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# OSCILLATION OF NONLINEAR DELAY DIFFERENTIAL EQUATION WITH NON-MONOTONE ARGUMENTS

ÖZKAN ÖCALAN[1,*], NURTEN KILIÇ[2], SERMIN ŞAHİN[3] AND UMUT MUTLU ÖZKAN[3]

Abstract. Consider the first-order nonlinear retarded differential equation

$$x'(t)+p(t)f\left(x\left(\tau(t)\right)\right)=0,\quad t\geq t_0$$

where $p(t)$ and $\tau(t)$ are function of positive real numbers such that $\tau(t)\leq t$ for $t\geq t_0$, and $\lim_{t\to\infty}\tau(t)=\infty$. Under the assumption that the retarded argument is non-monotone, new oscillation results are given. An example illustrating the result is also given.

**Keywords**: delay differential equation; non-monotone argument; oscillatory solutions; nonoscillatory solutions.

**2010 Mathematics Subject Classification**: 34K11, 34K06.

## 1. Introduction

Consider the nonlinear retarded differential equation

$$x'(t)+p(t)f\left(x\left(\tau(t)\right)\right)=0,\quad t\geq t_0 \tag{1.1}$$

where $p(t)$ and $\tau(t)$ are functions of nonnegative real numbers, and $\tau(t)$ is non-monotone or nondecreasing such that

$$\tau(t)\leq t \ \text{ for } \ t\geq t_0, \ \text{ and } \ \lim_{t\to\infty}\tau(t)=\infty, \tag{1.2}$$

and

$$f\in C(\mathbb{R},\mathbb{R}) \quad \text{and} \quad xf(x)>0 \ \text{ for } x\neq 0. \tag{1.3}$$

By a solution of (1.1) we mean a continuously differentiable function defined on $[\tau(T_0),\infty]$ for some $T_0\geq t_0$ and such that (1.1) is satisfied for $t\geq T_0$. Such a solution is called *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*.

Recently there has been an increasing interest in the study of the oscillatory behavior of the following special form of (1.1)

$$x'(t)+p(t)x\left(\tau(t)\right)=0,\quad t\geq t_0. \tag{1.4}$$

See, for example, [1−19] and the references cited therein. The first systematic study for the oscillation of all solutions of equation (1.4) was made by Myshkis. In 1950 [17] he proved that every solution of (1.4) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)]<\infty \ \text{ and } \ \liminf_{t\to\infty}[t-\tau(t)]\liminf_{t\to\infty}p(t)>\frac{1}{e}.$$

In 1972, Ladas, Lakshmikantham and Papadakis [16] proved that the same conclusion holds if, in addition, $\tau$ is a non-decreasing function and

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1. \tag{1.5}$$

In 1982, Koplatadze and Canturija [14] established the following result.

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If $\tau(t)$ is a non-monotone or nondecreasing and

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{1}{e}, \tag{1.6}$$

then all solutions of Eq.(1.4) oscillate, while if

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds < \frac{1}{e}, \tag{1.7}$$

then the equation (1.4) has a nonoscillatory solution.

To the best of our knowledge, there are few papers dealing with the oscillatory behavior of solutions of (1.1), see, for example, [9, 17]. The following theorem was given by Ladde et al. in [17].

**THEOREM A.** Assume that the $f$, $p$ and $\tau$ in Eq.(1.1) satisfy the following conditions:

*i*) The condition (1.2) holds and let $\tau(t)$ be strictly increasing on $\mathbb{R}_+$,

*ii*) $p(t)$ is locally integrable and $p(t) \geq 0$, a.e.;

*iii*) The condition (1.3) holds and let $f$ be nondecreasing, and

$$\lim_{x\to 0} \frac{x}{f(x)} = N < +\infty.$$

Assume further that

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > N,$$

or

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{N}{e}.$$

Then every solution of Eq.(1.1) is oscillatory.

The following theorem was given by Fukagai and Kusano in [9].

**THEOREM B.** Suppose that the conditions (1.2) and (1.3) hold. Suppose moreover that

$$\limsup_{x\to 0} \frac{|x|}{|f(x)|} = \lambda < \infty.$$

If

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{\lambda}{e},$$

then every solution of Eq.(1.1) is oscillatory.

Thus, in this paper, our aim is to obtain some oscillation criteria for all solutions of Eq.(1.1) under the assumption that $\tau(t)$ is non-monotone.

## 2. Main Results

In this section, we present a new sufficient conditions for the oscillation of all solutions of Eq.(1.1), under the assumption that the argument $\tau(t)$ is non-monotone or nondecreasing. Set

$$h(t) := \sup_{s\leq t} \tau(s), \quad t \geq 0. \tag{2.1}$$

Clearly, $h(t)$ is nondecreasing, and $\tau(t) \leq h(t)$ for all $t \geq 0$.

Assume that the $f$ in Eq.(1.1) satisfy the following condition:

$$\limsup_{x\to 0} \frac{x}{f(x)} = M, \quad 0 \leq M < \infty. \tag{2.2}$$

**Theorem 2.1.** *Assume that* (1.2), (1.3) *and* (2.2) *holds. If* $\tau(t)$ *is non-monotone or nondecreasing, and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > \frac{M}{e}, \tag{2.3}$$

*then all solutions of Eq.*(1.1) *oscillate.*

*Proof.* Assume, for the sake of contradiction, that there exists a nonoscillatory solution $x(t)$ of (1.1). Since $-x(t)$ is also a solution of (1.1), we can confine our discussion only to the case where the solution $x(t)$ is eventually positive. Then there exists $t_1 > t_0$ such that $x(t),\ x(\tau(t)) > 0$, for all $t \geq t_1$. Thus, from (1.1) we have

$$x'(t) = -p(t)f\left(x\left(\tau(t)\right)\right) \leq 0, \quad \text{for all } t \geq t_1.$$

Thus $x(t)$ is nonincreasing and has a limit $l \geq 0$ as $t \to \infty$.

Now, we claim that $l = 0$. Condition (2.3) implies that

$$\int_{a}^{\infty} p(t)dt = \infty. \tag{2.4}$$

In view of (2.4) and by the Theorem 3.1.5 in [17] that $\lim_{t\to\infty} x(t) = 0$. Suppose $M > 0$. Then, in view of (2.2) we can choose $t_2 > t_1$ so large that

$$f(x(t)) \geq \frac{1}{2M}x(t) \quad \text{for } t \geq t_2. \tag{2.5}$$

On the other hand, we know from Lemma 2.1.1 [7] that

$$\liminf_{t\to\infty} \int_{h(t)}^{t} p(s)ds = \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds. \tag{2.6}$$

Since $h(t) \geq \tau(t)$ and $x(t)$ is nonincreasing , by (1.1) and (2.5) we have

$$x'(t) + \frac{1}{2M}p(t)x(h(t)) \leq 0, \qquad t \geq t_3. \tag{2.7}$$

Also, from (2.3) and (2.6) it follows that there exists a constant $c > 0$ such that

$$\int_{h(t)}^{t} p(s)ds \geq c > \frac{M}{e}, \quad t \geq t_3 \geq t_2. \tag{2.8}$$

So, from (2.8), there exists a real number $t^* \in (h(t), t)$, for all $t \geq t_3$ such that

$$\int_{h(t)}^{t^*} p(s)ds > \frac{M}{2e} \quad \text{and} \quad \int_{t^*}^{t} p(s)ds > \frac{M}{2e}. \tag{2.9}$$

Integrating (2.7) from $h(t)$ to $t^*$ and using $x(t)$ is nonincreasing then we have

$$x(t^*) - x\left(h(t)\right) + \frac{1}{2M}\int_{h(t)}^{t^*} p(s)x\left(h(s)\right)ds \leq 0,$$

or

$$x(t^*) - x\left(h(t)\right) + \frac{1}{2M}x\left(h(t^*)\right)\int_{h(t)}^{t^*} p(s)ds \leq 0.$$

Thus, by (2.9), we have

$$-x\left(h(t)\right) + \frac{1}{2M}x\left(h(t^*)\right)\frac{M}{2e} < 0. \tag{2.10}$$

Integrating (2.7) from $t^*$ to $t$ and using the same facts , we get

$$x(t) - x(t^*) + \frac{1}{2M}\int_{t^*}^{t} p(s)x(h(s))\,ds \le 0.$$

Thus, by (2.9), we have

$$-x(t^*) + \frac{1}{2M}x(h(t))\frac{M}{2e} < 0. \tag{2.11}$$

Combining the inequalities (2.10) and (2.11), we obtain

$$x(t^*) > x(h(t))\frac{1}{4e} > x(h(t^*))\left(\frac{1}{4e}\right)^2,$$

and hence we have

$$\frac{x(h(t^*))}{x(t^*)} < (4e)^2 \quad \text{for } t \ge t_4.$$

Let

$$w = \frac{x(h(t^*))}{x(t^*)} \ge 1,$$

and because of $1 \le w < (4e)^2$ , $w$ is finite.

Now dividing (1.1) with $x(t)$ and then integrating from $h(t)$ to $t$ we obtain

$$\int_{h(t)}^{t} \frac{x'(s)}{x(s)}ds + \int_{h(t)}^{t} p(s)\frac{f(x(\tau(s)))}{x(s)}ds = 0$$

and

$$\ln\frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s)\frac{f(x(\tau(s)))}{x(\tau(s))}\frac{x(\tau(s))}{x(s)}ds = 0$$

Since $x(t)$ is nonincreasing, we get

$$\ln\frac{x(t)}{x(h(t))} + \int_{h(t)}^{t} p(s)\frac{f(x(\tau(s)))}{x(\tau(s))}\frac{x(h(s))}{x(s)}ds \le 0$$

and

$$\ln\frac{x(h(t))}{x(t)} \ge \frac{f(x(\tau(\xi)))}{x(\tau(\xi))}\frac{x(h(\xi))}{x(\xi)}\int_{h(t)}^{t} p(s)ds, \tag{2.12}$$

where $\xi$ is defined with $h(t) < \xi < t$, while $t \longrightarrow \infty$, $\xi \longrightarrow \infty$ and because of this $h(t) \longrightarrow \infty$. Then taking lower limit on both side of (2.12), we obtain $\ln w \ge \frac{w}{e}$. But this is impossible since $\ln x \le \frac{x}{e}$ for all $x > 0$. The case where $M = 0$ can be discussed similarly. The proof of the theorem is completed. □

**Theorem 2.2.** *Assume that* (1.2), (1.3), (2.2) *and* (2.4) *holds. If* $\tau(t)$ *is non-monotone, and*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds > 2M \tag{2.13}$$

*where* $h(t)$ *is defined by (2.1), then all solutions of Eq.*(1.1) *oscillate.*

*Proof.* Assume, for the sake of contradiction, that there exist a nonoscillatory solution $x(t)$ of (1.1). In view of (2.4), we know from Theorem 2.1 that $\lim_{t\to\infty} x(t) = 0$, for $t \geq t_1$.

Considering equation (1.1)

$$x'(t) + p(t)f(x(\tau(t))) = 0$$

by (2.5) we get

$$x'(t) + \frac{1}{2M}p(t)x(\tau(t)) \leq 0$$

Since $h(t) \geq \tau(t)$ and $x(t)$ is nonincreasing

$$x'(t) + \frac{1}{2M}p(t)x(h(t)) \leq 0 \tag{2.14}$$

Integrating (2.14) from $h(t)$ to $t$, and using the fact that the function $x(t)$ is nonincreasing and the function $h(t)$ is nondecreasing

$$x(t) - x(h(t)) + \frac{1}{2M}\int_{h(t)}^{t} p(s)x(h(s))ds \leq 0$$

or

$$x(t) - x(h(t)) + \frac{1}{2M}x(h(t))\int_{h(t)}^{t} p(s)ds \leq 0$$

This implies

$$x(t) - x(h(t)) + \left[1 - \frac{1}{2M}\int_{h(t)}^{t} p(s)ds\right] \leq 0$$

and hence

$$\int_{h(t)}^{t} p(s)ds < 2M$$

for sufficiently $t$. Therefore,

$$\limsup_{t\to\infty}\int_{h(t)}^{t} p(s)ds \leq 2M$$

This is a contradiction to (2.13). The proof is completed. □

Now, assume that $f$ is nondecreasing function then we have the following result.

**Theorem 2.3.** *Assume that (1.2), (1.3), (2.2) and (2.4) hold. If $\tau(t)$ is non-monotone, $f$ is nondecreasing function and*

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds > M \tag{2.15}$$

*where $h(t)$ is defined by (2.1), then all solutions of Eq.(1.1) oscillate.*

*Proof.* Assume, for the sake of contradiction, that there exist a nonoscillatory solution $x(t)$ of (1.1). In view of (2.4), we know from Theorem 2.1 that $\lim_{t\to\infty} x(t) = 0$, for $t \geq t_1$.

Considering equation (1.1)

$$x'(t) + p(t)f(x(\tau(t))) = 0$$

Since $\tau(t) \leq h(t)$, $x(t)$ is nonincreasing and $f$ is nondecreasing we have

$$x'(t) + p(t)f(x(h(t))) \leq 0 \tag{2.16}$$

Integrating (2.16) from $h(t)$ to $t$ and using the fact that $x(t)$ is nonincreasing and $f$, $h(t)$ are nondecreasing

$$x(t) - x(h(t)) + \int_{h(t)}^{t} p(s)f(x(h(s)))ds \leq 0$$

or

$$x(t) - x(h(t)) + f(x(h(t))) \int_{h(t)}^{t} p(s)ds \leq 0$$

and so

$$x(t) - x(h(t)) \left[1 - \frac{f(x(h(t)))}{x(h(t))} \int_{h(t)}^{t} p(s)ds\right] \leq 0$$

Therefore

$$\begin{aligned} 1 \quad &> \quad \frac{f(x(h(t)))}{x(h(t))} \int_{h(t)}^{t} p(s)ds \\ &\geq \quad \frac{1}{M} \limsup_{t\to\infty} \int_{h(t)}^{t} p(s)ds \end{aligned}$$

That is a contradiction. The proof is completed. □

We remark that if $\tau(t)$ is nondecreasing, then we have $\tau(t) = h(t)$ for all $t$, and the condition (2.13) and (2.15), respectively, reduce to

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > 2M \tag{2.16}$$

and

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds > M \tag{2.17}$$

Now, we have the following example.

**Example 2.1.** *Consider the nonlinear delay differential equation*

$$x'(t) + \frac{1}{e} x(\tau(t)) \ln(10 + |x(\tau(t))|) = 0, \quad t > 0, \tag{2.18}$$

*where*

$$\tau(t) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ -3t+12k+3, & \text{if } t \in [3k+1, 3k+2] \\ 5t-12k-13, & \text{if } t \in [3k+2, 3k+3] \end{cases}, \qquad k \in \mathbb{N}_0.$$

*By* (2.1)*, we see that*

$$h(t) := \sup_{s \le t} \tau(s) = \begin{cases} t-1, & \text{if } t \in [3k, 3k+1] \\ 3k, & \text{if } t \in [3k+1, 3k+2.6] \\ 5t-12k-13, & \text{if } t \in [3k+2.6, 3k+3] \end{cases}, \qquad k \in \mathbb{N}_0.$$

*If we put* $p(t) = \frac{1}{e}$ *and* $f(x) = x\ln(10+|x|)$. *Then, we have*

$$M = \limsup_{x\to 0} \frac{x}{f(x)} = \limsup_{x\to 0} \frac{x}{x\ln(10+|x|)} = \frac{1}{\ln 10}$$

*and*

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)ds = \frac{1}{e} > \frac{M}{e} = \frac{1}{e\ln 10}$$

that is, all conditions of Theorem 2.1 are satisfied and therefore all solutions of (2.18) oscillate.

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[1]Akdeniz University, Faculty of Science, Department of Mathematics, 07058, Antalya, Turkey

[2]Dumlupinar University, Faculty of Science and Arts, Department of Mathematics, 43000, Kütahya, Turkey

[3]Afyon Kocatepe University, Faculty of Science and Arts, Department of Mathematics, ANS Campus, 03200, Afyon, Turkey

*Corresponding author: ozkanocalan@akdeniz.edu.tr