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AN APPLICATION OF δ-QUASI MONOTONE SEQUENCE

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ABSTRACT. In this paper, a known theorem dealing with $|A,p_n|_k$ summability method of infinite series has been generalized to $|A,p_n;\delta|_k$ summability method. Also, some results have been obtained.

1. INTRODUCTION

A sequence (d_n) is said to be δ-quasi-monotone, if $d_n \to 0$, $d_n > 0$ ultimately and $\Delta d_n \geq -\delta_n$, where $\Delta d_n = d_n - d_{n+1}$ and $\delta = (\delta_n)$ is a sequence of positive numbers (see [1]). Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1)\,. \tag{1.1}$$

The sequence-to-sequence transformation

$$z_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.2}$$

defines the sequence (z_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said to be summable $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |\Delta z_{n-1}|^k < \infty, \tag{1.3}$$

where

$$\Delta z_{n-1} = -\frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} a_v, \quad n \geq 1.$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.4}$$

The series $\sum a_n$ is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [6])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} |\bar{\Delta} A_n(s)|^k < \infty, \tag{1.5}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

If we set $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|A, p_n|_k$ summability (see [8]). If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability.

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In the special case $\delta = 0$ and $p_n = 1$ for all n, $|A, p_n; \delta|_k$ summability is the same as $|A|_k$ summability (see [9]). Also, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n; \delta|_k$ summability (see [4]).

Before stating the main theorem we must first introduce some further notations.

Given a normal matrix $A = (a_{nv})$, we associate two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{1.6}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.7}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{1.8}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1.9}$$

2. Known Results

In [3], Bor has proved the following theorem dealing with $|\bar{N}, p_n|_k$ summability.

Theorem 2.1. *Let (X_n) be a positive non-decreasing sequence, $(\lambda_n) \to 0$ as $n \to \infty$ and (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \qquad as \quad n \to \infty. \tag{2.1}$$

Suppose that there exist a sequence of numbers (B_n) which is δ-quasi monotone with $\sum nX_n\delta_n < \infty$, $\sum B_n X_n$ is convergent and $|\Delta\lambda_n| \le |B_n|$ for all n. If

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |t_n|^k = O(X_m) \quad as \qquad m \to \infty, \tag{2.2}$$

then the series $\sum a_n \lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \ge 1$.

Later on, in [7], Özarslan and Şakar have proved the following theorem dealing with $|A, p_n|_k$ summability factors of infinite series.

Theorem 2.2. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0} = 1, \quad n = 0, 1, ..., \tag{2.3}$$

$$a_{n-1,v} \ge a_{nv} \quad for \quad n \ge v + 1, \tag{2.4}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{2.5}$$

$$|\hat{a}_{n,v+1}| = O\left(v\,|\Delta_v \hat{a}_{nv}|\right). \tag{2.6}$$

If (X_n) is a positive non-decreasing sequence and the conditions of Theorem 2.1 are satisfied, then the series $\sum a_n \lambda_n$ is summable $|A, p_n|_k$, $k \ge 1$.

3. Main Result

The purpose of this paper is to generalize Theorem 2.2 for $|A,p_n;\delta|_k$ summability. Now, we shall prove the following more general theorem.

Theorem 3.1. *Let $A=(a_{nv})$ be a positive normal matrix such that*

$$\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v\hat{a}_{nv}|=O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\}\quad as\quad m\to\infty. \tag{3.1}$$

If all conditions of Theorem 2.2 with condition (2.2) replaced by:

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|t_n|^k=O(X_m)\quad as\quad m\to\infty, \tag{3.2}$$

are satisfied, then the series $\sum a_n\lambda_n$ is summable $|A,p_n;\delta|_k$, $k\geq 1$ and $0\leq\delta<1/k$.

We require the following lemmas for the proof of Theorem 3.1.

Lemma 3.1. *([3]). Under the conditions of Theorem 3.1, we have that*

$$|\lambda_n|\,X_n=O\,(1)\quad as\quad n\to\infty. \tag{3.3}$$

Lemma 3.2. *([3]). Let (X_n) be a positive non-decreasing sequence. If (B_n) is δ-quasi monotone with $\sum nX_n\delta_n<\infty$ and $\sum B_nX_n$ is convergent, then*

$$nB_nX_n=O\,(1)\quad as\quad n\to\infty, \tag{3.4}$$

$$\sum_{n=1}^{\infty}nX_n|\Delta B_n|<\infty. \tag{3.5}$$

4. Proof of Theorem 3.1

Let (I_n) denotes A-transform of the series $\sum a_n\lambda_n$. Then, by (1.8) and (1.9), we have

$$\bar{\Delta}I_n=\sum_{v=0}^{n}\hat{a}_{nv}a_v\lambda_v=\sum_{v=1}^{n}\frac{\hat{a}_{nv}\lambda_v}{v}va_v.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}\bar{\Delta}I_n &= \sum_{v=1}^{n-1}\Delta_v\left(\frac{\hat{a}_{nv}\lambda_v}{v}\right)\sum_{r=1}^{v}ra_r+\frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n}ra_r\\ &= \sum_{v=1}^{n-1}\frac{v+1}{v}\Delta_v\left(\hat{a}_{nv}\right)\lambda_vt_v+\sum_{v=1}^{n-1}\frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_vt_v\\ &+ \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v}+\frac{n+1}{n}a_{nn}\lambda_nt_n\\ &= I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.\end{aligned}$$

To complete the proof of Theorem 3.1, by Minkowski's inequality, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,r}|^k<\infty,\quad for\quad r=1,2,3,4.$$

First, when $k>1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\ &\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}.\end{aligned}$$

By (1.6) and (1.7), we have that

$$\Delta_v(\hat{a}_{nv})=\hat{a}_{nv}-\hat{a}_{n,v+1}=\bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}=a_{nv}-a_{n-1,v}.$$

Thus using (1.6), (2.3) and (2.4)

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})\le a_{nn}.$$

Hence, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|\lambda_v||t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|t_r|^k\\ &+ O(1)|\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v+O(1)|\lambda_m|X_m\\ &= O(1)\sum_{v=1}^{m-1}B_vX_v+O(1)|\lambda_m|X_m\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.1.

Again, by using Hölder's inequality, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}v\,|\Delta_v(\hat{a}_{nv})|\,|B_v||t_v|^k\right)\\ &\times \left(\sum_{v=1}^{n-1}v\,|\Delta_v(\hat{a}_{nv})|\,|B_v|\right)^{k-1}.\end{aligned}$$

By using (3.4), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}v\,|\Delta_v(\hat{a}_{nv})|\,|B_v||t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}v|B_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}v|B_v||t_v|^k.\end{aligned}$$

Now, applying Abel's transformation to this sum, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{v=1}^{m-1}|\Delta\left(v|B_v|\right)|\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|t_r|^k\\ &+ O(1)m\,|B_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}v|\Delta B_v|X_v+O(1)\sum_{v=1}^{m-1}B_vX_v+O(1)mB_mX_m\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

by virtue of the hypotheses of Theorem 3.1 and Lemma 3.2.
Also, as in $I_{n,1}$, we have that

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,3}|^k &\le \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_{v+1}|^k|t_v|^k\right)\\ &\times \left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\ &= O(1)\sum_{v=1}^{m}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|\lambda_{v+1}||t_v|^k\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

by using (2.5), (2.6), (3.1), (3.2) and (3.3).

Finally, as in $I_{n,1}$, we have that

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|\lambda_n|^k|t_n|^k a_{nn}^k\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n||\lambda_n|^{k-1}|t_n|^k\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n||t_n|^k\\ &= O(1) \quad as \quad m\to\infty,\end{aligned}$$

by using (2.5), (3.1), (3.2) and (3.3). This completes the proof of Theorem 3.1.

It should be noted that if we take $\delta = 0$ in Theorem 3.1, then we get Theorem 2.2. In this case, condition (3.2) reduces to condition (2.2). Also, if we take $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then we get Theorem 2.1.

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