$ (see Definition 2).

2.2. **Continuous Point Evaluations.** A discussion concerning the point evaluations in the context of function spaces of polynomials on $\mathbb{R}^n$ can be found in [30], Section 4 (see also [31] for a more general framework). Some of the assertions of interest for us can be obtained from those in [30], with minor modifications. Nevertheless, in the following, some results will be discussed in a more general context.

Remark 2. Let $\mathcal{S}$ be a function space on $(\Omega, \mathfrak{S})$, endowed with a compatible semi-inner product $\langle *, *\rangle_0$, and let $\mathcal{H}$ be the Hilbert space associated to $(\mathcal{S}, \langle *, *\rangle_0)$ (see Subsection 2.1). For every point $\omega \in \Omega$, we denote by δ_ω the point evaluation at ω, that is, $\delta_\omega(f) = f(\omega)$, for every function $f \in \mathcal{S}$.

Of course, a point evaluation δ_ω is said to be *continuous* if there exists a constant $c_\omega > 0$ such that

$$|\delta_\omega(f)| \le c_\omega \|f\|_0,\ \ f \in \mathcal{S}. \tag{2.9}$$

A point evaluation is not necessarily continuous; see Example 2.

We denote by $\mathcal{Z}$ the subset of those points $\omega \in \Omega$ such that δ_ω is continuous.

Remark 3. We continue the discussion from Remark 1(3).

Let $\mathcal{H}^0 = \mathcal{S}/\mathcal{I}$ be endowed with the inner product (2.3). If $\omega \in \mathcal{Z}$, we must have $h(\omega) = 0$ for all functions $h \in \mathcal{I}$, as a consequence of (2.9). Consequently, we have the inclusion $\mathcal{Z} \subset \cap_{f\in\mathcal{I}}\mathcal{Z}(f)$.

Moreover, every point $\omega \in \mathcal{Z}$ induces a linear and continuous map $\hat{\delta}_\omega : \mathcal{H}^0 \mapsto \mathbb{C}$, given by $\hat{\delta}_\omega(\hat{f}) = f(\omega)$ for all $f \in \mathcal{S}$, with $\hat{f} = f + \mathcal{I}$, as the value $f(\omega)$ is constant on the equivalence class of f. In addition,

$$|\hat{\delta}_\omega(\hat{f})| = |f(\omega)| \le c_\omega \|f\|_0 = c_\omega \|\hat{f}\|,$$

via (2.9), where $\| * \|$ is the norm associated to the inner product (2.3).

The linear functional $\hat{\delta}_\omega$ can be extended to the completion $\mathcal{H}$ of $\mathcal{H}^0$, and keeping the same notation, we have

$$|\hat{\delta}_\omega(\phi)| \le c_\omega \|\phi\|,\ \phi \in \mathcal{H},\ \omega \in \mathcal{Z}.$$

The next result is a slight extension of Lemma 6 from [30].

Lemma 1. *If $\mathcal{S}$ is finite dimensional, we have*

$$\mathcal{Z} = \cap_{f\in\mathcal{I}} \mathcal{Z}(f),$$

and $\mathcal{Z} \in \mathfrak{S}$.

Proof. We already saw in Remark 3 that always $\mathcal{Z} \subset \cap_{f\in\mathcal{I}} \mathcal{Z}(f)$.

Assume now that $\mathcal{S}$ is finite dimensional, so $\mathcal{H} = \mathcal{H}^0$ is a finite dimensional Hilbert space. Let $\mathcal{B}$ be an algebraic basis of $\mathcal{I}_\Lambda$, which is a finite set. Clearly, $\mathcal{Z} \subset \cap_{b\in\mathcal{B}} \mathcal{Z}(b) = \cap_{f\in\mathcal{I}} \mathcal{Z}(f)$.

Conversely, if $\omega \in \cap_{b\in\mathcal{B}} \mathcal{Z}(b)$, then $\delta_\omega(b) = 0$ for all $b \in \mathcal{B}$, implying $\delta_\omega(f) = 0$ for all $f \in \mathcal{I}_\Lambda$. Therefore, δ_ω induces a linear functional on the Hilbert space $\mathcal{H}$, denoted by $\hat{\delta}_\omega$. The linear functional $\hat{\delta}_\omega$ is automatically continuous, and so δ_ω is continuous. This shows $\omega \in \mathcal{Z}$. Consequently, $\mathcal{Z} = \cap_{b\in\mathcal{B}} \mathcal{Z}(b)$, implying $\mathcal{Z} = \cap_{f\in\mathcal{I}} \mathcal{Z}(f)$.

Finally, $\mathcal{Z}(b) = b^{-1}(\{0\}) \in \mathfrak{S}$ for all $b \in \mathcal{B}$ showing that $\mathcal{Z} \in \mathfrak{S}$. □

Remark 4. (1) The previous argument shows that if $\mathcal{H}^0$ is finite dimensional, so $\mathcal{H} = \mathcal{H}^0$, we have the equality $\mathcal{Z} = \cap_{f\in\mathcal{I}} \mathcal{Z}(f)$. If, in addition, the space $\mathcal{I}$ has an at most countable algebraic basis, then $\mathcal{Z} \in \mathfrak{S}$.

(2) If $\mathcal{I} = \{0\}$ and $\mathcal{S} = \mathcal{H}^0$ is finite dimensional, we must have $\mathcal{Z} = \Omega$.

(3) The previous lemma shows that the set $\mathcal{Z}$ extends the concept of algebraic variety of a moment sequence, defined in the context of finite dimensional spaces of polynomials (see for instance (1.6) from [10]).

Remark 5. (1) Continuing the discussion from Remark 1(3), if $\mathcal{I} = \{f \in \mathcal{S}; f|\mathcal{Z} = 0\}$, and $\mathcal{S}$ is finite dimensional, the space $\mathcal{H} = \mathcal{H}^0$ may be regarded as a function space on $\mathcal{Z}$ because from $f|\mathcal{Z} = 0$, so $\|f\|_0 = 0$, we obtain $\hat{f} = 0$.

(2) Note that for every $\omega \in \mathcal{Z}$ there exists a vector $\nu_\omega \in \mathcal{H}$ such that $\hat{\delta}_\omega(\phi) = \langle \phi, \nu_\omega \rangle$ for all $\phi \in \mathcal{H}$, via a well-known theorem by Riesz. In fact, applying the Riesz theorem firstly on $\mathcal{RH}$, we deduce that $\nu_\omega \in \mathcal{RH}$.

Definition 2. Let $\mathcal{S}$ be a function space on $(\Omega, \mathfrak{S})$, endowed with a compatible semi-inner product $\langle *, * \rangle_0$. From now on, such a pair $(\mathcal{S}, \langle *, * \rangle_0)$ will be designated as a *quasi-Hilbert function space* (briefly, a *qHfs*). When the function space $\mathcal{S}$ on Ω is actually a Hilbert space, endowed with a compatible inner product $\langle *, * \rangle$, the pair $(\mathcal{S}, \langle *, * \rangle)$ it will be called a *Hilbert function space* (briefly, a *Hfs*).

We say that the semi-inner product $\langle *, * \rangle_0$ has a *representing measure* if there exists a probability measure μ on $\mathfrak{S}$ such that

$$\langle f, g \rangle_0 = \int_\Omega f(\omega)\overline{g(\omega)} d\mu(\omega),\ f, g \in \mathcal{S}. \tag{2.10}$$

We say that the semi-inner product $\langle *, * \rangle_0$ has an *atomic representing measure in* Ω if there exists a finite subset $\Omega_0 = \{\omega_1, \ldots, \omega_d\} \subset \Omega$ consisting of distinct points, and positive numbers $\lambda_1, \ldots, \lambda_d$, with $\lambda_1 + \cdots + \lambda_d = 1$, such that

$$\langle f, g \rangle_0 = \sum_{j=1}^{d} \lambda_j f(\omega_j)\overline{g(\omega_j)},\ f, g \in \mathcal{S}. \tag{2.11}$$

When the support of a given atomic representing measure consists of d points, it will be sometimes called a *d-atomic representing measure*. As usually, the points $\{\omega_1, \ldots, \omega_d\}$ are called the *nodes*, and the numbers $\{\lambda_1, \ldots, \lambda_d\}$ are called the *weights* of the measure μ.

A slightly more general question than Problem 1 is the following.

Problem 2. Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs on $(\Omega, \mathfrak{S})$. Find necessary and sufficient conditions to insure the existence of a representing measure of the semi-inner product $\langle *, * \rangle_0$.

Using direct arguments, as well as methods from the theory of moment problems in functions spaces of polynomials, we shall try to give some answers to Problem 2 in the next sections.

The following lemma is similar to Lemma 7 from [30] (see also [9] for a precursor of this result).

Lemma 2. *Suppose that the compatible semi-inner product $\langle *, * \rangle_0$ of the function space $\mathcal{S}$ on Ω has an atomic representing measure μ. Then* $\mathrm{supp}(\mu) \subset \mathcal{Z}$.

Proof. We use the notation from Definition 2. If $\omega_k \in \Omega_0$, we have:

$$|f(\omega_k)|^2 \le \frac{1}{\lambda_k} \sum_{j=1}^{d} \lambda_j |f(\omega_j)|^2 = \frac{1}{\lambda_k} \|f\|_0^2, \ f \in \mathcal{S},$$

for all $k = 1, \ldots, d$, showing that $\mathrm{supp}(\mu) \subset \mathcal{Z}$. $\square$

The next definition adapts some concepts from [14] to our context.

Definition 3. Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs. We say that the semi-inner product $\langle *, * \rangle_0$ is *weakly consistent* if whenever $f \in \mathcal{S}$ in null on $\mathcal{Z}$, it follows that $\langle f, 1 \rangle_0 = 0$.

We say that $\langle *, * \rangle_0$ is *consistent* if whenever $\sum_{k \in K} f_k g_k = 0$ on $\mathcal{Z}$ it follows $\sum_{k \in K} \langle f_k, g_k \rangle_0 = 0$, where $f_k, g_g \in \mathcal{RS}$, $k \in K$, K finite.

Remark 6. (1) Consider a qHfs $(\mathcal{S}, \langle *, * \rangle_0)$. It follows from Lemma 2 that a necessary condition for the existence of an atomic representing measure for the semi-inner product $\langle *, * \rangle_0$ on Ω is $\mathcal{Z} \neq \emptyset$.

(2) If the semi-inner product $\langle *, * \rangle_0$ is consistent, it is also weakly consistent.

(3) If the semi-inner product $\langle *, * \rangle_0$ has an atomic representing measure, then it is consistent. The converse is not true, in general (see [14]).

(4) Let $\mathcal{S}$ be a function space on a set Ω and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be a uspf. Of course, the previous results of this subsection applies to this case, when the semi-inner product is given by $\langle f, g \rangle_0 = \Lambda(f\bar{g})$, $f, g \in \mathcal{S}$. We recall that we use the notation $\mathcal{H}_\Lambda, \mathcal{H}_\Lambda^0, \mathcal{I}_\Lambda, \mathcal{Z}_\Lambda$ instead of $\mathcal{H}, \mathcal{H}^0, \mathcal{I}, \mathcal{Z}$, respectively. In this case (see also [14] for spaces of polynomials), if $\langle *, * \rangle_0$ is (weakly) consistent, we say that Λ is (*weakly*) *consistent*. Also note that Λ is consistent if and only if whenever $f \in \mathcal{S}^{(2)}$ satisfies $f|\mathcal{Z}_\Lambda = 0$, we have $\Lambda(f) = 0$. A similar property characterizes the weak consistency of Λ.

If $\mathcal{S}$ is finite dimensional, the uspf Λ is consistent if and only if $\Lambda \in \mathrm{span}\{\delta_\omega^{(2)}; \omega \in \mathcal{Z}_\Lambda\}$, where $\delta_\omega^{(2)}$ is the point evaluation at $\omega \in \Omega$, regarded as a functional on $\mathcal{S}^{(2)}$. Indeed, if $\Lambda \in \mathrm{span}\{\delta_\omega^{(2)}; \omega \in \mathcal{Z}_\Lambda\}$, then Λ is clearly consistent. Conversely, because $\mathrm{span}\{\delta_\omega^{(2)}; \omega \in \mathcal{Z}_\Lambda\}$ is in the dual of $\mathcal{S}^{(2)}$, which is finite dimensional, if $\Lambda \notin \mathrm{span}\{\delta_\omega^{(2)}; \omega \in \mathcal{Z}_\Lambda\}$, we can find a function $f_0 \in \mathcal{S}^{(2)}$ which is null on $\mathcal{Z}_\Lambda$ but with $\Lambda(f_0) \neq 0$, which is not possible.

Similarly, Λ is weakly consistent if and only if $\Lambda|\mathcal{S} \in \mathrm{span}\{\delta_\omega; \omega \in \mathcal{Z}_\Lambda\}$.

Finally, note that $\omega \notin \mathcal{Z}_\Lambda$ for some $\omega \in \Omega$, if and only if there exists a sequence $(f_n)_n$ in $\mathcal{S}$ such that $\Lambda(|f_n|^2) = 1$ and $|f_n(\omega)| \to \infty$ $(n \to \infty)$.

Example 2. We consider the the uspf $\Lambda : \mathcal{P}_4^1 \mapsto \mathbb{C}$, where $\mathcal{P}_4^1$ is the space of of polynomials in one real variable t, with complex coefficients, of degree ≤ 4, where $\Lambda(t^k) = 1$, $k = 0, 1, 2, 3$, $\Lambda(t^4) = 2$, extended by linearity. This is a simple example showing that there are truncated moment problems with no representing measure in $\mathbb{R}$ (see [30], Example 3; see also [13]). It can be also used in connection with Remark 5(1), and with Remark 6(4) as well.

As shown in [30], we have $\mathcal{I}_\Lambda = \{p(t) = a - at; a \in \mathbb{C}\}$. Then clearly, $\mathcal{Z}_\Lambda = \{1\}$, so the point evaluation $\delta_1(p) = p(1)$, $p \in \mathcal{P}_2^1$, must be continuous, by Lemma 1. In fact, if $p(t) = a + bt + ct^2$ with $a, b, c \in \mathbb{R}$ arbitrary,

$$p(1)^2 = (a + b + c)^2 \le \Lambda(p^2) = ((a + b + c)^2 + c^2),$$

and thus, when $a, b, c \in \mathbb{C}$, we deduce that $|p(1)|^2 \le \Lambda(|p|^2)$, $p \in \mathcal{P}_2^1$, showing, directly, that δ_1 is Λ-continuous.

On the other hand, if $\theta \in \mathbb{R}$, $\theta \neq 1$, setting $p_n(t) = 1 + n(1 - t)$, $n \ge 1$, we obtain $p_n(\theta) = 1 + n(1 - \theta) \to \pm\infty$ as $n \to \infty$, while $\Lambda(p_n^2) = 1$ for all $n \ge 1$, as in Remark 6(4).

Note that there are nonnul elements $\hat{f} \in \mathcal{H}_\Lambda = \mathcal{P}_2^1/\mathcal{I}_\Lambda$ such that $\hat{\delta}_1(\hat{f}) = 0$. Indeed, as one can easily see (see also [31], Example 3), we have $\mathcal{H}_\Lambda = \{u\hat{1} + v\widehat{t^2}; u, v \in \mathbb{C}\}$. Then $\hat{\delta}_1(\hat{1} - \widehat{t^2}) = 0$, while $\hat{1} - \widehat{t^2} \neq 0$. In other words, the Hilbert space associated to a qHfs $(\mathcal{S}, \langle *, * \rangle_0)$ is not a Hilbert function space.

Note also that, because $\mathcal{Z}_\Lambda = \{1\}$, the only possible atomic representing measure for Λ would be δ_1, via Lemma 2. But this is impossible because, for instance, $\delta_1(t^4) = 1$, while $\Lambda(t^4) = 2$.

As in the case of function spaces consisting of polynomials, an extremal situation associated with the consistency insure the existence of an atomic representing measure (see [14], Theorem 1.3).

Theorem 1. *Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a finite dimensional qHfs on Ω. Assume that $d := \mathrm{card}(\mathcal{Z}) = \dim(\mathcal{H})$. The semi-inner product $\langle *, * \rangle_0$ has a d-atomic representing measure if and only if it is consistent.*

Proof. The consistency of $\langle *, * \rangle_0$ allows us to replace the set Ω by the set $\mathcal{Z}$. Indeed, defining the function space $\mathcal{S}_\mathcal{Z}$ as the space $\{f|\mathcal{Z}; f \in \mathcal{S}\}$, and putting

$$\langle f|\mathcal{Z}, g|\mathcal{Z} \rangle_{0,\mathcal{Z}} := \langle f, g \rangle_0, \ \ f, g \in \mathcal{S},$$

we obtain a semi-inner product $\langle *, * \rangle_{0,\mathcal{Z}}$ on $\mathcal{S}_\mathcal{Z}$. Indeed, we note that if $f|\mathcal{Z} = 0$ or $g|\mathcal{Z} = 0$, we have $\langle f, g \rangle_0 = 0$ by the consistency of $\langle *, * \rangle_0$. This shows that the map $\langle *, * \rangle_{0,\mathcal{Z}}$ is well defined, and it is clearly a semi-inner product. In addition, if $f|\mathcal{Z} = 0$, the consistency implies $f \in \mathcal{I}$, showing that $\mathcal{I} = \{f \in \mathcal{S}, f|\mathcal{Z} = 0\}$. Hence we have a natural map

$$\mathcal{H} \ni \hat{f} \mapsto f|\mathcal{Z} \in C(\mathcal{Z}) := \{h : \mathcal{Z} \mapsto \mathbb{C}\},$$

which is a linear isomorphism, because it is clearly injective and surjective too, thanks to the assumption $\mathrm{card}(\mathcal{Z}) = \dim(\mathcal{H})$.

We write now $\mathcal{Z} = \{\zeta_1, \ldots, \zeta_d\}$, and put $\chi_j := \chi_{\{\zeta_j\}}$, that is, the characteristic function of the set $\{\zeta_j\}$, $j = 1, \ldots, d$. Let also $\hat{b}_j$ be the unique element of $\mathcal{H}$ with $b_j|\mathcal{Z} = \chi_j$. Because we have $b_j b_k|\mathcal{Z} = \chi_j\chi_k = 0$ if $j \neq k$, we must have $\langle b_j, b_k \rangle_0 = 0$. Similarly, $\chi_j^2 = \chi_j$ implies $\langle b_j, b_j \rangle_0 = \langle\ b_j, 1 \rangle_0$.

Note that $f|\mathcal{Z} = \sum_{j=1}^d f(\zeta_j)\chi_j$, so $f - \sum_{j=1}^d f(\zeta_j)b_j$ is null on $\mathcal{Z}$ for all $f \in \mathcal{S}$. Thus

$$\langle f, g \rangle_0 = \langle \sum_{j=1}^d f(\zeta_j)b_j, \sum_{k=1}^d \overline{g(\zeta_k)}b_k \rangle = \sum_{j=1}^d \lambda_j f(\zeta_j)\overline{g(\zeta_j)}, \ \ f, g \in \mathcal{S},$$

where $\lambda_j = \langle b_j, 1 \rangle$ for all j, which is a representation of the semi-inner product $\langle *, * \rangle_0$ via a d-atomic measure.

Conversely, if the semi-inner product $\langle *, * \rangle_0$ has a d-atomic representing measure with $d := \dim(\mathcal{H})$, then $\langle *, * \rangle_0$ is clearly consistent. □

Remark With the previous notation, we must have $\langle \hat{b}_j, \hat{b}_k \rangle = 0$, $\langle \hat{b}_j, \hat{b}_j \rangle = \langle \hat{b}_j, \hat{1} \rangle$, $j, k = 1, \ldots, d$, $j \neq k$, and $\hat{f} = \sum_{j=1}^d f(\zeta_j)\hat{b}_j$ for all $\hat{f} \in \mathcal{H}$. In other words, $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is an orthogonal basis of $\mathcal{H}$, consisting of idempotents (see Subsection 2.5 for details).

2.3. **Generators of Function Spaces.** Let $\mathcal{S}$ be a function space on $(\Omega, \mathfrak{S})$. We assume that there exist an n-tuple $\theta = (\theta_1, \ldots, \theta_n)$ of elements of $\mathcal{RS}$, and an integer $m \geq 1$, such that such that the family $\Theta_m := \{\theta^\alpha; |\alpha| \leq m, \alpha \in \mathbb{Z}_+^n\}$ spans the space $\mathcal{S}$.

When such a pair (θ, m) exists, we shortly say that the function space $\mathcal{S}$ is *m-generated by θ*. Clearly, in this case $\mathcal{S}$ is of finite dimension, and the family Θ_{2m} spans the space $\mathcal{S}^{(2)}$. In particular, $\mathcal{S}$ is 1-generated by θ if and only if $\mathcal{S}$ is the span of Θ_1.

Also note that if $\mathcal{S}$ be a function space that is m-generated by an n-tuple $\theta = (\theta_1, \ldots, \theta_n)$ of elements of $\mathcal{RS}$, and if $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is a uspf, then the Hilbert space $\mathcal{H}_\Lambda$ must be of finite dimension less or equal to the cardinal of the set Θ_m.

As a matter of fact, if $\mathcal{S}$ is a function space on Ω that is m-generated by an n-tuple $\theta = (\theta_1, \ldots, \theta_n)$ of elements of $\mathcal{RS}$, we must have the equality, $\mathcal{S} = \{p \circ \theta; p \in \mathcal{P}_m^n\}$, where θ is regarded as a function from Ω into $\mathbb{R}^n$.

In particular, $\mathcal{P}_m^n$ is a function space m-generated by by the n-tuple $t := (t_1, \ldots, t_n)$, where $t_1, \ldots, t_n$ are the independent variables of $\mathbb{R}^n$.

An important particular case of finitely generated function spaces is related to the so called *truncated K-moment problem* (see [12]), which means, for a fixed closed set $K \subset \mathbb{R}^n$, to look for a representing measure of a given uspf $\Lambda : \mathcal{P}^n_{2m,K} \mapsto \mathbb{C}$, where $\mathcal{P}^n_{m,K} := \{p|K; p \in \mathcal{P}^n_m\}$. Clearly, $\mathcal{P}^n_{m,K}$ is a function space on K, m-generated by n-tuple $t|K := (t_1|K, \dots, t_n|K)$.

To exhibit another example, fix an n-tuple $\theta = (\theta_1, \dots, \theta_n)$ in $\mathcal{P}^d$. Then $\mathcal{S} := \{p \circ \theta; p \in \mathcal{P}^n_m\}$ is a function space on $\mathbb{R}^d$. In fact, if $g = \max_{1 \le j \le n}\{\deg\theta_j\}$, we have $\mathcal{S} \subset \mathcal{P}^d_{mg}$.

Let again $\mathcal{S}$ be a function space on $(\Omega, \mathfrak{S})$, and let n-tuple $\theta = (\theta_1, \dots, \theta_n)$ of elements of $\mathcal{RS}$. When $\mathcal{S}$ spanned by the set $\{\theta^\alpha; \alpha \in \mathbb{Z}^n_+\}$, we say that $\mathcal{S}$ is ∞-*genarated by* θ. In thes case, $\mathcal{S}$ is actually a unital algebra. Of course, in this case we have $\mathcal{S} = \{p \circ \theta; p \in \mathcal{P}^n\}$.

2.4. **Reduction to Finite Dimensional Spaces.** We exhibit, in the following, a result allowing us to prove the existence of a representing measure for a uspf Λ, defined on a space $\mathcal{S}$ of continuous functions on a locally compact metrisable space, using representing measures of restrictions of Λ on some finite dimensional subspaces of $\mathcal{S}$. Such a reduction result goes back to the paper [23], where it is proved in the context of spaces of polynomials. It is also approached, in a more abstract framework, in [28].

As mentioned above, in this subsection the basic space Ω is supposed to be a locally compact metric space. We denote by $\Omega_\infty = \Omega \cup \{\infty\}$ the one-point compactification of Ω, which is compact and metrisable. If ρ is a fixed metric on Ω, a given sequence $(\omega_k)_{k\ge1}$ is said to tend to infinity, and we write $\lim_{k\to\infty} \omega_k = \infty$, if $\lim_{k\to\infty} \rho(\omega_0, \omega_k) = \infty$ for some (any) point $\omega_0 \in \Omega$.

Let $\mathcal{S}$ be a function space on Ω, consisting of continuous functions. We suppose that there exists an n-tuple $\theta = (\theta_1, \dots, \theta_n)$ of functions of $\mathcal{RS}$, separating the points of Ω and unbounded on Ω, that is, $\lim_{\omega\to\infty} \theta_j(\omega) = \infty$ for all $j = 1, \dots, n$. We also assume that the space $\mathcal{S}$ is ∞-generated by the θ, so $\mathcal{S}$ is a commutative unital algebra, closed under complex conjugation.

Next, we define the functions

$$q_k(\omega) = (1 + \theta_1(\omega)^2 + \cdots + \theta_n(\omega)^2)^{-k}$$

where $k \ge 0$ is an integer. Then we set

$$\mathcal{T}_k = \{f \in \mathcal{S}; \lim_{\omega\to\infty} q_k(\omega) f(\omega) \text{ exists}\},$$

which is a function space satisfying $1, q_k^{-1} \in \mathcal{T}_k$, and $\mathcal{T}_k \subset \mathcal{T}_{k+1}$, for all k. Moreover, as we have $\mathcal{S} = \{p\circ\theta; p \in \mathcal{P}^n\}$, and $|\theta_1^{\alpha_1} \cdots \theta_n^{\alpha_n}|^2 \le q_k(\omega)^{-1}$ when $|\alpha_1|+\cdots+|\alpha_n| \le k$, it follows that $\mathcal{S} = \cup_{k\ge0}\mathcal{T}_k$.

We now consider the algebra $C(\Omega_\infty)$, consisting of continuous functions on the compact set Ω_∞. Moreover, the set $\mathcal{Q} := \{q_k; k \ge 0\}$ is a multiplicative family in $C(\Omega_\infty)$, provided that each function q_k $(k \ge 1)$ is extended with 0 at ∞ (see [28] for details). In addition, we may regard $\mathcal{T}_k$ as a subspace of $q_k^{-1}C(\Omega_\infty)$.

Under the hypothesis on Ω and θ from above, we have the following.

Theorem 2. *A uspf $\Lambda : \mathcal{S} \mapsto \mathbb{C}$ has a representing measure with support in Ω if and only if $\Lambda(q_k^{-1}) > 0$ and*

$$|\Lambda(f)| \le \Lambda(q_k^{-1}) \sup_{\omega\in\Omega} |q_k(\omega) f(\omega)|, \;\; f \in \mathcal{T}_k, \;\; k \ge 0.$$

Proof. It Λ has a representing measure, say μ, because $q_k^{-1} \ge 1$ it follows $\Lambda(q_k^{-1}) \ge \Lambda(1) = 1$ for all $k \ge 0$. Therefore

$$|\Lambda(f)| \le \int_\Omega |f| d\mu \le \Lambda(q_k^{-1}) \sup_{\omega\in\Omega} |q_k(\omega) f(\omega)|,$$

for all $f \in \mathcal{T}_k$.

Conversely, we shall apply Theorem 3.7 from [28] to the space $\mathcal{S} \subset C(\Omega_\infty)/\mathcal{Q}$ and the map Λ. First of all, we note that $\Lambda(q_k^{-1}) \ge \Lambda(1) = 1$ for all $k \ge 0$ because q_k^{-1} is equal to 1 plus a sum of squares. Secondly, the space $C(\Omega_\infty)$ is separable. In fact, the functions $q_1(\theta), \theta_1 q_1(\theta), \dots, \theta_n q_1(\theta)$ separate the points of Ω_∞. Then the Weierstrass-Stone theorem implies the density of the unital algebra generated by this family in $C(\Omega_\infty)$.

Note that $q_k(\infty) = 0$ for all $k \ge 1$, and so $\{\infty\}$ is the union of the zeros of all functions from Q. We also have $1, q_k^{-1} \in \mathcal{T}_k \subset q_k^{-1}C(\Omega_\infty)$ for all $k \ge 0$. Moreover, $\mathcal{T}_{k_1} \subset \mathcal{T}_{k_2}$ whenever $k_1 \le k_2$, and this is equivalent to the fact that $q_{k_1}^{-1}$ divides $q_{k_2}^{-1}$.

If $\Lambda_k = \Lambda|\mathcal{T}_k$, putting $\|f\|_{\infty,k} = \sup_{t\in\Omega} |q_k(t)f(t)|, f \in C(\Omega_\infty)/q_k$, which is precisely the norm on $C(\Omega_\infty)/q_k$ (see [28]), the conditions from the statement imply the estimates $\|\Lambda_k\| \le \Lambda(q_k^{-1})$. Because $q_k^{-1} \in \mathcal{T}_k$, and its norm is one, we must have $\|\Lambda_k\| = \Lambda(q_k^{-1})$. According to Theorem 3.7 from [28], this implies the existence of a positive extension M of Λ to $C(\Omega_\infty)/\mathcal{Q}$. The proof of Theorem 3.7 from [28] shows the existence of a representing measure of M, whose support is precisely in Ω (see also Remark 3.8(1) from [28]). □

Remark 7. Theorem 2 can also be applied when the spaces $(\mathcal{T}_k)_{k\ge 0}$ are replaced by the simpler spaces $(\mathcal{S}_k)_{k\ge 0}$, with $\mathcal{S}_k = \{f \in \mathcal{S}; f = p\circ\theta, p \in \mathcal{P}_k^n\}$. In fact, assuming that for the uspf $\Lambda : \mathcal{S} \mapsto \mathbb{C}$ the restriction $\Lambda_k = \Lambda|\mathcal{S}_{2k}$ has a representing measure for each $k \ge 0$, we may obtain the assertion in the following way. The finite dimension of the involved spaces allows us to find, for every integer $k \ge 0$, an integer $r_k \ge 0$ such that $\mathcal{T}_k \subset \mathcal{S}_{2r_k}$. Then the representing measure of $\Lambda|\mathcal{S}_{2r_k}$ induces a representing measure for $\Lambda|\mathcal{T}_k$ for all $k \ge 0$, which allows the application of Theorem 2.

Using the preceding remark, we deduce easily the following:

Theorem 3. *Let Ω be a locally compact metric space, let $\mathcal{S}$ be a function space on Ω consisting of continuous functions, and let $\langle *,*\rangle_0$ be a semi-inner product on $\mathcal{S}$. Let also $\theta = (\theta_1,\ldots,\theta_n)$ be a tuple of functions from $\mathcal{RS}$, unbounded on Ω and separating its points, let $\mathcal{S}_m$ be the function space m-generated by θ and let $\langle *,*\rangle_{0m}$ be the restriction of the semi-inner product $\langle *,*\rangle_0$ to $\mathcal{S}_m$ $(m \ge 1)$. The semi-inner product $\langle *,*\rangle_0$ has a representing measure on Ω if and only if the semi-inner product $\langle *,*\rangle_{0m}$ has a representing measure on Ω for every $m \ge 1$.*

Theorem 3 shows that solving Problem 2 on finite dimensional function space leads to solutions of Problem 2 in a large class of infinite dimensional function spaces, including the classical ones in spaces of polynomials. For this reason, in the next sections we shall mainly deal with finite dimensional function spaces.

As noticed in several works (see for instance [9]), another important feature in the context of finite dimensional function spaces is that the existence of a representing measure of a given semi-inner product implies the existence an atomic representing measure, as presented in the following.

Theorem 4. *Let $(\mathcal{S},\langle *,*\rangle_0)$ be a qHfs on Ω, m-generated by the n-tuple $\theta = (\theta_1,\ldots,\theta_n)$ Suppose that $\langle *,*\rangle_0$ has a representing measure. Then $\langle *,*\rangle_0$ has an atomic representing measure.*

Proof. We consider the set $\mathbb{Z}_+^{n,2m} := \{\alpha \in \mathbb{Z}_+^n; |\alpha| \le 2m\}$, endowed with the lexicographic order. In addition, we assign to each integer $j \in \{1,2,\ldots,n_m\}$, where n_m is the cardinal of $\mathbb{Z}_+^{n,2m}$, a multi-index $\alpha(j) \in \mathbb{Z}_+^{n,2m}$ with $j \le k$ iff $\alpha(j) \le \alpha(k)$, and $\alpha(1) = 0$. In this way we have a (Borel measurable) map $\phi : \Omega \mapsto \mathbb{R}^{n_m}$ given by $\phi(\omega) = (\theta^{\alpha(1)}(\omega),\ldots,\theta^{\alpha(n_m)}(\omega)) \in \mathbb{R}^{n_m}$.

Now assume that $\langle *,*\rangle_0$ has a representing measure, so it has the form $\langle f,g\rangle_0 = \int_\Omega f\bar{g}d\mu,\ f,g \in \mathcal{S}$, where μ is a positive Borel measure on Ω, with $\mu(\Omega) = 1$. Let ν be the measure induced by the measure μ and the Borel map ϕ. Writing $\alpha(j) = \alpha'(j) + \alpha''(j)$ with $|\alpha'(j)|, |\alpha''(j)| \le m$, we have:

$$\int_{\mathbb{R}^{n_m}} |x_j| d\nu(x) = \int_\Omega |x_j\circ\phi| d\mu \le \int_\Omega |\theta^{\alpha(j)}| d\mu \le \|\theta^{\alpha'(j)}\|_0 \|\theta^{\alpha''(j)}\|_0 < \infty,$$

for all $j = 1,\ldots,n_m$, where $x_1,\ldots,x_{n_m}$ are the coordinate functions in $\mathbb{R}^{n_m}$. This shows that we may apply Corollary 2 from [4] to deduce the existence of a positive integer $d \le n_m$, a set of points $\omega_1,\ldots,\omega_d$ in the support of the measure μ, and positive numbers $\lambda_1,\ldots,\lambda_d$, such that $\sum_{j=1}^d \lambda_j = 1$, and

$$\int_\Omega \theta^\alpha d\mu = \sum_{j=1}^d \lambda_j \theta^\alpha(\omega_j),\ \alpha \in \mathbb{Z}_+^{n,2m}.$$

From this equality, we infer easily that $\langle *,*\rangle_0$ has an atomic representing measure. □

2.5. **Quasi-Hilbert Function Spaces and Idempotents.** The concepts of quasi-Hilbert function space (briefly, qHfs) and Hilbert function space (briefly, Hfs) are those introduced by Definition 2.

We have already noted that the Hilbert space associated to a qHfs $(\mathcal{S},\langle *,*\rangle_0)$ is not necessarily a Hilbert function space (see Example 2). We should mention that our concept of Hilbert function space is slightly different from the homonymous concept from [1]. In fact, the concept of Hilbert function space, as defined in [1], is often called a *reproducing kernel Hilbert space* (see for instance [2]). Unlike in [1], the point evaluations on a Hilbert function space in our sense are not necessarily well defined but the inner product of such a space must be compatible.

We introduce in the following the concept of element idempotent related to a given function space $\mathcal{S}$, endowed with a compatible semi-inner product $\langle *, * \rangle_0$. This is an extension of the concept of indempotent with respect to a uspf, introduced in [30].

Definition 4. Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs, and let $\mathcal{H}$ be the Hilbert space aassociated to $(\mathcal{S}, \langle *, * \rangle_0)$, whose inner product is denoted by $\langle *, * \rangle$, and whose norm is $\| * \|$. An element $\iota \in \mathcal{RH}$ is said to be an *idempotent* (*associated to* $\mathcal{S}$) if

$$\|\iota\|^2 = \langle \iota, \hat{1} \rangle. \tag{2.12}$$

We set $\mathcal{ID}(\mathcal{H}) := \{\iota \in \mathcal{RH}; \langle \iota, \hat{1} \rangle \neq 0\}$, that is, the family of all nonnull idempotents of $\mathcal{H}$.

Example 3. Let μ be a probability measure on $(\Omega, \mathfrak{S})$, and let $\mathcal{S}$ be a function space on $(\Omega, \mathfrak{S})$, consisting of square μ-integrable functions, so its semi-inner product $\langle *, * \rangle_0$ can be obtained by restricting the semi-inner product of $\mathcal{L}^2(\Omega, \mu)$ to $\mathcal{S}$. Clearly, the subspace $\mathcal{I}$ consists of those functions from $\mathcal{S}$, which are null μ-almost everywhere. As usually, let $L^2(\Omega, \mu)$ be the Hilbert space consisting of equivalence classes of square μ-integrable functions on Ω. Assume that the space $\mathcal{H}^0 = \mathcal{S}/\mathcal{I}$ is dense in $L^2(\Omega, \mu)$. Then $L^2(\Omega, \mu)$ is the Hilbert space associated to $(\mathcal{S}, \langle *, * \rangle_0)$. If χ_B denotes the characteristic function of a given set $B \in \mathfrak{S}$, the class of χ_B is clearly an idempotent in $L^2(\Omega, \mu)$ associated to $\mathcal{S}$, but χ_B does not necessarily belong to $\mathcal{S}$.

The following result is, in fact, an extension Lemma 4 from [30], with a proof in the present context.

Lemma 3. *Assume $\mathcal{H}$ to be separable, and let $\{\eta_j\}_{j \in J}$ be an orthonormal family in $\mathcal{RH}$ such that $\langle \eta_j, \hat{1} \rangle \neq 0$ for each $j \in J$, for some $J \subset \mathbb{N}$. Then the set $\{\iota_j\}_{j \in J}$ is an orthogonal family in $\mathcal{ID}(\mathcal{H})$, where $\iota_j = \langle \eta_j, \hat{1} \rangle \eta_j$ for all $j \in J$. If the set $\{\eta_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$, the set $\{\iota_j\}_{j \in \mathbb{N}}$ is an orthogonal basis of $\mathcal{H}$ in $\mathcal{ID}(\mathcal{H})$. Consequently,*

$$\phi = \sum_{j=1}^{\infty} \langle \iota_j, \hat{1} \rangle^{-1} \langle \phi, \iota_j \rangle \iota_j, \ \phi \in \mathcal{H},$$

where the series is convergent in $\mathcal{H}$. In particular $\hat{1} = \sum_{j=1}^{\infty} \iota_j$, where the series is convergent in $\mathcal{H}$.

Proof. Setting $\iota_j = \langle \eta_j, \hat{1} \rangle \eta_j$ for all $j \in J$, we have

$$\|\iota_j\|^2 = \langle \langle \eta_j, \hat{1} \rangle \eta_j, \langle \eta_j, 1 \rangle \eta_j \rangle = \langle \eta_j, \hat{1} \rangle^2 = \langle \iota_j, \hat{1} \rangle \neq 0,$$

showing that $\{\iota_j\}_{j \in J}$ is a family in $\mathcal{ID}(\mathcal{H})$. In addition,

$$\langle \iota_j, \iota_k \rangle = \langle \eta_j, \hat{1} \rangle \langle \eta_k, \hat{1} \rangle \langle \eta_j, \eta_k \rangle = 0, \ j, k \in J, \ j \neq k,$$

so the family $\{\eta_j\}_{j \in J}$ is orthogonal.

If $\{\eta_j\}_{j \in \mathbb{N}}$ is an orthonormal basis, we must have

$$\phi = \sum_{j=1}^{\infty} \langle \phi, \eta_j \rangle \eta_j = \sum_{j=1}^{\infty} \langle \iota_j, \hat{1} \rangle^{-1} \langle \phi, \iota_j \rangle \iota_j, \ f \in \mathcal{S},$$

and the series is clearly convergent in $\mathcal{H}$. In particular, we must have $\hat{1} = \sum_{j=1}^{\infty} \iota_j$, where the series is convergent in $\mathcal{H}$. □

Corollary 1. *If $\mathcal{H}$ is finite dimensional and $\{\eta_1, \ldots, \eta_d\} \subset \mathcal{RH}$ is an orthonormal basis with $\langle \eta_j, \hat{1} \rangle \neq 0$, $j = 1, \ldots, d$, the set $\{\langle \eta_1, \hat{1} \rangle \eta_1, \ldots \langle \eta_d, \hat{1} \rangle \eta_d\}$ is an orthogonal basis of $\mathcal{H}$ consisting of idempotents. Moreover,*

$$\langle \eta_1, \hat{1} \rangle \eta_1 + \cdots + \langle \eta_d, \hat{1} \rangle \eta_d = \hat{1}.$$

Corollary 2. *Assume that $\mathcal{H}$ is finite dimensional. Then there are functions $b_1, \ldots, b_d \in \mathcal{RS}$ such that $\|b_j\|_0^2 = \langle b_j, 1 \rangle_0 > 0$, $\langle b_j, b_k \rangle_0 = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, $\sum_{j=1}^{d} \langle b_j, 1 \rangle_0 - 1 \in \mathcal{I}$, and every $f \in \mathcal{S}$ can be uniquely represented as*

$$f = \sum_{j=1}^{d} \langle b_j, 1 \rangle_0^{-1} \langle f, b_j \rangle_0 b_j + f_0,$$

with $f_0 \in \mathcal{I}$ *and* $d = \dim\mathcal{H}$.

Using Lemma 3, we obtain the next result (which extends Theorem 1 from [30]).

Theorem 5. *Let* $(\mathcal{S}, \langle *, *\rangle_0)$ *be a qHfs, and let* $\mathcal{H}$ *be the associated Hilbert space. If* $\mathcal{H}$ *is separable and of dimension* ≥ 2*, it has infinitely many orthogonal bases consisting of idempotent elements.*

Proof. We may work in an abstract framework. Replacing $\mathcal{H}$ by $\mathcal{RH}$, we may assume, with no loss of generality, that $\mathcal{H}$ is a separable real Hilbert space, endowed with the real inner product $\langle *, *\rangle$, and with the corresponding norm $\| * \|$. Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be a sequence of positive numbers such $\sum_{j\in\mathbb{N}} \lambda_j^2 = 1$. Let also $\{\eta_j\}_{j\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$, and let $\eta = \sum_{j\in\mathbb{N}} \lambda_j\eta_j$, so $\|\eta\| = 1$. Then $\langle \eta, \eta_j\rangle = \lambda_j > 0$ for all j. Setting $\zeta_j := \langle \eta, \eta_j\rangle\eta_j$, $j \geq 1$, we obtain an orthogonal basis $\{\zeta_j\}_{j\in\mathbb{N}}$ such that $\|\zeta_j\|^2 = \langle \zeta_j, \eta\rangle$ for all $j \geq 1$.

Next, we fix an element $e \in \mathcal{H}$ such that $\|e\| = 1$, and choose an orthogonal transformation O of $\mathcal{H}$ such that $O\eta = e$. Putting, $e_j = O\eta_j$, and $\iota_j = \langle e, e_j\rangle e_j = O\zeta_j\,(j \geq 1)$, the family $\{\iota_j\}_{j\in\mathbb{N}}$ is an orthogonal basis of $\mathcal{H}$. Moreover, $\|\iota_j\|^2 = \langle \iota_j, e\rangle$ for all $j \geq 1$.

Going back to our initial case, for $e = \hat{1}$ we obtain an orthogonal basis $\{\iota_j\}_{j\in\mathbb{N}}$ of $\mathcal{H}$ consisting of idempotents. The construction from above shows that there are infinitely many possibilities to obtain such a basis $\{\iota_j\}_{j\in\mathbb{N}}$. ☐

Remark 8. Of course, the assertions from this subsection apply when the function space $\mathcal{S}$ is endowed with a usps $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, having the semi-inner product given by $\langle f, g\rangle_0 = \Lambda(f\bar{g})$, $f, g \in \mathcal{S}$. Note that if $\mathcal{S}$ is finite dimensional, an element $\hat{f} \in \mathcal{RH}_\Lambda$ is an idempotent if and only if $\Lambda(f^2) = \Lambda(f)$. For this reason, such an element may be called a Λ-*idempotent*, as in [30].

3. USPF's versus Semi-Inner Products

As mentioned above, the main aim of this work is to give necessary and sufficient conditions insuring the existence of integral representations of some given semi-inner products on function spaces of measurable functions; in other words, to look for solutions to Problem 2. One possible approach to Problem 2 is to adapt techniques from the theory of moment problems. We refer especially to [9,10] as well as to [29–31]. To this aim, it is necessary to clarify the connection between semi-inner products and unital square positive functionals, which is the main concern of this section.

Remark 9. Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs. We want to relate the semi-inner, product $\langle *, *\rangle_0$ with a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ in a natural and unique way. Roughly speaking, we want to have $\Lambda(f\bar{g}) = \langle f, g\rangle_0$, for all $f, g \in \mathcal{S}$. In fact, we have the following.

Lemma 4. *Let* $\mathcal{S}$ *be a function space on* Ω*. The following assertions are equivalent:*

(1) *the function space* $\mathcal{S}$ *has a compatible semi-inner product* $\langle *, *\rangle_0$ *satisfying*

$$\sum_{k\in K} f_k g_k = 0 \text{ on } \Omega \implies \sum_{k\in K} \langle f_k, g_k\rangle_0 = 0 \tag{3.1}$$

for all $f_k, g_k \in \mathcal{RS}$, $k \in K$, K *finite;*

(2) *the function space* $\mathcal{S}$ *has a uspf* $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$.

Proof. (1) $\implies$ (2). Choosing $f_k, g_k \in \mathcal{S}$, $k \in K$, K finite, we write $f_k = f_k' + if_k''$, $\bar{g}_k = g_k' - ig_k''$, with $f_k', f_k'', g_k', g_k'' \in \mathcal{RS}$, $k \in K$. Assuming $\sum_{k\in K} f_k\bar{g}_k = 0$ we must have $\sum_{k\in K}(f_k'g_k' + f_k''g_k'') = 0$ and $\sum_{k\in K}(f_k''g_k' - f_k'g_k'') = 0$ According to (3.1), we deduce that

$$\sum_{k\in K}(\langle f_k', g_k'\rangle_0 + \langle f_k'', g_k''\rangle_0) = 0 \quad \text{and} \quad \sum_{k\in K}(\langle f_k'', g_k'\rangle_0 - \langle f_k', g_k''\rangle_0) = 0.$$

Because the seminorm $\langle *, *\rangle_0$ is compatible, we infer that $\sum_{k\in K}\langle f_k, g_k\rangle_0 = 0$. Consequently, fixing an arbitrary element $F = \sum_{k\in k} f_k\bar{g}_k \in \mathcal{S}^{(2)}$, we put

$$\Lambda(F) = \sum_{k\in K}\langle f_k, g_k\rangle_0. \tag{3.2}$$

The previous argument shows that this definition does not depend on the particular representation of F, implying that the map $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ is linear. In addition, $\Lambda(1) = \langle 1, 1\rangle_0 = 1$,

$$\Lambda(\bar{F}) = \sum_{k\in K} \langle \bar{f}_k, \bar{g}_k\rangle_0 = \sum_{k\in K} \overline{\langle f_k, g_k\rangle_0} = \overline{\Lambda(F)},\ F = \sum_{k\in K} f_k\bar{g}_k \in \mathcal{S}^{(2)},$$

and $\Lambda(|f|^2) = \langle f, f\rangle_0 \geq 0$ for all $f \in \mathcal{S}$.

Conversely, given a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, the formula $\langle f, g\rangle_0 = \Lambda(f\bar{g})$, $f, g \in \mathcal{S}$ defines a semi-inner product compatible with the structure of $\mathcal{S}$, as noticed in Subsection 2.1. Therefore, we also have (2) $\Longrightarrow$ (1). □

Remark. Note that, in a qHfs $\mathcal{S}$ whose semi-inner product $\langle *, *\rangle_0$ has the property (3.1), taking $f, g \in \mathcal{S}$ and $h \in \mathcal{RS}$ such that $fh, gh \in \mathcal{S}$, we must have $\langle fh, g\rangle_0 = \langle f, gh\rangle_0$.

Proposition 1. *Let $\mathcal{S}$ be a function space on Ω. The following assertions are equivalent:*

(1) *the space $\mathcal{S}$ is a qfHs on Ω, whose compatible seminorm $\| * \|_0$ has the property*

$$\sum_{k\in K} f_k^2 = \sum_{l\in L} g_l^2 \text{ on } \Omega \implies \sum_{k\in K} \|f_k\|_0^2 = \sum_{l\in L} \|g_l\|_0^2, \tag{3.3}$$

for all $f_k, g_l \in \mathcal{RS}$, $k \in K$, $l \in L$, K, L finite;

(2) *the space $\mathcal{S}$ has a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$.*

Moreover, the uspf Λ is uniquely determined by the semi-inner product $\langle *, *\rangle_0$ with the property (3.3).

Proof. To prove the equivalence of the conditions (1) and (2) from the statement, it is enough to verify that condition (3.3) is equivalent to condition (3.1). To show that, we simply apply the obvious polarization formula:

$$\sum_{k\in K} u_k v_k = \frac{1}{4} \sum_{k\in K} [(u_k + v_k)^2 - (u_k - v_k)^2], \tag*{$(*)$,}$$

and its corresponding version

$$\sum_{k\in K} \langle u_k, v_k\rangle_0 = \frac{1}{4} \sum_{k\in K} \|u_k + v_k\|_0^2 - \|u_k - v_k\|_0^2], \tag*{$(**)$,}$$

valid for all $u_k, v_k \in \mathcal{RS}$, $k \in K$, K finite. We also note that in (3.3) we may always assume $K = L$, with no loss of generality.

The details of this verification, and the uniqueness of Λ as well, are left to the reader. □

Remark. Going back to Problem 2, Proposition 1 gives an answer to the question how to associate a qHfs with a uspf, in order to approach this problem as a moment problem.

Definition 5. Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs space. We say that the semi-inner product $\langle *, *\rangle_0$ is *expandable* if it has the property (3.3).

If $\langle *, *\rangle_0$ is expandable, the unique uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ given by Theorem 3.3 is said to be *associated to* $\langle *, *\rangle_0$.

Example 4. Let μ be a probability measure on $(\Omega, \mathfrak{S})$, and let $\mathcal{S} = \mathcal{L}^2(\Omega, \mu)$. Then $\mathcal{S}$ is a qHfs, whose natural semi-norm $\|f\|_0 = (\int_\Omega |f|^2 d\mu)^{1/2}$, $f \in \mathcal{S}$, is expandable. Moreover, it is easily seen that $\mathcal{S}^{(2)} = \mathcal{L}^1(\Omega, \mu)$, and the uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$, given by Theorem 1, is precisely $\Lambda(f) = \int_\Omega f d\mu$, $f \in \mathcal{L}^1(\Omega, \mu)$.

Example 5. The multidimensional Hamburger moment problem can be also related to Problem 2 in the following way. The basic function space is in this case $\mathcal{P}^n$ on $\mathbb{R}^n$, consistning of all polynomilas in $t = (t_1, \ldots, t_n)$. First of all, we fix a multi-sequence of real numbers $\gamma = (\gamma_\alpha)_{\alpha\in\mathbb{Z}_+^n}$ with the property

$$\sum_{\alpha,\beta} a_\alpha \bar{a}_\beta \gamma_{\alpha+\beta} \geq 0 \tag{3.4}$$

for all finite multi-sequences $(a_\alpha)_\alpha$ of complex numbers. In fact, we work in this example only with finite sums, and the multi-indices are from $\mathbb{Z}_+^n$, with the order $\alpha \leq \beta$ if $\beta - \alpha \in \mathbb{Z}_+^n$.

For any two polynomials $p(t) = \sum_\alpha c_\alpha t^\alpha$, $q(t) = \sum_\beta d_\beta t^\beta$ with complex coefficients we put

$$\langle p, q\rangle_0 = \sum_{\alpha,\beta} c_\alpha \bar{d}_\beta \gamma_{\alpha+\beta}. \tag{3.5}$$

The choice of the multi-sequence γ shows that the assignment (3.5) is a semi-inner product on $\mathcal{P}^n$. To illustrate Lemma 4, we shall verify directly that this semi-inner product is also expandable.

Let $\sum_j p_j\bar{q}_j = 0$, with $p_j(t) = \sum_\alpha c_{j,\alpha}t^\alpha$, $q_j(t) = \sum_\beta d_{j,\beta}t^\beta$. Then

$$\sum_j p_j\bar{q}_j = \sum_j \sum_{\alpha,\beta} c_{j,\alpha}\bar{d}_{j,\beta}t^{\alpha+\beta} = \sum_\sigma \left(\sum_{j,\alpha\le\sigma} c_{j,\alpha}\bar{d}_{j,\sigma-\alpha}\right) t^\sigma = 0.$$

Therefore, $\sum_{j,\alpha\le\sigma} c_{j,\alpha}\bar{d}_{j,\sigma-\alpha} = 0$ for all σ. On the other hand,

$$\sum_j \langle p_j, q_j\rangle_0 = \sum_j \sum_{\alpha,\beta} c_{j,\alpha}\bar{d}_{j,\beta}\gamma_{\alpha+\beta} = \sum_\sigma \left(\sum_{j,\alpha\le\sigma} c_{j,\alpha}\bar{d}_{j,\sigma-\alpha}\right) \gamma_\sigma = 0.$$

Although condition (3.4) for $n \ge 2$ does not necessarily impliy the existence of a representing measure for the semi-inner product (3.5), it suffices to show that (3.5) is expandable.

Example 6. For some details concerning this example we cite the work [3].

Let B be the open unit ball in $\mathbb{R}^n$, let S be the boundary of B, and let $h^2(B)$ be the real Hilbert space of all harmonic functions in B, which are Poisson transforms of real-valued functions from $\mathcal{L}^2(S,\sigma)$, where σ is the unique Borel probability measure on S that is rotation invariant. We denote by $\mathfrak{H}$ the space $h^2(B) + ih^2(B)$, which is a Hilbert space of complex-valued harmonic functions, and it is also a function space on B. The inner product of $\mathfrak{H}$ is given by

$$\langle f, g\rangle_{\mathfrak{H}} = \int_S f^\sharp\overline{g^\sharp}d\sigma,\ \ f, g \in \mathfrak{H},$$

where $f^\sharp$ denotes the unique element from $L^2(S,\sigma)$ whose Poisson transform is f. This inner product is clearly compatible with the function space $\mathfrak{H}$.

Assuming now that $\sum_{j\in J} f_j^2 = \sum_{k\in K} g_k^2$ for some $f_j, g_k \in \mathcal{R}\mathfrak{H}$, $j \in J, k \in K$, J, K finite, we infer that $\sum_{j\in J}(f_j^\sharp)^2 = \sum_{k\in K}(g_k^\sharp)^2$, so

$$\sum_{j\in J} \|f_j\|_{\mathfrak{H}}^2 = \int_S \sum_{j\in J}(f_j^\sharp)^2 d\sigma = \int_S \sum_{k\in K}(g_k^\sharp)^2 d\sigma = \sum_{k\in K}\|g_k\|_{\mathfrak{H}}^2.$$

In other words, the norm of $\mathfrak{H}$ is expandable, allowing us to define a uspf on $\mathfrak{H}^{(2)}$, in a natural way.

Example 7. Let $\mathcal{W}_1^2$ be the Sobolev space consisting of all complex-valued functions on the interval $[0,1]$, which are absolutely continuous and whose derivatives are square integrable, endowed with the norm

$$\|f\|_1^2 = \int_0^1 (|f(t)|^2 + |f'(t)|^2)dt,\ \ f \in \mathcal{W}_1^2.$$

This is a reproducing kernel Hilbert space, as shown in [1], Example 2.7. It is also a function space on $[0,1]$ and its inner product is compatible. Nevertheless, the norm is not expandable. Indeed, taking $f(t) = (2+2t^2)^{1/2}$, $f_1(t) = 1+t$, $f_2(t) = 1-t$, we have $f^2 = f_1^2 + f_2^2$, while $\|f\|_1^2 \ne \|f_1\|_1^2 + \|f_2\|_1^2$. To see this, it is enough to remark that $f'(t)^2 = 2t^2(1+t^2)^{-1}$, and so $\|f\|_1^2$ is an expression depending explicitly on π, while $\|f_1\|_1^2 + \|f_2\|_1^2$ is a rational number. Consequently, the norm of $\mathcal{W}_1^2$ is not expandable. In particular, Problem 2 has no solution for the Hfs $(\mathcal{W}_1^2, \|*\|_1)$.

Example 8. We now give a more abstract example.

Let $\mathcal{S}$ be a Hfs on Ω endowed with a norm $\|*\|$, given by a compatible inner product $\langle *, *\rangle$. Assume that there exists a family of functions $\{v_1,\dots,v_d\}$ in $\mathcal{R}\mathcal{S}$, and a family of points $\{\omega_1,\dots,\omega_d\}$ in Ω such that

$$\langle v_j, 1\rangle \ne 0,\ \ f(\omega_j) = \langle v_j, 1\rangle^{-1}\langle f, v_j\rangle,\ \text{ and }\ f = \sum_{j=1}^d f(\omega_j)v_j,\ \forall f \in \mathcal{S}.$$

If, moreover,

$$\langle v_k, v_l\rangle = \sum_{j=1}^d \langle v_j, 1\rangle^{-1}\langle v_k, v_j\rangle\langle v_l, v_j\rangle,\ \ k, l = 1,\dots,d,$$

the norm $\| * \|$ is expandable.

We can prove this assertion either directly or applying Theorem 7. In fact, Theorem 7 shows that, under conditions slightly larger than those from above, there exists a measure for the inner product of $\mathcal{S}$.

4. An Interpolation Approach

The existence of an atomic representing measure for a semi-inner product may be characterized in terms of an interpolation property. The next result is an extension of Proposition 3 in [31]. As in Theorem 1, this is an extreme situation in the sense that the number of nodes, usually larger than the dimension of the associated Hilbert space, is assumed to be equal to that dimension (see also [14], Theorem 1.3).

Theorem 6. *Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs on Ω, and let $\mathcal{H}$ be the Hilbert space associated to $(\mathcal{S}, \langle *, * \rangle_0)$, supposed to be finite dimensional. The semi-inner product $\langle *, * \rangle_0$ has a d-atomic representing measure in Ω with $d := \dim \mathcal{H}$ atoms if and only if there exist an orthogonal basis of $\mathcal{H}$ consisting of idempotents $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$, and a set $\mathfrak{Z} = \{\zeta_1, \ldots, \zeta_d\} \subset \mathcal{Z}$ such that $b_j(\zeta_j) = 1$ and $b_k(\zeta_j) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$. In addition, the set $\mathfrak{Z}$ is the support of the corresponding representing measure.*

Proof. The first part of the proof shares some arguments with that of Theorem 1.

To begin with, assume that the semi-inner product $\langle *, * \rangle_0$ has a representing measure in Ω, given by

$$\langle f, g \rangle_0 = \sum_{j=1}^{d} \lambda_j f(\zeta_j) \overline{g(\zeta_j)}, \ \ f, g \in \mathcal{S},$$

with $\lambda_j > 0$ for all $j = 1, \ldots, d$, and $\sum_{j=1}^{d} \lambda_j = 1$, where $d = \dim \mathcal{H}$, and the points $\zeta_1, \ldots, \zeta_d$ are distinct. Set $\mathfrak{Z} = \{\zeta_1, \ldots, \zeta_d\}$, which is a subset of $\mathcal{Z}$, via Lemma 2.

According to (2.2), we must have $\mathcal{I} = \{f \in \mathcal{S}; f|\mathfrak{Z}\}$. This shows that there exists a map $\rho : \mathcal{H} \mapsto C(\mathfrak{Z})$ given by $\hat{f} \mapsto f|\mathfrak{Z}$, which is correctly defined, linear and injective. This map is also surjective because we have $\dim(\mathcal{H}) = \dim(C(\mathfrak{Z}))$.

Let $\chi_k \in C(\mathfrak{Z})$ be the characteristic function of the set $\{\zeta_k\}$ and let $\hat{b}_k \in \mathcal{H}$ be the element with $\rho(\hat{b}_k) = \chi_k$, $k = 1, \ldots, d$. Note that the element b_k, representing the equivalence class $\hat{b}_k$, may be chosen in $\mathcal{RS}$, because $b_k|\mathfrak{Z} = \chi_k$ has real values, and we may replace, if necessary, the function b_k by its real part, for each $k = 1, \ldots, d$.

As $\langle b_j, b_k \rangle_0 = \sum_{l=1}^{d} \lambda_l (\chi_j \chi_k)(\zeta_l)$, and so $\langle b_j, b_k \rangle_0 = 0$, $\langle b_j, b_j \rangle_0 = \lambda_j = \langle b_j, 1 \rangle_0$ for all $j, k = 1, \ldots, d$, $j \neq k$, we deduce that the set $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is a family of orthogonal idempotents in $\mathcal{H} = \mathcal{H}^0$, which is actually a basis. Clearly, $b_j(\zeta_j) = 1$ and $b_k(\zeta_j) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, proving the necessity of the condition in the statement.

Conversely, if there exist an orthogonal basis of $\mathcal{H} = \mathcal{H}^0$ consisting of idempotents $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$, and a set $\mathfrak{Z} = \{\zeta_1, \ldots, \zeta_d\} \subset \mathcal{Z}$ such that $b_j(\zeta_j) = 1$ and $b_k(\zeta_j) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$, then $\langle *, * \rangle_0$ has a representing measure whose support is $\mathfrak{Z}$. Indeed, it follows from Corollary 2 that for every $f \in \mathcal{S}$ we have

$$f = \sum_{j=1}^{d} \langle b_j, 1 \rangle_0^{-1} \langle f, b_j \rangle_0 b_j + f_0,$$

with $f_0 \in \mathcal{I}$. Hence

$$f(\zeta_k) = \sum_{j=1}^{d} \langle b_j, 1 \rangle_0^{-1} \langle f, b_j \rangle_0 b_j(\zeta_k) = \langle b_k, 1 \rangle_0^{-1} \langle f, b_k \rangle_0$$

because $f_0(\zeta_k) = 0$, for all $k = 1, \ldots, d$. Taking another function $g \in \mathcal{S}$ and using the relations from above, we infer that

$$\langle f, g \rangle_0 = \sum_{j,k=1}^{d} \langle b_j, 1 \rangle_0^{-1} \langle b_k, 1 \rangle_0^{-1} \langle f, b_j \rangle_0 \langle \bar{g}, b_k \rangle_0 \langle b_j, b_k \rangle_0 =$$

$$\sum_{j=1}^{d}\langle b_j,1\rangle_0^{-1}\langle f,b_j\rangle_0\langle \bar g,b_j\rangle_0=\sum_{j=1}^{d}\langle b_j,1\rangle_0 f(\zeta_j)\overline{g(\zeta_j)}.$$

Because $\langle b_j,1\rangle_0>0$ for all $j=1,\dots,d$ and $\sum_{j=1}^d\langle b_j,1\rangle_0=1$, we have obtained the existence of a representing measure of $\langle *,*\rangle_0$ in Ω having d atoms, whose support is the set $\mathfrak{Z}$. $\square$

Proposition 3 from [31] is now a consequence of Theorem 6:

Corollary 3. *Let $\mathcal{S}$ be a finite dimensional function space on Ω. A uspf $\Lambda:\mathcal{S}^{(2)}\mapsto\mathbb{C}$ has a representing measure in Ω with $d:=\dim\mathcal{H}_\Lambda$ atoms if and only if there exist an orthogonal basis of $\mathcal{H}_\Lambda$ consisting of idempotents $\mathcal{B}=\{\hat b_1,\dots,\hat b_d\}$, and a set $\Omega_\Lambda=\{\omega_1,\dots,\omega_d\}\subset\mathcal{Z}_\Lambda$ such that $b_j(\omega_j)=1$ and $b_k(\omega_j)=0$ for all $j,k=1,\dots,d$, $j\neq k$.*

Proposition 2. *Let $(\mathcal{S},\langle *,*\rangle_0)$ be a qHfs on Ω, and let $\mathcal{H}$ be the Hilbert space associated to $(\mathcal{S},\langle *,*\rangle_0)$, supposed to be finite dimensional. Then there exists at most one d-atomic representing measure of the semi-inner product $\langle *,*\rangle_0$ with support in Ω, having $d:=\dim\mathcal{H}$ atoms.*

Proof. If the semi-inner product $\langle *,*\rangle_0$ has a d-atomic representing measure in Ω with $d:=\dim\mathcal{H}$ atoms, say μ, it follows from the proof of Theorem 6 that the map $\mathcal{H}\ni\hat f\mapsto f|\mathfrak{Z}\in L^2(\mathfrak{Z},\mu)$ is a unitary operator. Indeed, we have only to note that $L^2(\mathfrak{Z},\mu)$ can be identified with $C(\mathfrak{Z})$, so the map $\hat f\mapsto f|\mathfrak{Z}$ is bijective, and $\|\hat f\|^2=\int_{\mathfrak{Z}}|f|^2d\mu$ for all $\hat f$.

Now assume that there exists another d-atomic representing measure of $\langle *,*\rangle_0$ in Ω, with support $\Xi:=\{\xi_1,\dots,\xi_d\}$. As in the previus case, the map $\hat f\mapsto f|\Xi$ induces a unitary operator from $\mathcal{H}$ onto $L^2(\Xi,\nu)$.

We now extend μ (resp. ν) to Ω by setting $\mu(\Omega\setminus\mathfrak{Z})=0$ (resp. $\nu(\Omega\setminus\Xi)=0$). If $\zeta\in\mathfrak{Z}\setminus\Xi$, for the characteristic function χ of the set $\{\zeta\}$ (defined on Ω) we must have

$$0\neq\int_\Omega\chi d\mu=\langle\chi,1\rangle_0=\int_\Omega\chi d\nu=0,$$

which is impossible, so $\mathfrak{Z}\subset\Xi$. A similar argument shows that $\Xi\subset\mathfrak{Z}$. Therefore, $\mathfrak{Z}=\Xi$. In fact, this argument shows that the weights of both measures at a given point must be the same. $\square$

A more general form of Theorem 6 is given by the following. Unlike in Theorem 6, the number of nodes may be greater than the dimension of the associated Hilbert space (see also Theorem 5 from [30]).

Theorem 7. *Let $(\mathcal{S},\langle *,*\rangle_0)$ be a qHfs on Ω, and let $\mathcal{H}$ be the associated Hilbert space, supposed to be finite dimensional. The semi-inner product $\langle *,*\rangle_0$ has a d-atomic representing measure in Ω for some integer $d\geq 1$ if and only if $d\geq\dim\mathcal{H}$, and there exist a family of functions $\{v_1,\dots,v_d\}$ in $\mathcal{RS}$, and a family of points $\{\zeta_1,\dots,\zeta_d\}$ in Ω, such that*

$$\langle v_j,1\rangle_0\neq 0,\ f(\zeta_j)=\langle v_j,1\rangle_0^{-1}\langle f,v_j\rangle_0,\ f-\sum_{j=1}^{d}f(\zeta_j)v_j\in\mathcal{I},\ \forall f\in\mathcal{S},\tag{4.1}$$

and

$$\langle v_k,v_l\rangle_0=\sum_{j=1}^{d}\langle v_j,1\rangle_0^{-1}\langle v_k,v_j\rangle_0\langle v_l,v_j\rangle_0,\ k,l=1,\dots,d.\tag{4.2}$$

Proof. We assume first that the semi-inner product $\langle *,*\rangle_0$ has a d-atomic representing measure in Ω, say μ, and so we may proceed as in the first part of the proof of Theorem 6. In other words, there exist a set $\mathfrak{Z}:=\{\zeta_1,\dots,\zeta_d\}\subset\mathcal{Z}$ so that

$$\langle f,g\rangle_0=\sum_{j=1}^{d}\lambda_j f(\zeta_j)\overline{g(\zeta_j)},\ f,g\in\mathcal{S},$$

with $\lambda_j>0$ for all $j=1,\dots,d$, and $\sum_{j=1}^d\lambda_j=1$ for some integer $d\geq 1$. In fact, $\mu(\{\zeta_j\})=\lambda_j$, $j=1,\dots,d$. Moreover, $\mathcal{I}=\{f\in\mathcal{S};f|\mathfrak{Z}=0\}$. This shows that there exists a map $\rho:\mathcal{H}\mapsto L^2(\mathfrak{Z},\mu)$ given by $\hat f\mapsto f|\mathfrak{Z}$, which is a linear isometry. The image $\mathcal{H}_0:=\rho(\mathcal{H})\subset L^2(\mathfrak{Z},\mu)$ is a Hilbert subspace, and the map $\rho:\mathcal{H}\mapsto\mathcal{H}_0$ is a unitary operator.

Let $\chi_k \in L^2(\mathfrak{Z}, \mu)$ be the characteristic function of the set $\{\zeta_k\}$, $k = 1, \ldots, d$. Clearly, the family $\{\chi_1, \ldots, \chi_d\}$ is an orthogonal basis of the space $L^2(\mathfrak{Z}, \mu)$, and hence

$$d = \dim(L^2(\mathfrak{Z}, \mu)) \geq \dim(\mathcal{H}_0) = \dim(\mathcal{H}).$$

Let P_0 be the orthogonal projection of $L^2(\mathfrak{Z}, \mu)$ onto $\mathcal{H}_0$, and let $v_k|\mathfrak{Z} := P_0\chi_k$, $k = 1, \ldots, d$, with $v_k \in \mathcal{S}$ fixed. As for each $f = \bar{f} \in \mathcal{S}$ the number $\langle f, v_j\rangle_0 = \lambda_j f(\zeta_j)$ is real, the function v_j may be assumed to be real-valued for all $j = 1, \ldots, d$.

From the equality $\phi = \sum_{j=1}^d \phi(\zeta_j)\chi_j$, valid for all $\phi \in L^2(\mathfrak{Z}, \mu)$, we deduce that $f|\mathfrak{Z} = \sum_{j=1}^d f(\zeta_j)v_k|\mathfrak{Z}$, for all $f \in \mathcal{S}$. Therefore, $f - \sum_{j=1}^d f(\zeta_j)v_j \in \mathcal{I}$ for all $f \in \mathcal{S}$. Note also that

$$\langle v_j, 1\rangle_0 = \langle \chi_j, 1\rangle = \lambda_j > 0, \ j = 1, \ldots, d,$$

and

$$f(\zeta_j) = \lambda_j^{-1}\langle f|\mathfrak{Z}, \chi_j\rangle = \langle v_j, 1\rangle_0^{-1}\langle f, v_j\rangle_0, \ j = 1, \ldots, d,$$

and so equation(4.1) holds. In particular, we have the equality $(v_k v_l)(\zeta_j) = \lambda_j^{-2}\langle v_k, v_j\rangle_0\langle v_l, v_j\rangle_0$, whence we infer that

$$\langle v_k, v_l\rangle_0 = \sum_{j=1}^d \langle v_j, 1\rangle^{-1}\langle v_k, v_j\rangle_0\langle v_l, v_j\rangle_0, \ k, l = 1, \ldots, d,$$

which is precisely equation (4.2).

Conversely, assuming that there exists a family of functions $\{v_1, \ldots, v_d\}$ in $\mathcal{RS}$, and a family of points $\{\zeta_1, \ldots, \zeta_d\}$ in Ω, such that equations (4.1) and (4.2) hold, we can write

$$\langle f, g\rangle_0 = \langle \sum_{j=1}^d f(\zeta_j)v_j, \sum_{k=1}^d g(\zeta_k)v_k\rangle_0 =$$

$$\sum_{j=1}^d\sum_{k=1}^d f(\zeta_j)\overline{g(\zeta_k)}\langle v_j, v_k\rangle_0 = \sum_{j=1}^d\sum_{k=1}^d f(\zeta_j)\overline{g(\zeta_k)}\sum_{l=1}^d \langle v_l, 1\rangle_0^{-1}\langle v_j, v_l\rangle_0\langle v_k, v_l\rangle_0 =$$

$$\sum_{l=1}^d \langle v_l, 1\rangle_0^{-1}\sum_{j=1}^d f(\zeta_j)\langle v_j, v_l\rangle_0\sum_{k=1}^d \overline{g(\zeta_k)}\langle v_k, v_l\rangle_0 = \sum_{l=1}^d \langle v_l, 1\rangle_0^{-1}\langle f, v_l\rangle_0\langle \bar{g}, v_l\rangle_0 =$$

$$\sum_{l=1}^d \langle v_l, 1\rangle_0 f(\zeta_l)\overline{g(\zeta_l)}, \ f, g \in \mathcal{S},$$

showing that the inner product of $\mathcal{S}$ has a representing measure. $\square$

Corollary 4. *Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs on Ω, and let $\mathcal{H}$ be the associated Hilbert space, supposed to be finite dimensional. If the semi-inner product $\langle *, *\rangle_0$ has an atomic representing measure μ in Ω with support $\mathfrak{Z}$, then* $\operatorname{card}(\mathfrak{Z}) \geq \dim(\mathcal{H})$*, and the map $\mathcal{H} \ni \hat{f} \mapsto f|\mathfrak{Z} \in L^2(\Xi, \mu)$ is a linear isomatry.*

Remark 10. Let $(\mathcal{S}, \langle *, *\rangle)$ be a finite dimensional Hfs on Ω. Then every point evaluation is automatically continuous. Consequently, the space $\mathcal{S}$ has a reproducing kernel denoted by $K(*, *)$, that is, $f(\omega) = \langle f, K_\omega\rangle$ for all $f \in \mathcal{S}$ and $\omega \in \Omega$, where $K_\omega(*) = K(*, \omega)$ (see [1,2,18] etc. for details). In the present framework, the function $K(*, *)$ must be real valued and therefore symmetric.

The next result is an application of Theorem 6.

Proposition 3. *Let $(\mathcal{S}, \langle *, *\rangle)$ be a finite dimensional Hilbert function space on Ω, and let $K(*, *)$ be its kernel. The inner product $\langle *, *\rangle$ has a d-atomic representing measure, with $d = \dim \mathcal{S}$, if and only if there are d distinct points $\zeta_1, \ldots, \zeta_d$ in Ω such that $K(\zeta_j, \zeta_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$.*

Proof. Assume first that there are d distinct points $\zeta_1, \ldots, \zeta_d$ in Ω such that $K(\zeta_j, \zeta_k) = 0$ for all $j, k = 1, \ldots, d$, $j \neq k$. Set $e_j(\omega) = K(\zeta_j, \zeta_j)^{-1/2}K(\zeta_j, \omega)$, $j = 1, \ldots, d$, $\omega \in \Omega$. Since we have

$$\langle e_j, e_k\rangle = K(\zeta_j, \zeta_j)^{-1/2}K(\zeta_k, \zeta_k)^{-1/2}\langle K(\zeta_j, *), K(\zeta_k, *)\rangle =$$

$$K(\zeta_j, \zeta_j)^{-1/2}K(\zeta_k, \zeta_k)^{-1/2}K(\zeta_j, \zeta_k) = 0$$

if $j \neq k$, and

$$\langle e_j, e_j\rangle = K(\zeta_j,\zeta_j)^{-1}\langle K(\zeta_j,*), K(\zeta_j,*)\rangle = 1,$$

the family $\{e_1,\dots,e_d\}$ is an orthonormal basis of $\mathcal{S}$. Moreover, $\langle e_j, 1\rangle = K(\zeta_j,\zeta_j)^{-1/2} > 0$ for all j, and so, setting $b_j = K(\zeta_j,\zeta_j)^{-1/2}e_j$, we obtain a family $\{b_1,\dots,b_d\}$, which is an orthonormal basis of $\mathcal{S}$ consisting of idempotents. Clearly, $b_j(\zeta_j) = 1$ and $b_j(\zeta_k) = 0$ if $j \neq k$. Using Theorem 6, we infer the existence of an d-atomic representing measure for $\langle *, *\rangle$.

Conversely, if the inner product $\langle *, *\rangle$ has a d-atomic representing measure, that is, $\langle f, g\rangle = \sum_{j=1}^d \lambda_j f(\zeta_j)\overline{g(\zeta_j)},\ f, g \in \mathcal{S}$, for some distinct points $\zeta_1,\dots,\zeta_d$ in Ω, with $\lambda_j > 0$ for all $j = 1,\dots,d$, $\sum_{j=1}^d \lambda_j = 1$, as in the first part of the proof of Theorem 6 we find a basis $\{b_1,\dots,b_d\}$ of $\mathcal{S}$ consisting of orthogonal idempotents. Therefore, $f(\omega) = \sum_{j=1}^d \langle b_j, 1\rangle^{-1}\langle f, b_j\rangle b_j(\omega)$ for all $f \in \mathcal{S}$ and $\omega \in \Omega$. Moreover, $\lambda_j = \langle b_j, 1\rangle$, $b_j(\zeta_j) = 1$ and $b_k(\zeta_j) = 0$ for all $j,k = 1,\dots,d$, $j \neq k$. Setting, $K(\zeta,\omega) = \sum_{j=1}^d \lambda_j^{-1} b_j(\zeta)b_j(\omega)$, $\zeta,\omega \in \Omega$, we deduce that

$$\langle f, K_\omega\rangle = \sum_{k=1}^d \lambda_k f(\zeta_k)\sum_{j=1}^d \lambda_j^{-1} b_j(\zeta_k)b_j(\omega) =$$

$$\sum_{j=1}^d \lambda_j^{-1}\left(\sum_{k=1}^d \lambda_k f(\zeta_k)b_j(\zeta_k)\right)b_j(\omega) = f(\omega),\ f \in \mathcal{S},\ \omega \in \Omega,$$

because $\sum_{k=1}^d \lambda_k f(\zeta_k)b_j(\zeta_k) = \langle f, b_j\rangle$, showing that $K(*,*)$ is the kernel of $\mathcal{S}$. In addition, we clearly have $K(\zeta_j,\zeta_k) = 0$ for all $j,k = 1,\dots,d$, $j \neq k$. □

5. Relative Multiplicativity

As done in [30] and [31] for uspf's, we can also characterize the existence of a representing measure of a semi-inner product in terms of idempotents. The following is a basic concept, which generalizes a corresponding one from [30], Definition 3.

Definition 6. Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs m-generaterd by the n-tuple θ. Let also $\mathcal{B} = \{\hat b_1,\dots,\hat b_d\}$ be an orthogonal basis consisting of idempotent elements of the associated Hilbert space. We say that the basis $\mathcal{B}$ is *multiplicative* (*with respect to* θ) if

$$\langle \theta^\alpha, b_j\rangle_0\langle \theta^\beta, b_j\rangle_0 = \langle b_j, 1\rangle_0\langle \theta^{\alpha+\beta}, b_j\rangle_0 \tag{5.1}$$

whenever $|\alpha| + |\beta| \le m$, $j = 1,\dots,d$.

The next result is an extension of Theorem 3 from [31], which in turn is an extension of Theorem 2 from [30]. In addition, the present proof is simpler and more transparent.

Theorem 8. *Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs on Ω, m-generaterd by the n-tuple θ. Assume that the associated Hilbert space $\mathcal{H}$ is finite dimensional.*

The inner product of $\mathcal{H}$ has a representing measure on Ω consisting of $d := \dim\mathcal{H}$ atoms if and only if there exists an orthogonal basis $\mathcal{B} = \{\hat b_1,\dots,\hat b_d\}$ of $\mathcal{H}$, consisting of idempotent elements, which is multiplicative with respect to θ, and $\delta(\hat\theta) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$.

Proof. Let $\mathcal{B} = \{\hat b_1,\dots,\hat b_d\}$ be an orthogonal basis of $\mathcal{H}$ consisting of idempotent elements. Every element $\hat f \in \mathcal{H}$ has a unique representation of the form $\hat f = \sum_{j=1}^d \langle \hat b_j, \hat 1\rangle^{-1}\langle \hat f, \hat b_j\rangle \hat b_j$, via Lemma 3.

We consider on $\mathcal{H}$ the linear functionals $\delta_j(\hat f) = \langle \hat b_j, \hat 1\rangle^{-1}\langle \hat f, \hat b_j\rangle$, $j = 1,\dots,d$, so $\hat f = \sum_{j=1}^d \delta_j(\hat f)\hat b_j$ for all $\hat f \in \mathcal{H}$. In particular, $\delta_j(\hat b_j) = 1$ and $\delta_j(\hat b_k) = 0$ for all $j,k = 1,\dots,d$, $j \neq k$. In other words, the set $\Delta := \{\delta_1,\dots,\delta_d\}$ is the dual basis of $\mathcal{B}$.

Next, we define the functions $\hat f_\Delta : \Delta \mapsto \mathbb{C}$ by $\hat f_\Delta(\delta) = \delta(\hat f)$ for all $\hat f \in \mathcal{H}$ and $\delta \in \Delta$. Setting $\mathcal{H}_\Delta := \{\hat f_\Delta; \hat f \in \mathcal{H}\}$, we have a linear map $\mathcal{H} \ni \hat f \mapsto \hat f_\Delta \in \mathcal{H}_\Delta$, which is surjective by definition, and injective because $\hat f_\Delta = 0$ implies $\hat f = 0$. In other words, the map $\mathcal{H} \ni \hat f \mapsto \hat f_\Delta \in \mathcal{H}_\Delta$ is a linear isomorphism. In addition, $\hat f_\Delta = \sum_{k=1}^d \hat f_\Delta(\delta_j)\widehat{b_k}_\Delta$ for all $\hat f \in \mathcal{H}$.

As a matter of fact, the function $\widehat{b_k}_\Delta$ is the characteristic function of the set $\{\delta_k\}$, $k = 1,\dots,d$. This shows that $\mathcal{H}_\Delta = C(\Delta)$ and the spaces $\mathcal{H}$, $\mathcal{H}_\Delta = C(\Delta)$, have the same dimension. (Here, as before,

$C(\Delta) := \{\phi : \Delta \mapsto \mathbb{C}\}$, is regarded as a finite dimensional C^*-algebra.) In fact, $\mathcal{H}_\Delta$ and $C(\Delta)$ are isomorphic as C^*-algebras. Indeed, the product of two functions from $\mathcal{H}_\Delta$, say $\hat{f}_\Delta = \sum_{j=1}^d \delta_j(\hat{f})\widehat{b_j}_\Delta$, $\hat{g}_\Delta = \sum_{j=1}^d \delta_j(\hat{g})\widehat{b_j}_\Delta$, is given by

$$\hat{f}_\Delta \hat{g}_\Delta = \sum_{j=1}^d \delta_j(\hat{f})c_j(\hat{g})\widehat{b_j}_\Delta,$$

which coincides with the product of $C(\Delta)$. In particular, $\hat{f}_\Delta \hat{g}_\Delta$ is an element of $\mathcal{H}_\Delta$, and the C^*-algebra structure of $C(\Delta)$ is inherited by $\mathcal{H}_\Delta$.

We now assume that $\mathcal{B}$ is multiplicative with respect to θ, and that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$.

We note that the space $\mathcal{H}$ is spanned by the family $\{\widehat{\theta^\alpha}; |\alpha| \le m\}$, by hypothesis, so the vector space $\mathcal{H}_\Delta$ is spanned by the family $\{\widehat{\theta^\alpha}_\Delta; |\alpha| \le m\}$, while the C^*-algebra $\mathcal{H}_\Delta$ is generated by the family $\{\widehat{\theta_1}_\Delta \dots \widehat{\theta_n}_\Delta\}$. We need a more explicit relation between these families, obtained by using (5.2), which will be proved in the following. Because we have

$$\langle \theta^\alpha, b_j\rangle_0 \langle \theta^\beta, b_j\rangle_0 = \langle \hat{b}_j, \hat{1}\rangle^2 \delta_j(\widehat{\theta^\alpha})\delta_j(\widehat{\theta^\beta}) =$$
$$\langle b_j, 1\rangle_0 \langle \theta^{\alpha+\beta}, b_j\rangle_0 = \langle \hat{b}_j, \hat{1}\rangle^2 \delta_j(\widehat{\theta^{\alpha+\beta}})$$

whenever $|\alpha| + |\beta| \le m$ and $j = 1, \dots, d$, via (5.1), we infer that

$$\widehat{\theta^\alpha}_\Delta \widehat{\theta^\beta}_\Delta = \widehat{\theta^{\alpha+\beta}}_\Delta,$$

whenever $|\alpha| + |\beta| \le m$. Hence, by recurrence, we deduce that

$$\widehat{\theta^\alpha}_\Delta = (\hat{\theta}_\Delta)^\alpha \quad \text{if} \quad |\alpha| \le m. \tag{5.2}$$

The hypothesis $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, allows us to find a point $\zeta_j \in \Omega$ such that $\theta(\zeta_j) = \delta_j(\hat{\theta})$ for each $j = 1, \dots, d$.

Let $f \in \mathcal{S}$ be a fixed element. As $\mathcal{S}$ is m-generated by θ, there exists a polynomial $p \in \mathcal{P}_m^n$ such that $f = p \circ \theta$. Then we have $\hat{f} = p \circ \hat{\theta}$, and so $\hat{f}_\Delta = p \circ \hat{\theta}_\Delta$, via (5.2). Hence, we must have

$$\delta_j(\hat{f}) = \hat{f}_\Delta(\delta_j) = p(\hat{\theta}_\Delta(\delta_j)) = (p \circ \theta)(\zeta_j) = f(\zeta_j),\ j = 1, \dots, d,$$

This equality leads to

$$\langle f, g\rangle_0 = \langle \hat{f}, \hat{g}\rangle = \sum_{j=1}^d \langle b_j, 1\rangle f(\zeta_j)\overline{g(\zeta_j)},\ f, g \in \mathcal{S}, \tag{5.3}$$

where $\langle b_j, 1\rangle > 0$ for all j and $\sum_{j=1}^d \langle b_j, 1\rangle = 1$, via Lemma 3. Consequently, the inner product of has a representing measure on Ω.

Conversely, assume that there exists a finite family $\{\zeta_1, \dots, \zeta_d\} \subset \Omega$, consisting of distinct points, such that

$$\langle f, g\rangle_0 = \sum_{j=1}^d \lambda_j f(\zeta_j)\overline{g(\zeta_j)},\ f, g \in \mathcal{S},$$

where $\lambda_j > 0$ for all $j, \sum_{j=1}^d \lambda_j = 1$, and $d = \dim \mathcal{H}$.

We proceed as in the proof of Theorem 6. Set $\mathfrak{Z} = \{\zeta_1, \dots, \zeta_d\}$, which is a subset of $\mathcal{Z}$, via Lemma 2. As we must have $\mathcal{I} = \{f \in \mathcal{S}; f|\mathfrak{Z} = 0\}$, there exists a map $\rho : \mathcal{H} \mapsto C(\mathfrak{Z})$ given by $\hat{f} \mapsto f|\mathfrak{Z}$, which is correctly defined, linear and bijective. We denote by $\chi_k \in C(\mathfrak{Z})$ the characteristic function of the set $\{\zeta_k\}$, and by $\hat{b}_k \in \mathcal{RH}$ the element with $\rho(\hat{b}_k) = \chi_k$, $k = 1, \dots, d$. Then the set $\mathcal{B} := \{\hat{b}_1, \dots, \hat{b}_d\}$ is a family of orthogonal idempotents in $\mathcal{H}$, which is actually a basis. Moreover, $b_j(\zeta_j) = 1$ and $b_k(\zeta_j) = 0$ for all $j, k = 1, \dots, d$, $j \ne k$.

Setting $\delta_j(\hat{f}) = f(\zeta_j)$, $f \in \mathcal{S}$, $j = 1, \dots, d$, and $\Delta := \{\delta_1, \dots, \delta_d\}$, we infer that Δ is the dual basis of $\mathcal{B}$, and we have

$$\delta_j(\widehat{\theta^\alpha}) = \theta^\alpha(\zeta_j) = (\theta_1(\zeta_j)^{\alpha_1} \cdots \theta_n(\zeta_j)^{\alpha_n}) = \delta_j(\hat{\theta}^\alpha),$$

whenever $|\alpha| \le m$ and $j = 1, \dots, d$, showing that $\mathcal{B}$ is a multiplicative basis (with respect to θ), as in (5.2). In addition, the obvious equality $\delta_j(\hat{\theta}) = \theta(\zeta_j)$, $j = 1, \dots, d$, concludes the proof of Theorem 8. □

Corollary 5. *Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs on Ω, 1-generated by the n-tuple θ. The semi-inner product $\langle *, * \rangle_0$ has a representing measure on Ω consisting of $d := \dim\mathcal{H}$ atoms if either*

(1) *there exists an orthogonal basis $\mathcal{B}$ of $\mathcal{H}$ consisting of idempotent elements such that $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$, or*

(2) $\theta(\Omega) = \mathbb{R}^n$.

Proof. Because $(\mathcal{S}, \langle *, * \rangle_0)$ is 1-generated, property (5.1) is automatically fulfilled. To get the assertion (1) from the statement we need the inclusion $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$, in order to apply the previous theorem, while to get (2), such an inclusion is always true, for an arbitrary orthogonal basis consisting of idempotents. □

6. Dimensional Stability and Consequences

In this section we intend to extend and recapture, in the present context, some results regarding the dimensional stability, developed in [29]. We also recall that the concept of dimensional stability in function spaces of polynomials, as approached in [29], is equivalet to that of *flatness*, due to Curto and Fialkow (see [9, 10]).

Remark 11. Let $\mathcal{S} = \mathcal{S}_m$ be a function space m-generated by the n-tuple $\theta := (\theta_1, \dots, \theta_n)$ for some integer $m > 0$. For every positive integer $k \le m$, we denote by $\mathcal{S}_k$ the function space k-generated by θ. We fix a semi-inner product $\langle *, * \rangle_0 = \langle *, * \rangle_{0m}$, compatible with $\mathcal{S}$ and let $\langle *, * \rangle_{0k}$ be the semi-inner product induced by $\langle *, * \rangle_{0m}$ on $\mathcal{S}_k$. Clearly, $(\mathcal{S}_k, \langle *, * \rangle_{0k})$ is a qHfs, and we denote by $(\mathcal{H}_k, \langle *, * \rangle_k)$ its associated Hilbert space, so $\mathcal{H}_k = \mathcal{S}_k/\mathcal{I}_k$, where $\mathcal{I}_k = \{f \in \mathcal{S}_k; \|f\|_{0k} = 0\}$, whenever $0 < k \le m$. If $k < m$, we have $\mathcal{I}_k \subset \mathcal{I}_{k+1} \subset \cdots \subset \mathcal{I}_m$, and

$$\langle f + \mathcal{I}_k, g + \mathcal{I}_k \rangle_k = \langle f + \mathcal{I}_{k+1}, g + \mathcal{I}_{k+1} \rangle_{k+1} = \cdots = \langle f + \mathcal{I}_m, g + \mathcal{I}_m \rangle_m$$

for all $f, g \in \mathcal{S}_k$. In particular, if $0 < k \le l \le m$, we have a natural linear map $J_{k,l} : \mathcal{H}_k \mapsto \mathcal{H}_l$ given by $J_{k,l}(f + \mathcal{I}_k) = f + \mathcal{I}_l$, $f \in \mathcal{S}_k$, which is an isometry.

When $l = k+1$, we write sometimes J_k instead of $J_{k,k+1}$.

We also put $\mathcal{S}_0 = \mathbb{C}$, endowed with its natural inner product, so $\mathcal{I}_0 = \{0\}$, and $\mathcal{H}_0 = \mathbb{C}$.

For a given qHfs $(\mathcal{S}, \langle *, * \rangle_0)$ which is m-generated by an n-tuple $\theta := (\theta_1, \dots, \theta_n)$, we keep the notation from above, if not otherwise specified. The family $(\mathcal{H}_k)_{k=0}^n$ will be designated as *the sequence of Hilbert spaces associated to* $(\mathcal{S}, \langle *, * \rangle_0, \theta)$.

When the semi-inner product $\langle *, * \rangle_0$ is expandable, so there exists a uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ $\langle *, * \rangle_0$, the family $(\mathcal{H}_k)_{k=0}^n$ will be also called *the sequence of Hilbert spaces associated to* $(\mathcal{S}, \Lambda, \theta)$.

Definition 7. Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs m-generated by the n-tuple $\theta := (\theta_1, \dots, \theta_n)$ for some integer $m \ge 1$. Let also $k \in \{0, \dots, m-1\}$. We say that the sequence of Hilbert spaces $(\mathcal{H}_k)_{k=0}^m$ associated to $(\mathcal{S}, \langle *, * \rangle_0, \theta)$ is *stable at* k if $\dim(\mathcal{H}_k) = \dim(\mathcal{H}_{k+1})$.

When the semi-inner product $\langle *, * \rangle_0$ is expandable, so there exists a unique uspf $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ associated to $\langle *, * \rangle_0$, and if $(\mathcal{S}, \langle *, * \rangle_0, \theta)$ is stable at k, we say shortly that Λ *is stable at* k.

Definition 7 implies that the isometry $J_k : \mathcal{H}_k \mapsto \mathcal{H}_{k+1}$ is actually a unitary operator whenever the sequence $(\mathcal{H}_l)_{l=0}^n$ is stable at k. In fact, J_k unitary means that it is surjective, so for each $g \in \mathcal{S}_{k+1}$ we can find an $f \in \mathcal{S}_k$ such that $g - f \in \mathcal{I}_{k+1}$. In particular, if $g \in \mathcal{R}\mathcal{S}_{k+1}$, we can find an $f \in \mathcal{R}\mathcal{S}_k$ such that $g - f \in \mathcal{R}\mathcal{I}_{k+1}$.

Example 9. The stability introduced by Definition 7 is a rather strong condition. Let us illustrate it by an example. Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a function space, with $\langle *, * \rangle_0$ expandable, and m-generated by the n-tuple $\theta := (\theta_1, \dots, \theta_n)$ for some integer $m \ge 3$. Let $\mathcal{S}_k$ be the subspace k-generated by θ, and let $\langle *, * \rangle_{0k}$ be the restriction of $\langle *, * \rangle_0$ to the space $\mathcal{S}_k$, which is a compatible semi-inner product on for $\mathcal{S}_k$ for all $k = 1, 2, \dots, m$. Let also $\mathcal{H}_0 = \{0\}$. Assume that the sequence $(\mathcal{H}_k)_{k=0}^m$ is stable at 0, so the space $\mathcal{H}_1$ is unitarily equivalent to the space $\mathcal{H}_0 = \mathbb{C}$. Then $\mathcal{S}_1 = \mathcal{I}_1 + \mathbb{C}$, and $\mathcal{H}_1 = \mathbb{C}\hat{1}$. Let us show that we also have $\mathcal{H}_2 = \mathbb{C}\hat{1}$. For, note that $\theta_j = \tau_j + h_j$, with $\tau_j \in \mathbb{C}$ and $h_j \in \mathcal{I}_1$, $j = 1, \dots, n$. To go further, we need to show that if $f \in \mathcal{S}_1$ and $h \in \mathcal{I}_1$, then $fh \in \mathcal{I}_2$. Indeed, because $\langle *, * \rangle_0$ is expandable, and $fh\overline{fh} - h\overline{\bar{f}fh} = 0$, we must have

$$\|fh\|_{02}^2 = |\langle h, \bar{f}fh \rangle_{03}| \le \|h\|_{03} \|\bar{f}fh\|_{03} = \|h\|_{01}^2 \|\bar{f}fh\|_{03}^2 = 0,$$

by Lemma 4 and the Cauchy-Schwarz inequality. In particular, with the notation from above,

$$\theta_j\theta_k = \tau_j\tau_k + \tau_j h_k + \tau_k h_j + h_j h_k \in \mathbb{C} + \mathcal{I}_2,\ j,k = 1,\dots,n,$$

which implies the equality $\mathcal{H}_2 = \mathbb{C}\hat{1}$.

A more general situation, and under weaker conditions, will be presented in the following (see Theorem 10 and Remark 15).

The dimensional stability in the case of function spaces of polynomials implies the existence of representing measures for uspf's (see [9, 10, 29]). Nevertheless, in the present context, it is not always the case.

Example 10. Let Ω be a nonempty set and let θ_1 be a real-valued function on Ω. Let also $\mathcal{S} = \{c_0 + c_1\theta_1; c_0, c_1 \in \mathbb{C}\}$, which is a function space 1-generated by $\{\theta_1\}$. Set $\Lambda(1) = 1$, $\Lambda(\theta_1) = \alpha$, $\Lambda(\theta_1^2) = \alpha^2$ for some $\alpha > 0$, and extend this map to $\mathcal{S}^{(2)}$ by linearity. Because we have $\Lambda(|c_0+c_1\theta_1|^2) = |c_0+c_1\alpha|^2 \geq 0$, it follows that Λ is a uspf. Moreover,

$$\mathcal{I}_\Lambda = \{c_0 + c_1\theta_1; c_0 + c_1\alpha = 0, c_0, c_1 \in \mathbb{C}\}.$$

In fact, if $f = c_0 + c_1\theta_1$ is arbitrary in $\mathcal{S}$, it can be uniquely written as

$$f = (-c_1\alpha + c_1\theta_1) + (c_0 + c_1\alpha) \in \mathcal{I}_\Lambda + \mathbb{C}.$$

Therefore, the Hilbert space $\mathcal{H}_\Lambda$ is isomorphic to $\mathbb{C}$. In addition, $\hat{\theta}_1 = \alpha\hat{1}$.

To check whether the uspf Λ has a representing measure, we shall apply Corollary 5. Clearly $\{\hat{1}\}$ is an idempotent and $\mathcal{B} := \{\hat{1}\}$ is a basis of $\mathcal{H}_\Lambda$, which is multiplicative with respect to $\theta := (\theta_1)$, by Corollary 5. If $\delta_1(c\hat{1}) = c$ for all $c \in \mathbb{C}$, it clear that $\Delta := (\delta_1)$ is the dual basis of $\mathcal{B}$. Then Corollary 5 insures the existence of a 1-atomic representing measure for Λ if and only if $\delta_1(\hat{\theta}_1) = \alpha \in \theta_1(\Omega)$. Assuming $\theta_1(\omega_0) = \alpha$ for some $\omega_0 \in \Omega$, we obtain $\Lambda(h) = h(\omega_0)$ for all $h \in \mathcal{S}^{(2)}$. In other words, the uspf Λ is represented by the Dirac measure at ω_0. Nevertheless, this representation is not necessarily unique because there might exist several points ω with the property $\theta_1(\omega) = \theta_1(\omega_0)$. Moreover, if $\alpha \notin \theta_1(\Omega)$, we have no representation measure of Λ with support in Ω.

The next result extends Lemma 2.3 from [30].

Lemma 5. *Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs m-generated by $\theta = (\theta_1,\dots,\theta_n)$, with $\langle *, *\rangle_0$ expandable. If the sequence of Hilbert spaces $(\mathcal{H}_k)_{k=0}^m$ associated to $(\mathcal{S}, \langle *, *\rangle_0, \theta)$ is stable at $m-1$, then $(\sum_{j=1}^n \theta_j\mathcal{I}_m)\cap\mathcal{S} \subset \mathcal{I}_m$. In particular, $\theta_j\mathcal{I}_{m-1} \subset \mathcal{I}_m$ for all $j = 1,\dots,n$.*

Proof. Let $f = \sum_{j=1}^n \theta_j f_j \in \mathcal{S}$ with $f_j \in \mathcal{I}_m$ for all $j = 1,\dots,n$, and let $g \in \mathcal{S}_{m-1}$. Then

$$|\langle f, g\rangle_0| \leq \sum_{j=1}^n |\langle \theta_j f_j, g)\rangle_0| \leq \sum_{j=1}^n \|f_j\|_0\|\theta_j g\|_0 = 0$$

by the Cauchy-Schwarz inequality.

Now, let $h \in \mathcal{S}_{m-1}$ be such that $f - h \in \mathcal{I}_m$, which exists because of the stability of $(\mathcal{H}_k)_{k=0}^m$ at $m-1$. Then

$$\|f\|_0^2 = \langle f, h\rangle_0 + \langle f, f-h\rangle_0 = 0,$$

by the previous computation and the Cauchy-Schwarz inequality. Therefore $f \in \mathcal{I}_m$.

The last assertion is obvious. □

Remark 12. Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs on Ω, m-generated $\theta = (\theta_1,\dots,\theta_n)$, with $\langle *, *\rangle_0$ expandable, and let $\Lambda : \mathcal{S}^{(2)} \mapsto \mathbb{C}$ be the uspf associated to $\langle *, *\rangle_0$. We have $\mathcal{S} = \{f = p\circ\theta, p \in \mathcal{P}_m^n\}$, and set $\mathcal{S}_\theta := \{p|\theta(\Omega); p \in \mathcal{P}_m^n\}$, which is a function space on $\Omega_\theta := \theta(\Omega)$. We define a map $\Lambda_\theta : \mathcal{S}_\theta^{(2)} \mapsto \mathbb{C}$ by the equality $\Lambda_\theta(\phi) = \Lambda(p\circ\theta)$, $\phi = p|\Omega_\theta$, $p \in \mathcal{P}_{2m}^n$. The definition is correct because $p|\Omega_\theta = 0$ implies $p\circ\theta = 0$, and so $\Lambda(p\circ\theta) = 0$. In fact, Λ_θ is a uspf. In addition, the space $\mathcal{S}_\theta$ is m-generated by $\tau = (\tau_1,\dots,\tau_n)$, with $\tau_j := t_j|\Omega_\theta$, $j = 1,\dots,n$.

Assume now that $(\mathcal{S}, \langle *, *\rangle_0, \theta)$ is stable at $m-1$. Choosing a function $\phi = p|\Omega_\theta$, $p \in \mathcal{P}_m^n$, so $f := p\circ\theta \in \mathcal{S}_m$, we can find a function $g \in \mathcal{S}_{m-1}$ such that $h := f - g \in \mathcal{I}_m$. As $g = q\circ\theta$ for some $q \in \mathcal{P}_{m-1}^n$, and $h = r\circ\theta$ for some $r \in \mathcal{P}_m^n$, we obtain the equality $\phi = \psi + \iota$, where $\psi := q|\Omega_\theta \in \mathcal{S}_{\theta,m-1}$ and $\iota \in \mathcal{I}_{\theta,m}$ because $\Lambda_\theta(|\iota|^2) = \Lambda(|r\circ\theta|^2) = 0$. In other words, Λ_θ is stable at $m-1$.

Finding an atomic representing measure for Λ_θ means to solve a Ω_θ-moment problem, whose solution, when it exists, leads to a representing measure for Λ. For a closed $K \subset \mathbb{R}^n$, the K-moment problem has been approached in [12]. Because our framework is slightly larger and our results are generally not covered by the contents of [12], we present in the following a direct approach, independent of [12], but following the lines of [29].

Remark 13. From now on, if not otherwise specified, we fix a qHfs $(\mathcal{S}, \langle *, * \rangle_0)$ m-generated by $\theta = (\theta_1, \dots, \theta_n)$, with $\langle *, * \rangle_0$ expandable.

Next, assume that the sequence of Hilbert spaces $(\mathcal{H}_k)_{k=0}^m$ associated to $(\mathcal{S}, \langle *, * \rangle_0, \theta)$ is stable at $m-1$. Lemma 5 allows us to define correctly the map $M_j : \mathcal{H}_{m-1} \mapsto \mathcal{H}_m$ by the equality $M_j(f + \mathcal{I}_{m-1}) = \theta_j f + \mathcal{I}_m$ for all $j = 1, \dots, m$. Setting $J = J_{m-1}$ (see Remark 11), we may consider on the Hilbert space $\mathcal{H}_m$ the linear operators $T_j = M_j J^{-1}$ for all $j = 1, \dots, n$. Note that, fixing $f \in \mathcal{S}_m$ and choosing $g \in \mathcal{S}_{m-1}$ such that $f - g \in \mathcal{I}_m$, we have $T_j(f + \mathcal{I}_m) = \theta_j g + \mathcal{I}_m$ for all j. As mentioned after Definition 7, if $f \in \mathcal{RS}_m$ we can choose $g \in \mathcal{RS}_{m-1}$ such that $f - g \in \mathcal{RI}_m$. Therefore, $T_j(\mathcal{RH}_m) \subset \mathcal{RH}_m$ for all $j = 1, \dots, n$.

With this notation, we have the following.

Proposition 4. *The linear maps T_j, $j = 1, \dots, m$, are self-adjoint operators, and $T = (T_1, \dots, T_n)$ is a commuting tuple on $\mathcal{H}_m$.*

Proof. Let $f_k \in \mathcal{S}_m$ and $g_k \in \mathcal{S}_{m-1}$ be such that $f_k - g_k \in \mathcal{I}_m$ $(k = 1, 2)$. Then

$$\begin{aligned} \langle T_j^*(f_1 + \mathcal{I}_m), f_2 + \mathcal{I}_m \rangle &= \langle f_1 + \mathcal{I}_m, \theta_j g_2 + \mathcal{I}_m \rangle = \langle f_1, \theta_j g_2 \rangle_0 \\ &= \langle \theta_j g_1, f_2 \rangle_0 = \langle T_j(f_1 + \mathcal{I}_m), f_2 + \mathcal{I}_m \rangle, \end{aligned}$$

via Lemma 5 and Remark 13. Hence $T_1, \dots, T_n$ are self-adjoint.

We prove now that $T_1, \dots, T_n$ mutually commute. It suffices to show that $M_j J^{-1} M_k = M_k J^{-1} M_j$ for all $j, k = 1, \dots, n$. To show this, fix a function $f \in \mathcal{S}_{m-1}$. Thus $M_j(f + \mathcal{I}_{m-1}) = \theta_j f + \mathcal{I}_m$. We can choose $g_j \in \mathcal{S}_{m-1}$ such that $\theta_j f - g_j \in \mathcal{I}_m$. Therefore, $J^{-1}(\theta_j f + \mathcal{I}_m) = g_j + \mathcal{I}_{m-1}$, and $M_k(g_j + \mathcal{I}_{m-1}) = \theta_k g_j + \mathcal{I}_m$.

Similarly, $M_k(f + \mathcal{I}_{m-1}) = \theta_k f + \mathcal{I}_m$, and we can choose $g_k \in \mathcal{S}_{m-1}$ such that $\theta_k f - g_k \in \mathcal{I}_m$, so $M_j(g_k + \mathcal{I}_{m-1}) = \theta_j g_k + \mathcal{I}_m$. To complete the proof, it suffices to show that $\theta_k g_j - \theta_j g_k \in \mathcal{I}_m$. Indeed, note that $\theta_j \theta_k f - \theta_j g_k \in \theta_j \mathcal{I}_m$ and $\theta_k \theta_j f - \theta_k g_j \in \theta_k \mathcal{I}_m$. Consequently,

$$\theta_k g_j - \theta_j g_k \in (\theta_k \mathcal{I}_m + \theta_j \mathcal{I}_m) \cap \mathcal{S}_m \subset \mathcal{I}_m,$$

via Lemma 5. Consequently, $T_1, \dots, T_n$ mutually commute. □

Remark 14. With the notation from Proposition 4, if α, β are multi-indices with $|\alpha + \beta| \le m$, then $T^\alpha(\theta^\beta + \mathcal{I}_m) = \theta^{\alpha+\beta} + \mathcal{I}_m$. Indeed, if $|\beta| < m$, we have $T_j(\theta^\beta + \mathcal{I}_m) = (\theta_j \theta^\beta + \mathcal{I}_m)$, as in Remark 13. The general formula can be obtained by recurrence. In particular, if $|\alpha| \le m$, then $T^\alpha(1 + \mathcal{I}_m) = \theta^\alpha + \mathcal{I}_m$, and so $p(T)(1 + \mathcal{I}_m) = p(\theta) + \mathcal{I}_m$ for all $p \in \mathcal{P}_m^n$. Moreover, fixing $p \in \mathcal{P}_m^n$, as we have $p(\theta) = q(\theta) + h$, with $q \in \mathcal{P}_{m-1}^n$ and $h \in \mathcal{I}_m$, we obtain $p(T)(1 + \mathcal{I}_m) = q(T)(1 + \mathcal{I}_m)$.

The following assertion is now obtained as an application of Theorem 8. See also Theorem 2.11 and Corollary 2.13 from [29] (proved in a different way), as well as Corollary 7.11 from [9].

Theorem 9. *Let $(\mathcal{S}, \langle *, * \rangle_0)$ be a qHfs on Ω, m-generated by $\theta = (\theta_1, \dots, \theta_n)$, with $\langle *, * \rangle_0$ expandable. We assume that the sequence of Hilbert spaces $(\mathcal{H}_k)_{k=0}^m$ associated to $(\mathcal{S}, \langle *, * \rangle_0, \theta)$ is stable at $m - 1\,(m \ge 1)$. Then we have:*

(1) there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \dots, \hat{b}_d\}$ of $\mathcal{H} := \mathcal{H}_m$ consisting of idempotent elements, which is multiplicative with respect to θ;

(2) the semi-inner product $\langle *, * \rangle_0$ has a d-atomic representing measure with support in Ω, where $d := \dim \mathcal{H}$, if and only $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, where Δ is the dual basis of $\mathcal{B}$;

(3) if the semi-inner product $\langle *, * \rangle_0$ has an atomic representing measure with support in Ω, this atomic measure is uniquely determined.

Proof. (1) First of all, note that $\mathcal{H} = \{p(T)\hat{1}; p \in \mathcal{P}_m^n\}$. Indeed, if $\hat{f} \in \mathcal{H}$ is an arbitrary element, as $\mathcal{S}$ is m-generated by θ, we can find a polynomial $p \in \mathcal{P}_m^n$ such that $\hat{f} = p \circ \theta + \mathcal{I}_m$, and so $\hat{f} = p(T)(1 + \mathcal{I}_m)$, via Remark 14.

Next, we want to apply Theorem 8 to show that there exists an orthogonal basis $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}$ consisting of idempotent elements, which is multiplicative with respect to θ.

We first consider the commuting n-tuple $T = (T_1, \ldots, T_n)$, consisting of self-adjoint operators, acting in $\mathcal{H}$, given by Proposition 4. The spectral theorem for n-tuples of commuting self-adjoint operators (see for instance [6], [24], [26] etc.) implies the existence of commuting self-adjoint projections $E_j = E(\{\xi^{(j)}\})$, $j = 1, \ldots, d$, such that $h(T) = \sum_{j=1}^d h(\xi^{(j)})E_j$ for every function $h : \sigma(T) \mapsto \mathbb{C}$, where $\sigma(T) := \{\xi^{(1)}, \ldots, \xi^{(d)}\}$ is the joint spectrum of T, which coincides with the support of E. Moreover, if the function h is real-valued, the operator $h(T)$ is self-adjoint. In addition, because the space $\mathcal{RH}$ is invariant under $T_1, \ldots, T_n$ (see Remark 13), it must be also invariant under $h(T)$, whenever h is real-valued. In particular, $\mathcal{RH}$ is invariant under E_j, $j = 1, \ldots, d$,.

We now construct an orthogonal family $\{\hat{b}_1, \ldots, \hat{b}_d\}$ of $\mathcal{H}$ consisting of idempotents. Because $\sum_{j=1}^d E_j$ is the identity on $\mathcal{H}$, setting $\hat{b}_j = E_j\hat{1} \in \mathcal{RH}, j = 1, \ldots, d$, we obtain a decomposition $\hat{1} = \sum_{j=1}^d \hat{b}_j$. As $E_j \neq 0$, we must have $E_j\hat{g} = \hat{g} \neq 0$ for some $\hat{g} = q \circ \theta + \mathcal{I}_m = q(T)(1 + \mathcal{I}_m)$, with $q \in \mathcal{P}_m^n$, via Remark 14. Assuming $\hat{b}_j = 0$, we would obtain $E_j\hat{g} = \hat{g} = q(T)\hat{b}_j = 0$, which is not possible. Therefore, $\hat{b}_j \neq 0$ for all $j = 1, \ldots, d$. Note also that $\langle \hat{b}_j, \hat{1} \rangle = \langle \hat{b}_j, \hat{b}_j \rangle > 0$, so $\hat{b}_j$ is an idempotent for all $j = 1, \ldots, d$. In other words, $\{\hat{b}_1, \ldots, \hat{b}_d\}$ is an an orthogonal family in $\mathcal{H}$ consisting of idempotent elements.

To show that $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is a basis of $\mathcal{H}$ it suffices to show that $\dim(\mathcal{H}) = d$. For, we consider the sub-C^*-algebra $\mathcal{C}_T$ generated by T in the C^*-algebra of all linear (automatically bounded) operators acting in $\mathcal{H}$. Because $\mathcal{C}_T$ is finite dimensional, we must have $\mathcal{C}_T = \{p(T); p \in \mathcal{P}_s^n\}$, for some integer $s \geq m$. In fact, choosing an element $p(T)$ with $p \in \mathcal{P}_s^n$, we may replace p by a polynomial $q \in \mathcal{P}_m^n$, such that $p(T) = q(T)$. We prove this assertion by recurrence. Let $p_j(t) = t_jp(t)$, with $p \in \mathcal{P}_m^n$. Then there exists $q \in \mathcal{P}_{m-1}^n$ such that $p(T)\hat{1} = q(T)\hat{1}$, by Remark 14. Therefore, $p_j(T)\hat{1} = q_j(T)\hat{1}$, where $q_j(t) = t_jq(t) \in \mathcal{P}_m^n$. Using the fact that every $\hat{h} \in \mathcal{H}$ is of the form $g(T)\hat{1}$ for some polynomial g, we deduce the equality $p_j(T) = q_j(T)$. An induction argument shows that for $p(T)$ with $p \in \mathcal{P}_s^n$, $s > m$, we have the equality $p(T) = q(T)$ for some polynomial $q \in \mathcal{P}_m^n$. Particularly, $\mathcal{C}_T = \{p(T); p \in \mathcal{P}_m^n\}$.

As mentioned above, the spectral theorem allows us to write

$$p(T) = \sum_{j=1}^d p(\xi^{(j)})E_j,\, p \in \mathcal{P}_m^n.$$

In particular, $\{E_1, \ldots, E_d\}$, which is clearly a linearly independent family of operators, is actually an algebraic basis of (the linear space) $\mathcal{C}_T$. Note also that

$$p(T)\hat{1} = \sum_{j=1}^d p(\xi^{(j)})\hat{b}_j,\, p \in \mathcal{P}_m^n.$$

Consequently, using the equality $\mathcal{H} = \{p(T)\hat{1}; p \in \mathcal{P}_m^n\}$ mentioned above, we deduce that $\dim(\mathcal{H}) = \dim(\mathcal{C}_T) = d$. In particular, $\mathcal{B} = \{\hat{b}_1, \ldots, \hat{b}_d\}$ is an orthogonal basis of $\mathcal{H}$, consisting of idempotents. In addition, considering the measure $\nu(*) = \langle E(*)\hat{1}, \hat{1} \rangle$ on $\sigma(T)$, and putting $\lambda_j = \langle E_j\hat{1}, \hat{1} \rangle = \langle \hat{b}_j, \hat{1} \rangle$, we have

$$\langle \theta^\alpha, b_j \rangle_0 \langle \theta^\beta, b_j \rangle_0 = \langle T^\alpha \hat{1}, E_j\hat{1} \rangle \langle T^\beta \hat{1}, E_j\hat{1} \rangle =$$
$$\int_{\{\xi^{(j)}\}} t^\alpha d\nu(t) \int_{\{\xi^{(j)}\}} t^\beta d\nu(t) = \lambda_j^2 (\xi^{(j)})^\alpha (\xi^{(j)})^\beta =$$
$$\lambda_j \int_{\{\xi^{(j)}\}} t^{\alpha+\beta} d\nu(t) = \lambda_j \langle \theta^{\alpha+\beta}, b_j \rangle_0$$

whenever $|\alpha| + |\beta| \leq m$, $j = 1, \ldots, d$. In other words, the basis $\mathcal{B}$ is multiplicative with respect to θ, which concludes the asserion (1) from the statement.

To obtain the assertion (2) from the statement, we recall that the dual basis $\Delta := \{\delta_1, \ldots, \delta_d\}$ of $\mathcal{B}$ is given by $\delta_j(\hat{f}) = \langle \hat{b}_j, \hat{1} \rangle^{-1} \langle \hat{f}, \hat{b}_j \rangle$, $j = 1, \ldots, d$. In particular,

$$\delta_j(\widehat{\theta_k}) = \lambda_j^{-1} \langle E_jT_k\hat{1}, \hat{1} \rangle = \int_{\{\xi^{(j)}\}} t_k d\nu(t) = \xi_k^{(j)},\, j, k = 1, \ldots, d.$$

Theorem 8 shows that the inner product of $\mathcal{H}$ has a representing measure on Ω consisting of $d := \dim\mathcal{H}$ atoms if and only $\delta(\hat{\theta}) \in \theta(\Omega)$, $\delta \in \Delta$, which concludes the proof (2).

(3) This assertion is not a direct consequence of Proposition 2, becuase we may consider a priori two atomic measures whose supports have different cardinals. To apply Proposition 2, we need a supplementary argument.

An explicit form of the integral representation whose existence is given in (2) is obtained as for equation (5.3). Specifically, choosing $\zeta_j \in \Omega$ such that $\xi^{(j)} = \delta_j(\zeta_j)$, $j = 1, \dots, d$, we deduce the equality

$$\langle f, g\rangle_0 = \sum_{j=1}^{d} \lambda_j f(\zeta_j)\overline{g(\zeta_j)},$$

providing a (d-atomic) representing measure for the semi-inner product of $\mathcal{S}$.

Let μ be this representing measure of the inner product $\langle *, *\rangle_0)$, with support $\mathfrak{Z} = \{\zeta_1, \dots \zeta_d\}$ and weights $\lambda_j = \mu(\xi^{(j)})$, $j = 1, \dots, d$. Assume that the semi-inner product $\langle *, *\rangle_0$ has another atomic representing measure in Ω, say ν, with support $\Sigma := \{\sigma_1, \dots, \sigma_g\} \subset \Omega$. Then necessarily, $g \ge d = \dim(\mathcal{H})$, and the map $\mathcal{H} \ni \hat{f} \mapsto f|\Sigma \in L^2(\Xi, \nu)$ is an isometry (see Corollary 4). Moreover, $\mathcal{I}_m = \{f \in \mathcal{S}; f|\Sigma = 0\}$.

Let B_j be the linear operator on $L^2(\Sigma, \nu)$ given by $B_j h = \theta_j h$ for all $j = 1, \dots, n$ and $h \in L^2(\Sigma, \nu)$. Then $B := (B_1, \dots, B_n)$ is an n-tuple of commuting self-adjoint operators. With $T = (T_1, \dots, T_n)$ as before, fixing $\hat{f} \in \mathcal{H}$ with $f \in \mathcal{S}_m$, and choosing $g \in \mathcal{S}_{m-1}$ with $h := f - g \in \mathcal{I}_m$, so $T_j \hat{f} = \widehat{\theta_j g}$, $(f - g)|\Sigma = 0$, and $(\theta_j g)|\Xi = (\theta_j f)|\Sigma = B_j(f|\Sigma)$. In other words, identifying the Hilbert space $\mathcal{H}$ with the (Hilbert) subspace $\{f|\Sigma; f \in \mathcal{S}\}$, we see that B_j is an extension of the operator T_j for all $j = 1, \dots, n$. In particular, the spectral measure E of T is the restriction of the spectral measure E_B of B to $\mathcal{H}$.

We now consider the elements $E_B(\{\sigma_j\})(1|\Sigma)$, which must belong to $\mathcal{H}$, because $\mathcal{H}$ is invariant under E_B. Therefore, setting $\hat{c}_j = E_B(\{\sigma_j\})(1|\Sigma) = E(\{\sigma_j\})\hat{1}$, $j = 1, \dots, g$, as in the second part of the proof, $\{\hat{c}_1, \dots, \hat{c}_g\}$ is an orthogonal family of nonnull idempotent elements of $\mathcal{H}$. Consequently, we must have $g = d$, and so $\dim(L^2(\Xi, \nu)) = d$. We may now apply Proposition 2, to get the asserton (3).

The next result is an extension of Theorem 2.6 from [29] (see also Theorem 7.8 and Corollary 7.9 from [9]).

Theorem 10. *Let $(\mathcal{S}, \langle *, *\rangle_0)$ be a qHfs on Ω, m-generated by $\theta = (\theta_1, \dots, \theta_n)$, with $\langle *, *\rangle_0$ expandable. We assume that the sequence of Hilbert spaces $(\mathcal{H}_k)_{k=0}^m$ associated to $(\mathcal{S}, \langle *, *\rangle_0, \theta)$ is stable at $m - 1\,(m \ge 1)$. Then the semi-inner product $\langle *, *\rangle_0$ can be uniquely extended to an expandable semi-inner product of $\mathcal{S}_\infty$, which has a d-atomic measure in Ω, where $d = \dim(\mathcal{H}_m)$.*

Proof. Using the notation and arguments from (the proof of) Theorem 9, we have

$$\langle f, g\rangle_0 = \sum_{j=1}^{d} \lambda_j f(\zeta_j)\overline{g(\zeta_j)},\ f, g \in \mathcal{S},$$

which is an integral representation of $\langle *, *\rangle_0$ by a d-atomic measure. A direct extension of this formula allows us to define

$$\langle f, g\rangle_{0\infty} = \sum_{j=1}^{d} \lambda_j f(\zeta_j)\overline{g(\zeta_j)},\ f, g \in \mathcal{S}_\infty,$$

which is an expandable semi-inner product on $\mathcal{S}_\infty$. We want to show that $\langle *, *\rangle_{0\infty}$ is uniquely determined.

Let $\langle *, *\rangle'_{0\infty}, \langle *, *\rangle''_{0\infty}$ be two expandable semi-inner products on $\mathcal{S}_\infty$, both of them extending $\langle *, *\rangle_0$. For $k \ge m + 1$, let $\mathcal{S}_k = \{p \circ \theta; p \in \mathcal{P}_k^n\}$, $\mathcal{I}'_k = \{f \in \mathcal{S}_k; \langle f, f\rangle'_{0\infty} = 0\}$, $\mathcal{I}''_k = \{f \in \mathcal{S}_k; \langle f, f\rangle''_{0\infty} = 0\}$. Clearly, $\mathcal{I}_m \subset \mathcal{I}'_k \cap \mathcal{I}''_k$ for all $k \ge m + 1$.

We shall show by induction that for every multi-index α, with $|\alpha| \ge m$, there exists an element $f_\alpha \in \mathcal{S}_{m-1}$, such that $\theta^\alpha - f_\alpha \in \mathcal{I}'_{|\alpha|} \cap \mathcal{I}''_{|\alpha|}$. The assertion is obvious for $|\alpha| = m$, via the stability at $m - 1$. Assume the property true for all multi-indices of length k, for a $k \ge m$, and let us prove it for multi-indices of length $k+1$. If $|\alpha| = k+1$, there exists a number $j \in \{1, \ldots, n\}$ and a multi-index β with $|\beta| = k$ such that $\theta^\alpha = \theta_j \theta^\beta$. By the induction hypothesis, we can find a function $f_\beta \in \mathcal{S}_{m-1}$ such that $\theta^\beta - f_\beta \in \mathcal{I}'_k \cap \mathcal{I}''_k$. Therefore, $\theta^\alpha - \theta_j f_\beta \in \mathcal{I}'_{k+1} \cap \mathcal{I}''_{k+1}$, by the Cauchy-Schwarz inequality. Further, $\theta_j f_\beta \in \mathcal{S}_m$ and so we can find a function $f_{j,\beta} \in \mathcal{S}_{m-1}$ such that $\theta_j f_\beta - f_{j,\beta} \in \mathcal{I}_m$, via the stability at $m-1$. Consequently,

$$\theta^\alpha - f_\alpha = \theta^\alpha - \theta_j f_\beta + \theta_j f_\beta - f_{j,\beta} \in \mathcal{I}'_{k+1} \cap \mathcal{I}''_{k+1} + \mathcal{I}_m = \mathcal{I}'_{k+1} \cap \mathcal{I}''_{k+1},$$

where $f_\alpha = f_{j,\beta}$.

Extending the property from above to arbitrary functions from $\mathcal{S}_\infty$, we deduce, in particular, that for every pair of function $f_1, f_2 \in \mathcal{S}_k$, with $k \geq m$, we can find a pair $g_1, g_2 \in \mathcal{S}_{m-1}$ such that $f_j - g_j \in \mathcal{I}'_k \cap \mathcal{I}''_k$, $j = 1, 2$. Therefore,

$$\langle f_1, f_2 \rangle'_{0\infty} = \langle g_1, g_2 \rangle_0 = \langle f_1, f_2 \rangle''_{0\infty},$$

showing the uniqueness of the natural extension $\langle *, * \rangle_{0\infty}$ of the semi-inner product $\langle *, * \rangle_{0\infty}$. □

Remark 15. From the proof of the previous theorem, we deduce that for all $k \geq m$ and $f \in \mathcal{S}_k$ there exists $g \in \mathcal{S}_{m-1}$ such that $f - g \in \mathcal{I}_k$, where $\mathcal{I}_k := \{h \in \mathcal{S}_k; \langle h, h \rangle_{0\infty} = 0\}$. This implies that all spaces $\mathcal{H}_k := \mathcal{S}_k / \mathcal{I}_k$ are unitarily equivalent Hilbert spaces. This assertion is true even for $k = \infty$.

7. An Example

Example 11. This is an example related to the paper [16](see also [14]). Specifically, we look for atomic representing measures of a given semi-inner products of the space, $\mathcal{P}^2_m$ $(m \geq 1)$, whose suport lies in the curve $\Gamma := \{(t, t^3) \in \mathbb{R}^2; t \in \mathbb{R}\}$. In what follows, we want to solve a Γ-moment problem (see Subsection 2.3), trying to use some of our techniques. The basic function space will be $\mathcal{S}_m = \{P|\Gamma; P \in \mathcal{P}^2_m\} = \mathcal{P}^2_{m,\Gamma}$. Because the representation of an element $P|\Gamma \in \mathcal{S}_m$ is, in general, not unique, let us characterize the subspace $\mathcal{J} := \{P \in \mathcal{P}^2_m; P|\Gamma = 0\}$. Given a polynomial $P(x_1, x_2) = \sum_{0 \leq k+l \leq m} a_{k,l} x_1^k x_2^l$ in $\mathcal{P}^2_m$, we have $P|\Gamma = 0$ if and only if $P(t, t^3) = \sum_{0 \leq k+l \leq m} a_{k,l} t^{k+3l} = 0$ for all $t \in \mathbb{R}$. Explicitly, we must have

$$P(t, t^3) = \sum_{j=0}^{3m} \left(\sum_{l \in I(j)} a_{j-3l,l} \right) t^j = 0,\ t \in \mathbb{R},$$

with $I(j) = \{l \geq 0; 3l \leq j \leq m + 2l\}$, which happens if and only if $\sum_{l \in I(j)} a_{j-3l,l} = 0$ whenever $0 \leq j \leq 3m$. In other words,

$$\mathcal{J} = \{P(x_1, x_2) = \sum_{0 \leq k+l \leq m} a_{k,l} x_1^k x_2^l;\ \sum_{l \in I(j)} a_{j-3l,l} = 0,\ 0 \leq j \leq 3m\}.$$

Moreover, the space $\mathcal{S}_m$ is isomorphic to the space $\mathcal{P}^2_m / \mathcal{J}$, the elements of $\mathcal{S}_m$ may be regarded as equivalence classes of $\mathcal{P}^2_m$ modulo $\mathcal{J}$, and they will be denoted by $\tilde{P} = P + \mathcal{J}$, $P \in \mathcal{P}^2_m$.

Let $\theta_1, \theta_2 : \mathbb{R} \mapsto \mathbb{R}$ be given by $\theta_1(t) = t$, $\theta_2(t) = t^3$, $t \in \mathbb{R}$. We have a natural map of $\mathcal{S}_m$ into $\mathcal{P}^1_{3m}$ given by $\tilde{P} \mapsto P \circ \theta$. Since $P \in \mathcal{J}$ is equivalent to $P \circ \theta = 0$, this map is correctly defined, linear, and injective; it is surjective too. Indeed, we may define a linear map from $\mathcal{P}^1_{3m}$ into $\mathcal{P}^2_m$ in the following way. If $p_0(t) = \sum_{k=0}^{3m} a_k t^k$, we use the representation

$$p_0(t) = \sum_{l=0}^{m} a_{3l} t^{3l} + \sum_{l=0}^{m-1} a_{3l+1} t^{3l+1} + \sum_{l=0}^{m-1} a_{3l+2} t^{3l+2}.$$

We now make a certain choice, obviously not unique. Namely, we replace the monomial t^{3l} by the monomial x_2^l, the monomial t^{3l+1} by the monomial $x_1 x_2^l$, and the monomial t^{3l+2} by the monomial $x_1^2 x_2^l$, so we obtain the polynomial

$$P_0(x_1, x_2) = \sum_{l=0}^{m} a_{3l} x_2^l + \sum_{l=0}^{m-1} a_{3l+1} x_1 x_2^l + \sum_{l=0}^{m-1} a_{3l+2} x_1^2 x_2^l,$$

we clearly have $p_0 = P_0 \circ \theta$, showing that the image of $\tilde{P}_0$ is precisely p_0.

Let $\langle *, * \rangle_0$ be an expandable semi-inner product of $\mathcal{S}_m$. We want to define an expandable semi-inner product of $\mathcal{P}^1_{3m}$. If $p, q \in \mathcal{P}^1_{3m}$, as we have $p = P \circ \theta, q = Q \circ \theta$ for some $P, Q \in \mathcal{P}^2_m$, we set

$$\langle p, q \rangle_0 := \langle \tilde{P}, \tilde{Q} \rangle_0.$$

The definition does not depend on the choice of $P, Q \in \mathcal{P}^2_m$ with $p = P \circ \theta, q = Q \circ \theta$ because, if either $P \circ \theta = 0$ or $Q \circ \theta = 0$, we must have $\langle \tilde{P}, \tilde{Q} \rangle_0 = 0$. In addition, it clearly provides an expandable semi-inner product of $\mathcal{P}^1_{3m}$.

The existence of atomic measures for expandable semi-inner products of spaces of polynomials in one variable, in particular for the space $\mathcal{P}^1_{3m}$, treated as a moment problem, can be explicitly described (see for instance [8]). We restrict ourselves to a particular case of Theorem 9, applied to $\mathcal{P}^1_{3m}$, endowed with the expandable inner product $\langle p, q \rangle_0$, $p, q \in \mathcal{P}^1_{3m}$. The dimensional stability at $3m - 1$ is given by a condition of the form

$$\|t^{3m} - \sum_{k=0}^{3m-1} c_k t^k\|_0 = 0,$$

for some polynomial $\sum_{k=0}^{3m-1} c_k t^k$, which is equivalent to the condition

$$\left\| x_2^{3m} - \sum_{l=0}^{m-1} c_{3l} x_2^l - \sum_{l=0}^{m-1} c_{3l+1} x_1 x_2^l - \sum_{l=0}^{m-1} c_{3l+2} x_1^2 x_2^l + \mathcal{J} \right\|_0,$$

expressed only in the given terms.

A solution of this moment problem means the existence of (distinct) points $\tau_1, \ldots, \tau_r \in \mathbb{R}$ and positive numbers $\lambda_1, \ldots, \lambda_r$ with $\lambda_1 + \cdots + \lambda_r = 1$ such that

$$\langle p, q \rangle_0 = \sum_{j=1}^{d} \lambda_j p(\tau_j) \overline{q(\tau_j)},\ p, q \in \mathcal{P}^1_{3m}.$$

Therefore, if $p, q \in \mathcal{S}_m$ are written under the form $p = P \circ \theta, q = Q \circ \theta$ with $P, Q \in \mathcal{P}^2_m$, we have

$$\langle \tilde{P}, \tilde{Q} \rangle_0 = \langle p, q \rangle_0 = \sum_{j=1}^{d} \lambda_j P(\tau_j, \tau_j^3) \overline{Q(\tau_j, \tau_j^3)},$$

which is a solution of our Γ-moment problem.

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