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ON THE $(p,q)-$STANCU GENERALIZATION OF A GENUINE BASKAKOV-DURRMEYER TYPE OPERATORS

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Abstract. In this paper, we introduce a Stancu generalization of a genuine Baskakov- Durrmeyer type operators via $(p,q)-$ integer. We investigate approximation properties of these operators. Furthermore, we study on the linear positive operators in a weighted space of functions and obtain the rate of these convergence using weighted modulus of continuity.

1. Introduction

In the field of approximation theory, the quantum calculus has been studied for a long time. The generalization of $(p,q)-$ calculus was introduced by Sahai and Yadav in [15]. Recently, a series of papers giving $(p,q)-$ generalizations a sequence of linear positive operators have been published in [3,4,9–13]. Our aim is to give Stancu type generalization, via $(p,q)-$ integer, defined by Agrawal and Thamer as follows

$$B_n(f,x) = (n-1)\sum_{k=1}^{\infty} b_{n,k}(x) \int_0^{\infty} b_{n,k-1}(t) f(t)\, dt + f(0)\,(1+x)_q^{-n}, \tag{1.1}$$

where

$$b_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}.$$

in [5].

We refer reader to [2] for unexplained terminologies and notations.

2. Preliminaries and notations

Let's give a table of some basic formulas, motivated from $q-$calculus, used in $(p,q)-$calculus as the following Table1

$(p,q)-$calculus	Relation with $q-$calculus
$[n]_{p,q} = \frac{p^n - q^n}{p-q}$	$[n]_{p,q} = p^{n-1}[n]_{q/p}$
$[n]_{p,q}! = [1]_{p,q}[2]_{p,q}...[n]_{p,q}$	$[n]_{p,q}! = p^{\binom{n}{2}}[n]_{q/p}!$
$(a \oplus b)_{p,q}^n = (a+b)(ap+bq)...(ap^{n-1}+bq^{n-1})$	$(a \oplus b)_{p,q}^n = p^{\binom{n}{2}}(a+b)_{q/p}^n$
$d_{p,q}f(x) = f(px) - f(qx)$	$d_q f(x) = f(x) - f(qx)$

Table 1

Recall that the beta function, introduced [14], in $q-$ calculus is defined by

$$B_q(n,k) = K(A,n) \int_0^{\infty/A} \frac{t^{k-1}}{(1+t)_q^{n+k}} d_q t, \tag{2.1}$$

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where

$$K(A,n)=\frac{A^n}{1+A}\left(1+\frac{1}{A}\right)_q^n(1+A)_q^{1-s},\ A>0. \tag{2.2}$$

In the formula (2.2), $K(A,n)=q^{n(n-1)/2}$ and $K(A,0)=1$ for $n\in\mathbb{N}$. Inspiring the formula (2.1), we introduce (p,q)-beta functions $B_{p,q}(n,k)$, as a generalization of $B_q(n,k)$, $A>0$ and $n,k\in\mathbb{N}\setminus\{0\}$, defined by

$$B_{p,q}(k,n)=p^{\binom{n}{2}}q^{\binom{k}{2}}\int_0^{\infty/A}\frac{t^{k-1}}{(1\oplus t)_{p,q}^{n+k}}d_{p,q}t. \tag{2.3}$$

If $p=1$ is replaced in (2.3), then the formula is reduced to (2.1).

3. Genuine type Stancu generalization via $(p,q)-$integer

Let's start to give a Stancu type $(p,q)-$generalization of these operators in (1.1). For $0\leq\alpha,\beta$ and $0<q<p\leq 1$, these operators are defined as follows;

$$\begin{aligned}B_{p,q,n}^{\alpha,\beta}(f,x) &= [n-1]_{p,q}\sum_{k=1}^{\infty}\Bigg\{b_{n,k}(p,q,x)p^{(n-1)^2+k}q^{k(k-1)}\\ &\quad\times\int_0^{\infty/A}b_{n,k-1}(p,q,t)f\left(\frac{[n]_{p,q}t+\alpha}{[n]_{p,q}+\beta}\right)d_{p,q}t\Bigg\}\\ &\quad+f\left(\frac{\alpha}{[n]_{p,q}+\beta}\right)p^{\binom{n}{2}}(1\oplus x)_{p,q}^{-n},\end{aligned} \tag{3.1}$$

where

$$b_{n,k}(p,q,t)=\begin{bmatrix} n+k-1\\ k\end{bmatrix}_{p,q}\frac{t^k}{(1\oplus t)_{p,q}^{n+k}}.$$

If one replaces $p=q=1$ and $\alpha=\beta=0$ in (3.1), then the operators $B_{p,q,n}^{\alpha,\beta}$ are reduced to the operators $B_n$in (1.1). Similar type operators studied in [1,7,16].

To obtain our main results, we need calculating the values of Korovkin monomial functions.

Lemma 3.1. *The following equalities are satisfied for* $e_m(t)=t^m$, $m=0,1,2$ *and* $n>3$

$$\begin{aligned}B_{p,q,n}^{\alpha,\beta}(1,x) &= 1,\\ B_{p,q,n}^{\alpha,\beta}(t,x) &= \frac{[n]_{p,q}^2}{pq\left([n]_{p,q}+\beta\right)[n-2]_{p,q}}x+\frac{\alpha}{[n]_{p,q}+\beta},\\ B_{p,q,n}^{\alpha,\beta}(t^2,x) &= \frac{[n]_{p,q}^3[n+1]_{p,q}}{p^2q^4\left([n]_{p,q}+\beta\right)^2[n-2]_{p,q}[n-1]_{p,q}}x^2\\ &\quad+\left(\frac{p^{n-4}[2]_{p,q}[n]_{p,q}^3}{q^3\left([n]_{p,q}+\beta\right)^2[n-2]_{p,q}[n-1]_{p,q}}+\frac{2\alpha[n]_{p,q}^2}{pq\left([n]_{p,q}+\beta\right)^2[n-2]_{p,q}}\right)x\\ &\quad+\frac{\alpha^2}{\left([n]_{p_n,q_n}+\beta\right)^2}.\end{aligned}$$

Proof. By the definition $(p,q)-$beta functions in (2.3), we obtain the estimate,

$$\begin{aligned}\int_0^{\infty/A}b_{n,k-1}(p,q;t)t^m d_{p,q}t &= \begin{bmatrix} n+k-2\\ k-1\end{bmatrix}_{p,q}\int_0^{\infty/A}\frac{t^{k+m-1}}{(1\oplus t)_{p,q}^{n+k-1}}d_{p,q}t\\ &= \frac{[k+m-1]_{p,q}!\,[n-m-2]_{p,q}!}{[k-1]_{p,q}!\,[n-1]_{p,q}!p^{\binom{n-m-1}{2}}q^{\binom{k+m}{2}}}.\end{aligned} \tag{3.2}$$

If we apply the operators in (3.1) to the equality (3.2) for $m=0$, we get

$$\begin{aligned}
&B^{\alpha,\beta}_{p,q,n}(1,x)\\
=\ &[n-1]_{p,q}\sum_{k=1}^{\infty}\Big\{b_{n,k}(p,q;x)p^{(n-1)^2+k}q^{k(k-1)}\\
&\times\int_0^{\infty/A} b_{n,k-1}(p,q;t)d_{p,q}t\Big\}+p^{\binom{n}{2}}(1\oplus x)^{-n}_{p,q}\\
=\ &[n-1]_{p,q}\sum_{k=1}^{\infty}b_{n,k}(p,q;x))\frac{p^{(n-1)^2+k}q^{k(k-1)}}{[n-1]_{p,q}p^{\binom{n-1}{2}}q^{\binom{k}{2}}}\\
&+p^{\binom{n}{2}}(1\oplus x)^{-n}_{p,q}\\
=\ &\sum_{k=0}^{\infty}\frac{[n+k-1]_{p,q}!}{[k]_{p,q}!\,[n-1]_{p,q}!}\frac{p^{\binom{n}{2}}q^{\binom{k}{2}}\,(px)^k}{(1\oplus x)^{n+k}_{p,q}}\\
=\ &\sum_{k=0}^{\infty}p^{\binom{n}{2}}q^{\binom{k}{2}}b_{n,k}(p,q;px)\\
=\ &1.
\end{aligned}$$

And the proof of (i) is finished. With the direct computation, we obtain (ii) as follows:

$$\begin{aligned}
B^{\alpha,\beta}_{p,q,n}(t,x) &= [n-1]_{p,q}\sum_{k=1}^{\infty}\Big\{b_{n,k}(p,q;x)p^{(n-1)^2+k}q^{k(k-1)}\\
&\quad\times\int_0^{\infty/A} b_{n,k-1}(p,q;t)\left(\frac{[n]_{p,q}t+\alpha}{[n]_{p,q}+\beta}\right)d_{p,q}t\Big\}\\
&\quad+\frac{\alpha}{[n]_{p,q}+\beta}p^{\binom{n}{2}}(1\oplus x)^{-n}_{p,q}\\
&= \frac{[n]_{p,q}}{[n]_{p,q}+\beta}\sum_{k=1}^{\infty}b_{n,k}(p,q;x)\frac{p^{(n-1)^2+k}q^{k(k-1)}[k]_{p,q}}{[n-2]_{p,q}p^{\binom{n-2}{2}}q^{\binom{k+1}{2}}}\\
&\quad+\frac{\alpha}{[n]_{p,q}+\beta}B^{\alpha,\beta}_{p,q,n}(1,x)\\
&= \frac{[n]^2_{p,q}x}{pq\big([n]_{p,q}+\beta\big)[n-2]_{p,q}}\sum_{k=0}^{\infty}p^{\binom{n+1}{2}}q^{\binom{k}{2}}b_{n+1,k}(p,q;px)\\
&\quad+\frac{\alpha}{[n]_{p,q}+\beta}\\
&= \frac{[n]^2_{p,q}}{pq\big([n]_{p,q}+\beta\big)[n-2]_{p,q}}x+\frac{\alpha}{[n]_{p,q}+\beta}.
\end{aligned}$$

For (iii),

$$\begin{aligned}
B^{\alpha,\beta}_{p,q,n}(t^2,x) &= [n-1]_{p,q}\sum_{k=1}^{\infty}\Big\{b_{n,k}(p,q;x)p^{(n-1)^2+k}q^{k(k-1)}\\
&= \times\int_0^{\infty/A} b_{n,k-1}(p,q;t)\left(\frac{[n]_{p,q}t+\alpha}{[n]_{p,q}+\beta}\right)^2 d_{p,q}t\Big\}\\
&\quad+\left(\frac{\alpha}{[n]_{p,q}+\beta}\right)^2 p^{\binom{n}{2}}(1\oplus x)^{-n}_{p,q}.
\end{aligned}$$

Using the equality

$$[k]_{p,q} = q^{k-s}[s]_{p,q} + p^s[k-s]_{p,q},\ 0 \le s \le k, \tag{3.3}$$

we have

$$\begin{aligned}
&B^{\alpha,\beta}_{p,q,n}(t^2,x)\\
=\ &\frac{[n]^2_{p,q}}{\left([n]_{p,q}+\beta\right)^2}\sum_{k=2}^{\infty}\left\{\frac{[k]_{p,q}\left(p^2[k-1]_{p,q}+q^{k-1}[2]_{p,q}\right)p^{\frac{n^2+3n+2k-10}{2}}q^{\frac{k^2-5k-2}{2}}}{[n-2]_{p,q}[n-3]_{p,q}}\right.\\
&\left.\times\, b_{n,k}(p,q;x)\right\}\\
&+\frac{2\alpha\,[n]_{p,q}}{[n]_{p,q}+\beta}\sum_{k=1}^{\infty}\frac{[k]_{p,q}}{[n-2]_{p,q}}p^{\frac{n^2+n+2k-4}{2}}q^{\frac{k^2-3k}{2}}b_{n,k}(p,q;x)\\
&+\frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2}B^{\alpha,\beta}_{p,q,n}(1,x)
\end{aligned}$$

Then

$$\begin{aligned}
&B^{\alpha,\beta}_{p,q,n}(t^2,x)\\
=\ &\frac{[n]^3_{p,q}[n+1]_{p,q}x^2}{p^2q^4\left([n]_{p,q}+\beta\right)^2[n-2]_{p,q}[n-3]_{p,q}}\sum_{k=0}^{\infty}p^{\binom{n+2}{2}}q^{\binom{k}{2}}b_{n+2,k}(p,q;px)\\
&+\left(\frac{p^{n-4}[2]_{p,q}[n]^2_{p,q}}{q^3\left([n]_{p,q}+\beta\right)^2[n-2]_{p,q}[n-1]_{p,q}}+\frac{2\alpha[n]^2_{p,q}}{pq([n]_{p,q}+\beta)^2[n-2]_{p,q}}\right)x\\
&\times\sum_{k=0}^{\infty}p^{\binom{n+1}{2}}q^{\binom{k}{2}}b_{n+1,k}(p,q;x)\\
&+\frac{\alpha^2}{\left([n]_{p,q}+\beta\right)^2}B^{\alpha,\beta}_{p,q,n}(1,x).
\end{aligned}$$

And so we have completed the proof of (iii). □

Now we consider that $B[0,\infty)$ denotes the set of all bounded functions from $[0,\infty)$ to $\mathbb{R}$, $B[0,\infty)$ is a normed space with the norm $\|f\|_B = \sup\{|f(x)| : x \in [0,\infty)\}$ and $C_B[0,\infty)$ denotes the subspace of all continuous functions in $B[0,\infty)$. We denote first modulus of continuity on finite interval $[0,b]$, $b>0$

$$\omega_{[0,b]}(f,\delta) = \sup_{0<h\le\delta,x\in[0,b]}|f(x+h)-f(x)|\,. \tag{3.4}$$

The Peetre's $K-$functional is defined by

$$K_2(f,\delta) = \inf\left\{\|f-g\|_B + \delta\,\|g''\|_B : g \in W^2_\infty\right\},\ \delta>0$$

where $W^2_\infty = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By [6, p. 177, Theorem 2.4], there exists a positive constant C such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta})$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}}\ \sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)-f(x)|\,. \tag{3.5}$$

The weighted Korovkin- type theorems were proved by Gadzhiev [8]. We give the Gadzhiev's results in weighted spaces. $B_\rho[0,\infty)$ denotes the set of all functions f, from $[0,\infty)$ to $\mathbb{R}$, satisfying growth condition $|f(x)| \le N_f\rho(x)$, where $\rho(x) = 1+x^2$ and N_f is a constant depending only on f. $B_\rho[0,\infty)$ is a normed space with the norm $\|f\|_\rho = \sup\left\{\dfrac{|f(x)|}{\rho(x)} : x \in \mathbb{R}\right\}$. $C_\rho[0,\infty)$ denotes the subspace of all continuous functions in $B_\rho[0,\infty)$ and $C^*_\rho[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ for which $\lim\limits_{|x|\to\infty}\dfrac{|f(x)|}{\rho(x)}$ exists finitely.

We define a genuine type generalization to the operators $B_{p,q,n}^{\alpha,\beta}$ given in (3.1) as follows

$$\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}(f,x):=\begin{cases} B_{p_n,q_n,n}^{\alpha,\beta}(f,\sigma(\alpha,\beta,p_n,q_n,n;x)) & , \quad x>\frac{\alpha}{[n]_{p_n,q_n}+\beta} \\ f(x) & , \quad 0\le x\le \frac{\alpha}{[n]_{p_n,q_n}+\beta}\end{cases}$$

where

$$\sigma(\alpha,\beta,p_n,q_n,n;x)=\frac{p_nq_n\left([n]_{p_n,q_n}+\beta\right)[n-2]_{p_n,q_n}}{[n]_{p_n,q_n}^2}\left(x-\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right).$$

Notice that these operators $\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}$ are defined from the space $C_\rho^*[0,\infty)$ into $B_\rho[0,\infty)$. To satisfy hypothesis of Korovkin's Theorem, we assume that $\lim_{n\to\infty} p_n^n$ and $\lim_{n\to\infty} q_n^n$ are real numbers when $\lim_{n\to\infty} p_n=1$ and $\lim_{n\to\infty} q_n=1$ for $0<q_n<p_n\le 1$ and $n>3$. On the other hand, since the operators $\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}(f,x)$ are defined as $f(x)$ on the interval $\left[0,\frac{\alpha}{[n]_{p_n,q_n}+\beta}\right]$, it is enough to examine the approximation properties of these operators at the interval $\left(\frac{\alpha}{[n]_{p_n,q_n}+\beta},\infty\right)$. The following lemma can be obtained with the help of Lemma 3.1.

Lemma 3.2. *The operators $\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}$ satisfy the following equalities for $x>\frac{\alpha}{[n]_{p_n,q_n}+\beta}$ and $e_m(t)=t^m$, $m=0,1,2$*

$$\begin{aligned}
\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}(1,x) &= 1,\\
\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}(t,x) &= x,\\
\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}(t^2,x) &= \frac{[n+1]_{p_n,q_n}[n-2]_{p_n,q_n}}{q_n^2[n-1]_{p_n,q_n}[n]_{p_n,q_n}}x^2\\
&\quad+\left(\frac{-2\alpha[n+1]_{p_n,q_n}[n-2]_{p_n,q_n}}{q_n^2\left([n]_{p_n,q_n}+\beta\right)[n-1]_{p_n,q_n}[n]_{p_n,q_n}}+\frac{p_n^{n-3}[2]_{p_n,q_n}[n]_{p_n,q_n}}{q_n^2\left([n]_{p_n,q_n}+\beta\right)[n-1]_{p_n,q_n}}\right.\\
&\quad\left.+\frac{2\alpha}{\left([n]_{p_n,q_n}+\beta\right)}\right)x-\frac{p_n^{n-3}[2]_{p_n,q_n}[n]_{p_n,q_n}\alpha}{q_n^2\left([n]_{p_n,q_n}+\beta\right)^2[n-1]_{p_n,q_n}}.
\end{aligned}$$

We need computing the second moment before giving our main results .

Lemma 3.3. *We have the following inequality*

$$\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}\left((t-x)^2,x\right)\le\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1)$$

for $x>\frac{\alpha}{[n]_{p_n,q_n}+\beta}$.

Proof. By Lemma 3.2, we write the second moment as

$$\begin{aligned}
&\widetilde{B}_{p_n,q_n,n}^{\alpha,\beta}\left((t-x)^2,x\right)\\
=&\left(\frac{[n+1]_{p_n,q_n}[n-2]_{p_n,q_n}}{q_n^2[n-1]_{p_n,q_n}[n]_{p_n,q_n}}-1\right)x^2\\
&+\left(\frac{-2\alpha[n+1]_{p_n,q_n}[n-2]_{p_n,q_n}}{q_n^2\left([n]_{p_n,q_n}+\beta\right)[n-1]_{p_n,q_n}[n]_{p_n,q_n}}+\frac{p_n^{n-3}[2]_{p_n,q_n}[n]_{p_n,q_n}}{q_n^2\left([n]_{p_n,q_n}+\beta\right)[n-1]_{p_n,q_n}}+\frac{2\alpha}{\left([n]_{p_n,q_n}+\beta\right)}\right)x\\
&-\frac{p_n^{n-3}[2]_{p_n,q_n}[n]_{p_n,q_n}\alpha}{q_n^2\left([n]_{p_n,q_n}+\beta\right)^2[n-1]_{p_n,q_n}}\\
\le&\left|\frac{[n+1]_{p_n,q_n}[n-2]_{p_n,q_n}}{q_n^2[n-1]_{p_n,q_n}[n]_{p_n,q_n}}-1\right|x^2\\
&+\left(\frac{p_n^{n-3}[2]_{p_n,q_n}[n]_{p_n,q_n}}{q_n^2\left([n]_{p_n,q_n}+\beta\right)[n-1]_{p_n,q_n}}+\frac{2\alpha}{\left([n]_{p_n,q_n}+\beta\right)}\right)x.
\end{aligned}$$

Considering the following equalities

$$\begin{aligned}
[n+1]_{p_n,q_n} &= p_n^3\,[n-2]_{p_n,q_n}+q_n^{n-3}\,[3]_{p_n,q_n}\,,\\
[n]_{p_n,q_n} &= p_n^2\,[n-2]_{p_n,q_n}+q_n^{n-2}\,[2]_{p_n,q_n}\,,\\
[n-1]_{p_n,q_n} &= p_n\,[n-2]_{p_n,q_n}+q_n^{n-1}\,[1]_{p_n,q_n}\,,
\end{aligned}$$

we get

$$\begin{aligned}\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}((t-x)^2,x) &\leq \left(\frac{(p_n^3-p_n^3q_n^2)}{q_n^2}+\frac{[3]_{p_n,q_n}}{[n]_{p_n,q_n}}\right)x^2 \\ &\quad+\left(\frac{p_n^{n-3}[2]_{p_n,q_n}[n]_{p_n,q_n}}{q_n^2\left([n]_{p_n,q_n}+\beta\right)[n-1]_{p_n,q_n}}+\frac{2\alpha}{\left([n]_{p_n,q_n}+\beta\right)}\right)x \\ &\leq \left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1).\end{aligned}$$

And the proof of the Lemma 3.3 is now finished. □

Thus we are ready to give direct results.

Lemma 3.4. *We have the inequality for every $x\in[0,\infty)$ and $f''\in C_B[0,\infty)$*

$$\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x)-f(x)\right|\leq \delta^{\alpha,\beta}_{p_n,q_n,n}(x)\left\|f''\right\|_B,$$

where $\delta^{\alpha,\beta}_{p_n,q_n,n}(x):=\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1).$

Proof. Using Taylor's expansion

$$f(t)=f(x)+(t-x)f'(x)+\int_x^t(t-u)f''(u)du$$

and the Lemma 3.2, we have the following equality

$$\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x)-f(x)=\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}\left(\int_x^t(t-u)f''(u)du;x\right).$$

On the other hand, combining the inequality

$$\left|\int_x^t(t-u)f''(u)du\right|\leq\|f''\|_B\frac{(t-x)^2}{2},$$

and Lemma 3.3, we get

$$\begin{aligned}\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x)-f(x)\right| &= \left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}\left(\int_x^t(t-u)f''(u)du,x\right)\right| \\ &\leq \frac{\|f''\|_B}{2}\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}((t-x)^2,x) \\ &\leq \frac{\|f''\|_B}{2}\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1),\end{aligned}$$

as desired. □

Theorem 3.1. *We have the following inequality*

$$\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x)-f(x)\right|\leq 2C\omega_2\left(f,\sqrt{\delta^{\alpha,\beta}_{p_n,q_n,n}(x)}\right),$$

where $\delta^{\alpha,\beta}_{p_n,q_n,n}(x)=\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1).$

Proof. For any $g\in W_\infty^2$, we obtain the inequality

$$\begin{aligned}&\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x)-f(x)\right| \\ &\leq \left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f-g,x)-(f-g)(x)+\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(g,x)-g(x)\right|.\end{aligned}$$

Then, Lemma 3.4, we have

$$\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x)-f(x)\right|\leq 2\left\|f-g\right\|_B+\delta^{\alpha,\beta}_{p_n,q_n,n}(x)\left\|g''\right\|_B.$$

Taking infimum over $g\in W^2_\infty$ on the right side of the above inequality and using the inequality (3.5), we reach the desired result. □

Theorem 3.2. *For every $f\in C^*_\rho[0,\infty)$, we have the following the limit*

$$\lim_{n\to\infty}\left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f)-f\right\|_\rho=0.$$

Proof. From Lemma 3.2 , it is obvious that $\left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(e_0)-e_0\right\|_\rho=0$ and $\left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(e_1)-e_1\right\|_\rho=0$. We have

$$\begin{aligned}\left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(e_2)-e_2\right\|_\rho &= \frac{\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}\left((t-x)^2,x\right)}{1+x^2}\\ &\leq \sup_{x\in[0,\infty)}\left\{\frac{\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1)}{1+x^2}\right\}\\ &\leq 2\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right).\end{aligned}$$

Then we get $\lim\limits_{n\to\infty}\left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(e_2)-e_2\right\|_\rho=0.$

Thus, from A. D. Gadzhiev's Theorem in [8], we obtain the proof of Theorem 3.2. □

Lemma 3.5. *We assume that $\frac{\alpha}{[n]_{p_n,q_n}+\beta}<b$. Then we have the following inequality*

$$\left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f;x)-f(x)\right\|_{C[0,b]}\leq N\left\{(1+b)^2\delta^{\alpha,\beta}_{p_n,q_n,n}(b)+\omega_{[0,b+1]}\left(f;\sqrt{\delta^{\alpha,\beta}_{p_n,q_n,n}(b)}\right)\right\},$$

where $\delta^{\alpha,\beta}_{p_n,q_n,n}(x)=\left(\frac{p_n^3-p_n^3q_n^2}{q_n^2}+\frac{10(\alpha+\beta+1)}{q_n^2\left([n]_{p_n,q_n}+\beta\right)}\right)x(x+1)$, for every $f\in C^*_\rho[0,\infty)$.

Proof. Let $x\in\left[\frac{\alpha}{[n]_{p_n,q_n}+\beta},b\right]$ and $t>b+1$. Since $t-x>1$, we have

$$\begin{aligned}|f(t)-f(x)| &\leq K_f(2+(t-x+x)^2+x^2)\\ &\leq 3K_f(1+b)^2(t-x)^2.\end{aligned}\tag{3.6}$$

Let $x\in\left[\frac{\alpha}{[n]_{p_n,q_n}+\beta},b\right]$, $t<b+1$ and $\delta>0$. Then, we have

$$|f(t)-f(x)|\leq\left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta).\tag{3.7}$$

Due to (3.6) and (3.7), we can write

$$|f(t)-f(x)|\leq 3K_f(1+b)^2(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{[0,b+1]}(f,\delta).$$

After, using Cauchy- Schwarz' s inequality , we get

$$\begin{aligned}&\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f;x)-f(x)\right|\\ \leq\ & 3K_f(1+b)^2B^{\alpha,\beta}_{p_n,q_n,n}\left((t-x)^2,x\right)\\ &+\omega_{[0,b+1]}(f;\delta)\left[1+\frac{1}{\delta}\left(\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}\left((t-x)^2,x\right)\right)^{1/2}\right]\\ \leq\ & 3K_f(1+b)^2\delta^{\alpha,\beta}_{p_n,q_n,n}(x)+\omega_{[0,b+1]}(f;\delta)\left[1+\frac{1}{\delta}\left(\delta^{\alpha,\beta}_{p_n,q_n,n}(x)\right)^{1/2}\right].\end{aligned}$$

Considering Lemma 3.3 and choosing
$$\delta^2 := \delta^{\alpha,\beta}_{p_n,q_n,n}(b)$$
and $N = \max\{3K_f, 2\}$. We reach the proof of Lemma 3.5. □

Theorem 3.3. *For every $f \in C^*_\rho[0,\infty)$ and $\gamma > 0$, we have the limit*
$$\lim_{n\to\infty} \sup_{x\geq 0} \frac{\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x) - f(x)\right|}{1+x^{2+\gamma}} = 0.$$

Proof. For $\gamma > 0$, $f \in C^*_\rho[0,\infty)$ and $b \geq 1$, using (3.4) the following inequality is satisfied
$$\begin{aligned}
&\sup_{x\geq 0} \frac{\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x) - f(x)\right|}{1+x^{2+\gamma}} \\
\leq\ & \sup_{0\leq x<b} \frac{\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x) - f(x)\right|}{1+x^{2+\gamma}} + \sup_{b\leq x} \frac{\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x) - f(x)\right|}{1+x^{2+\gamma}} \\
\leq\ & \left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f) - f\right\|_{C[0,b]} + \sup_{b\leq x} \frac{\left|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f,x) - f(x)\right|}{1+x^{2}} \\
\leq\ & \left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f) - f\right\|_{C[0,b]} + \left\|\widetilde{B}^{\alpha,\beta}_{p_n,q_n,n}(f) - f\right\|_{\rho}.
\end{aligned}$$
Using Lemma 3.5 and Theorem 3.2 , we get the proof of Theorem 3.3 . □

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