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ON HYERS-ULAM STABILITY FOR NONLINEAR DIFFERENTIAL EQUATIONS OF NTH ORDER

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ABSTRACT. This paper considers the stability of nonlinear differential equations of nth order in the sense of Hyers and Ulam. It also considers the Hyers-Ulam stability for superlinear Emden-Fowler differential equation of nth order. Some illustrative examples are given.

1. INTRODUCTION

In 1940, Ulam [1] posed the stability problem of of functional equations: Given a group $G_1$and a metric group G_2 with metric $\rho(.,.)$. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(x), h(x)) < \delta$ for all $x, y \in G_1$,then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \varepsilon$,for all $x, y \in G_1$? The problem for approximately additive mappings, on Banach spaces, was solved by Hyers [2]. The result obtained by Hyers was generalized by Rassias [3].

During the last two decades many mathematicians have extensively investigated the stability problems of functional equations (see [4-11]).

Alsina and Ger [12] were the first mathematicians who investigated the Hyers-Ulam stability of the differential equation $g' = g$.They proved that if a differentiable function $y : I \to R$ satisfies $|y' - y| \leq \varepsilon$ for all $t \in I$,then there exists a differentiable function $g : I \to R$ satisfying $g'(t) = g(t)$ for any $t \in I$ such that $|g - y| \leq 3\varepsilon$,for all $t \in I$. This result of alsina and Ger has been generalized by Takahasi et al [13] to the case of the complex Banach space valued differential equation $y' = \lambda y$.

Furthermore, the results of Hyers-Ulam stability of differential equations of first order were also generalized by Miura *et al.* [14], Jung [15] and Wang *et al.* [16].

Li [17] established the stability of linear differential equation of second order in the sense of the Hyers and Ulam $y'' = \lambda y$. Li and Shen [18] proved the stability of nonhomogeneous linear differential equation of second order in the sense of the Hyers and Ulam $y'' + p(x)y' + q(x)y + r(x) = 0$, while Gavruta *et al.* [19] proved the Hyers-Ulam stability of the equation $y'' + \beta(x)y = 0$ with boundary and initial conditions.

The author in his study [20] estabilshed the Hyers-Ulam stability of the equations of the second order

$$z'' = F(x, z)$$

with the initial conditions $z(x_0) = 0 = z'(x_0)$.

In this paper we investigate the Hyers-Ulam stability of the following nonlinear differential equation of nth order

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$$y^{(n)} = f(t, y, y', y'', ..., y^{(n-1)}) \tag{1}$$

with the initial conditions

$$y(t_0) = y_0\ ,\ y'(t_0) = y_1,\ ...\ ,\ y^{(n-1)}(t_0) = y_{n-1} \tag{2}$$

where $y \in C^{(n)}(I), I = [t_0, t_1], (t, [y]) \equiv (t, y, y', y'', ..., y^{(n-1)})\ \in D,\ t \in I,\ -\infty < t_0 < t_1 < \infty$, and $f(t, y, y', y'', ..., y^{(n-1)})$ is defined on a closed bounded set $D \subset \mathbb{R}^{n+1}$ that satisfies the condition

$$|f(t, [y]) - f(t, [z])| \leq g(t)\frac{|y(t) - z(t)|}{(t_1 - t_0)^{n-1}} \tag{3}$$

where $g(t) : I \to (0, \infty)$ is integrable function.

Moreover we establish the Hyers-Ulam stability of the problem (1),(2) for f satisfying the Lipschitz condition

$$|f(t, [y]) - f(t, [z])| \leq A_0 \sum_{i=0}^{n-1} \left|y^{(i)}(t) - z^{(i)}(t)\right| \tag{4}$$

where $A_0 > 0$.

Definition 1 We will say that the equation (1) has the Hyers -Ulam stability if there exists a positive constant $K > 0$ with the following property:

For every $\varepsilon > 0,\ y \in C^{(n)}(I)$, if

$$|y^{(n)} - f(t, [y])|\ \leq \varepsilon \tag{5}$$

with the initial condition (2), then there exists a solution $z(t) \in C^{(n)}(I)$ of the equation (1), such that $|y(t) - z(t)| \leq K\varepsilon$, where K is a constant that does not depend on ε nor on $y(t)$.

2. MAIN RESULTS ON HYERS-ULAM STABILITY

Theorem 1 If $y \in C^{(n)}(I)$ and $f(t, y, y', y'', ..., y^{(n-1)})$ satisfies condition (3) on a closed bounded set $D \subset \mathbb{R}^{n+1}$, then the initial value problem(1),(2) is stable in the sense of Hyers and Ulam.

Proof. Let $\varepsilon > 0$ and $y(t)$ be an approximate solution of the initial value problem (1),(2).We will show that there exists a function $z(t) \in C^{(n)}(I)$ satisfying the equation (1) and the initial condition (2) such that

$$|y(t) - z(t)| \leq K\varepsilon$$

From the inequality (5) we have

$$-\varepsilon \leq y^{(n)} - f(t, [y]) \leq \varepsilon \tag{6}$$

Integrating the last inequality n times, we obtain

$$\left|y(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^k y_k}{k!} - \frac{1}{(n-1)!}\int_{t_0}^{t}(t - s)^{n-1} f(s, [y])ds\right| \leq \frac{(t - t_0)^n \varepsilon}{n!} \tag{7}$$

It is clear that

$$z(t) = \sum_{k=0}^{n-1} \tfrac{(t-t_0)^k y_k}{k!} + \int_{t_0}^{t} f(s,[z]) \tfrac{(t-s)^{n-1}}{(n-1)!} ds$$

satisfies equation (1) and the initial condition (2).

Consider the difference

$$\begin{aligned} |y(t)-z(t)| &\le \left| y(t) - \sum_{k=0}^{n-1} \tfrac{(t-t_0)^k y_k}{k!} - \tfrac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f(s,[y]) ds \right| \\ &\quad + \left| \int_{t_0}^{t} f(s,[y]) - f(s,[z])) \frac{(t-s)^{n-1}}{(n-1)!} ds \right| \\ &\le \frac{(t-t_0)^n \varepsilon}{n!} + \frac{1}{(n-1)!} \int_{t_0}^{t} \frac{g(s)\,|y(s)-z(s)|}{(t_1-t_0)^{n-1}} (t_1-t_0)^{n-1}\, ds \qquad (8) \end{aligned}$$

Applying Gronwall's inequality, we obtain from inequalities (7) and (8)

$$|y(t)-z(t)| \le \frac{(t_1-t_0)^n \varepsilon}{n!} \exp\left(\frac{1}{(n-1)!} \int_{t_0}^{t} g(t) ds \right)$$

Whence

$$\max_{t_0 \le t \le t_1} |y(t)-z(t)| \le K\varepsilon$$

Hence the initial value problem (1),(2) is stable in the sense of Hyers and Ulam.

Theorem 2 If $y \in C^{(n)}(I)$ and $f(t,y,y',y'',...,y^{(n-1)})$ satisfies the Lipschitz condition (4) on a closed bounded set $D \subset \mathbb{R}^{n+1}$, then the initial value problem(1),(2) is stable in the sense of Hyers and Ulam.

Proof. Given $\varepsilon > 0$, assume that y is an approximate solution of Eq. (1). We will show that there exists a function $z(t) \in C^{(n)}(I)$ satisfying equation (1) such that

$$|y(t)-z(t)| \le K\varepsilon$$

From the inequality (5) we have

$$-\varepsilon \le y^{(n)} - f(t,[y]) \le \varepsilon \qquad (9)$$

By integrating the inequality (9) k times, we obtain

$$\left| y^{(n-k)}(t) - \sum_{j=0}^{k-1} \tfrac{(t-t_0)^j y_{n-k+j}}{j!} - \tfrac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f(s,[y]) ds \right| \le \frac{(t-t_0)^k \varepsilon}{k!} \qquad (10)$$

where $1 \le k \le n$.

We can easily verify that the function $z(t)$

$$z^{(n-k)}(t) = \sum_{j=0}^{k-1} \tfrac{(t-t_0)^j y_{n-k+j}}{j!} + \int_{t_0}^{t} f(s,[z]) \tfrac{(t-s)^{k-1}}{(k-1)!} ds$$

must satisfy the initial value problem (1),(2) Now let $\Delta^{(n-k)} \equiv \left| y^{(n-k)} - \ z^{(n-k)} \right|$.

Then, using the inequalities (4),(10), we get the estimation

$$\Delta^{(n-k)} \leq \left| y^{(n-k)}(t) - \sum_{j=0}^{k-1} \tfrac{(t-t_0)^j y_{n-k+j}}{j!} - \tfrac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} f(s,[y]) ds \right|$$

$$+ \frac{1}{(n-1)!} \int_{t_0}^{t} |f(s,[y]) - f(s,[z])| \, (t-s)^{n-1} ds$$

$$\leq \frac{(t_1-t_0)^k \varepsilon}{k!} + \frac{A_0 n}{(n-1)!} \int_{t_0}^{t} \left| y^{(n-k)}(s) - z^{(n-k)}(s) \right| (t-s)^{n-1} ds \tag{11}$$

Thus, according to (4),(10) and (11), from Gronwall's inequality it follows that

$$\left| y^{(n-k)}(t) - z^{(n-k)}(t) \right| \leq \frac{(t_1-t_0)^k \varepsilon}{k!} \exp\left(\frac{A_0 (t_1-t_0)^n}{(n-1)!} \right)$$

Consequently for $k = n$, we have

$$\max_{t_0 \leq t \leq t_1} |y(t) - z(t)| \leq \frac{(t_1-t_0)^n \varepsilon}{n!} \exp\left(\frac{A_0 (t_1-t_0)^n}{(n)!} \right)$$

Hence the initial value problem (1),(2) is stable in the sense of Hyers and Ulam.

Remark 1 Suppose that $y \in C^{(n)}(I)$ satisfies the inequality (9) with the zero initial condition $y(t_0) = 0,\ y'(t_0) = 0,\ \ldots\ ,\ y^{(n-1)}(t_0) = 0$. If $f(t,[z])$ satisfies Lipschitz condition (4) and $f(t,0,,...,0) \equiv 0$, then one can similarly show that the zero solution $z_0 \equiv 0$ of equation (1) is stable in the sense of Hyers and Ulam.

3. HYERS-ULAM STABILITY FOR SUPERLINEAR NTH ORDER DIFFERENTIAL EQUATION

In this section we investigate the Hyers Ulam stability of solutions for superlinear nth order differential equation

$$y^{(n)} = h(t) \, |y|^{\alpha} \, sgny \quad , \qquad \alpha > 1 \tag{12}$$

with the initial condition

$$y(t_0) = y_0 \ ,\ y'(t_0) = y_1,\ \ldots\ ,\ y^{(n-1)}(t_0) = y_{n-1} \tag{13}$$

where $y \in C^{(n)}(I), I = [t_0, t_1],\ -\infty < t_0 < t_1 < \infty$, and $h(t) : I \to \mathbb{R}$ is continuous.

Theorem 3 If $y \in C^{(n)}(I)$,and $h(t) : I \to \mathbb{R}$ is continuous, then the initial value problem (12),(13) is stable in the sense of Hyers and Ulam.

Proof. Given $\varepsilon > 0$,Suppose $y(t)$ is an approximate solution of the initial value problem (12),(13).We show that there exists an exact solution $z(t) \in C^{(n)}(I)$ satisfying the equation (12) such that

$$|y(t) - z(t)| \leq K\varepsilon$$

where k is a constant that does not explicitly depend on ε nor on $y(t)$.

From the inequality (5) we have

$$-\varepsilon \leq y^{(n)} - h(t) \, |y|^{\alpha} \, sgny \leq \varepsilon \tag{14}$$

By integrating the last inequality n times, we obtain

$$(15)\qquad \left| y(t) - \sum_{k=0}^{n-1} \tfrac{(t-t_0)^k y_k}{k!} - \tfrac{1}{(n-1)!} \int_{t_0}^{t} h(s)\,|y|^{\alpha}\, sgny.(t-s)^{n-1} ds \right| \leq \frac{(t-t_0)^n \varepsilon}{n!}$$

where $1 \leq k \leq n$.

We can easily verify that the function $z(t)$

$$z(t) = \sum_{k=0}^{n-1} \tfrac{(t-t_0)^k y_k}{k!} + \tfrac{1}{(n-1)!} \int_{t_0}^{t} h(s)\,|z|^{\alpha}\, sgnz.(t-s)^{n-1} ds$$

must satisfy the initial value problem (12),(13).

Now since the derivative $\left| \frac{\partial (h(t) y^{\alpha})}{\partial y} \right|$ is bounded on S, then the function $f(t,y) = h(t)\,|y|^{\alpha}\, sgny$ satisfies Lipschitz condition

$$|f(t,y) - f(t,z)| \leq L\,|y(t) - z(t)| \qquad ,(t,y),(t,z) \in S$$

where $S = [t_0, t_1] \times [-M, M] \subset \mathbb{R}^2$, and $M = \max\limits_{t_0 \leq t \leq t_1} |y(t)|$.

Since h is continuous on I, then $\exists\ B_0 > 0$, $|h(t)| \leq B_0$, and from the inequality (15), we get the estimation

$$|y(t) - z(t)| \leq \frac{(t_1 - t_0)^n \varepsilon}{n!} + \frac{B_0 L}{(n-1)!} \int_{t_0}^{t} |y(s) - z(s)|\,(t-s)^{n-1} ds$$

From Gronwall's inequality it follows that

$$|y(t) - z(t)| \leq \frac{(t_1 - t_0)^n \varepsilon}{n!} \exp\left(\frac{B_0 L (t_1 - t_0)^n}{n!} \right)$$

Consequently, we have

$$\max_{t_0 \leq t \leq t_1} |y(t) - z(t)| \leq \frac{(t_1 - t_0)^n \varepsilon}{n!} \exp\left(\frac{(t_1 - t_0)^n}{(n)!} \right)$$

Hence the initial value problem (12),(13) is stable in the sense of Hyers and Ulam.

Remark 2 Suppose that $y \in C^{(n)}(I)$ satisfies the inequality (6) with the zero initial condition $y(t_0) = 0,\ y'(t_0) = 0,\ \ldots,\ y^{(n-1)}(t_0) = 0$. If the function $h : I \to \mathbb{R}$ is continuous, then one can similarly establish the Hyers-Ulam stability of zero solution $z_0 \equiv 0$ of (12).

Example1 Consider the problem

$$(16)\qquad y^{(5)} = 8y^2 \sin t + e^t$$

$$(17)\qquad y^{(k)}(t_0) = 0, \quad k = \overline{0,4}$$

and the inequality

$$(18)\qquad -\varepsilon \leq y^{(5)} - y^2 \sin t + e^t \leq \varepsilon$$

where $(t,y) \in [t_0, t_1] \times [-M_1, M_1]$, $M_1 = \max\limits_{t_0 \leq t \leq t_1} |y(t)|$.

Integrating the inequality (18) five times and using the initial condition (17), we get that

$$\left| y(t) - \frac{1}{3}\int_{t_0}^{t} (y^3 \sin t + e^t)(t-s)^4 ds \right| \leq \frac{(t-t_0)^5 \varepsilon}{5!}$$

One can easily show that $z(t)$

$$z(t) = \frac{1}{3}\int_{t_0}^{t} (z^3 \sin t + e^t)(t-s)^4 ds$$

has to satisfy the initial value problem (16),(17).

Now Let us estimate the difference:

$$\begin{aligned} |y(t) - z(t)| &\leq \left| y(t) - \frac{1}{3}\int_{t_0}^{t} (t-s)^4 (y^3 \sin t + e^t) ds \right| \\ &+ \frac{1}{3}\int_{t_0}^{t} (t-s)^4 \left| y^3 - z^3 \right| |\sin t|\, ds \\ &\leq \frac{(t_1 - t_0)^5 \varepsilon}{5!} + M_1^2 \int_{t_0}^{t} (t-s)^4 |y - z|\, ds \end{aligned}$$

Therefore, we obtain

$$\max_{t_0 \leq t \leq t_1} |y(t) - z(t)| \leq \frac{(t_1 - t_0)^5 \varepsilon}{5!} \exp\left(\frac{M_1^2 (t_1 - t_0)^5}{5!} \right)$$

Hence the initial value problem (16),(17) is stable in the sense of Hyers and Ulam.

4. A SPECIAL CASE OF EQUATION (11)

Consider the equation

$$y^{(n)} = h(t)y \tag{19}$$

with the initial conditions

$$y(t_0) = y_0\ ,\ y'(t_0) = y_1,\ \dots\ ,\ y^{(n-1)}(t_0) = y_{n-1} \tag{20}$$

where $y \in C^{(n)}(I), I = [t_0, t_1],\ -\infty < t_0 < t_1 < \infty$, and $h(t) : I \to \mathbb{R}$ is continuous.

Theorem 4 If $y \in C^{(n)}(I)$,and $h(t) : I \to \mathbb{R}$ is continuous, then the initial value problem(19),(20) is stable in the sense of Hyers and Ulam.

Proof. assume that $\varepsilon > 0$ and that y is n times continuously differentiable real-valued function on $I = [t_0, t_1]$. We will show that there exists a function $z(t) \in c^2(I)$ satisfying equation (19) such that

$$|y(t) - z(t)| \leq K\varepsilon$$

We have

$$-\varepsilon \leq y^{(n)} - h(t)y \leq \varepsilon \tag{21}$$

By integrating the inequality (21) n times, we obtain

$$\left| y(t) - \sum_{k=0}^{n-1} \tfrac{(t-t_0)^k y_k}{k!} - \tfrac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} h(s) y ds \right| \leq \frac{(t-t_0)^n \varepsilon}{n!}$$

where $1 \leq k \leq n$.

It is easily to verify that the function $z(t)$

$$z(t) = \sum_{k=0}^{n-1} \tfrac{(t-t_0)^k y_k}{k!} + \tfrac{1}{(n-1)!} \int_{t_0}^{t} (t-s)^{n-1} h(s) z ds$$

satisfies the initial value problem (19),(20).

One can get the estimation

$$|y(t) - z(t)| \leq \frac{(t_1 - t_0)^n \varepsilon}{n!} + \frac{B_0}{(n-1)!} \int_{t_0}^{t} |y(t) - z(t)| (t-s)^{n-1} ds \tag{22}$$

Using Gronwall's inequality we have

Hence

$$\max_{t_0 \leq t \leq t_1} |y(t) - z(t)| \leq \frac{(t_1 - t_0)^n \varepsilon}{n!} \exp\left(\frac{B_0 (t_1 - t_0)^n}{(n)!} \right)$$

Therefore, the initial value problem (19),(20) is stable in the sense of Hyers and Ulam.

Example 2 Consider the equation

$$y^{(4)} - (1 + \cos t)\, y = 0 \tag{23}$$

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = -1,\ y'''(0) = 0 \tag{24}$$

and the inequality

$$\left| \ y^{(4)} - (1 + \cos t)\, y \ \right| \leq \varepsilon$$

where $0 \leq t \leq b, \quad b \in \mathbb{R}$.

Integrating the last inequality four times, we get

$$\left| y(t) - t + \frac{t^{\,2}}{2} - \frac{1}{6} \int_{0}^{t} (t-s)^3 (1 + \cos t)\, y ds \right| \leq \frac{t^3 \varepsilon}{6}$$

One can easily find that $z(t)$

$$z(t) = t - \frac{t^{\,2}}{2} + \frac{1}{6} \int_{0}^{t} (t-s)^3 (1 + \cos t)\, z ds$$

satisfies the equation (23) and initial condition (24)

Then, we obtain an estimation

$$|y(t) - z(t)| \leq \frac{b^3 \varepsilon}{6} \exp(b^4/12)$$

Hence Eq. (23) has the Hyers -Ulam stability.

References

[1] Ulam S.M., (1964), Problems in Modern Mathematics, John Wiley & Sons, New York, NY, USA, Science edition.

[2] Hyers D. H., (1941), On the stability of the linear functional equation, Proceedings of the National Academy of Sciences of the United States of America, vol. 27, pp. 222–224.

[3] Rassias T. M. (1978), On the stability of the linear mapping in Banach spaces, Proceedings of the American Mathemaical Society, vol. 72, no. 2, pp. 297–300.

[4] Miura T., Takahasi S.-E., Choda H., (2001), On the Hyers-Ulam stability of real continuous function valued differentiable map, Tokyo Journal of Mathematics 24, pp. 467-476.

[5] Jung S. M., (1996), On the Hyers-Ulam-Rassias stability of approximately addi- tive mappings, Journal of Mathematics Analysis and Application 204, pp. 221-226.

[6] Park C. G., (2002), On the stability of the linear mapping in Banach modules,Journal of Mathematics Analysis and Application 275, pp. 711-720.

[7] Gavruta P., (1994), A generalization of t he Hyers-Ulam-Rassias stability of approximately additive mappings, Journal of Mathematical Analysis and Applications, vol. 184, no. 3 , pp. 431–436.

[8] Jun K .-W., Lee Y. -H., (2004), A generalization of the Hyers-Ulam-Rassias stability of the Pexiderized quadratice quations, Journal of Mathematcal Analysis and Applications , vol. 297, no. 1, pp. 70– 86.

[9] Jung S.-M. (2001), Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, Fla, U SA.

[10] Park C. , (2005), Homomorphisms between Poisson JC*-algebras, Bulletin of the Brazilian Mathematical Society, vol. 36, no. 1 , pp. 79–97.

[11] C. Park, Cho Y.-S. and Han M., (2007), Functional inequalities associated with Jordan-von Neumanntype additive functional equations, Journal of Inequalities and Applications , vol. 2007, Article ID 41820, 13 pages.

[12] Alsina C., Ger R., (1998), On some inequalities and stability results related to the exponential function, Journal of Inequalities and Application 2, pp 373-380.

[13] Takahasi E., Miura T., and Miyajima S., (2002), On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$, Bulletin of the Korean Mathematical Society, vol. 39, no. 2, pp 309–315.

[14] Miura T., Miyajima S., Takahasi S.-E., (2007), A characterization of Hyers-Ulam stability of first order linear differential operators, Journal of Mathematics Analysis and Application 286. pp. 136-146.

[15] Jung S. M., (2005), Hyers-Ulam stability of linear differential equations of first order, Journal of Mathematics Analysis and Application 311, pp. 139-146.

[16] Wang G., Zhou M. and Sun L. (2008), Hyers-Ulam stability of linear differential equations of first order, Applied Mathematics Letters 21, pp 1024-1028.

[17] Li Y., (2010), Hyers-Ulam Stability of Linear Differential Equations,Thai Journal of Mathematics,Vol. 8 No 2, pp 215–219.

[18] Li Y. and Shen Y., (2009), Hyers-Ulam Stability of Nonhomogeneous Linear Differential Equations of Second Order, International Journal of Mathematics and Mathematical Sciences, Vol. 2009, Article ID 576852, pp. 7.

[19] Gavruta P., Jung S. and Li Y. , (2011), Hyers-Ulam Stability For Second-Order Linear Differential Equations With Boundary Conditions, EJDE, Vol.2011, No. 80, pp1-7, .http://ejde.math.txstate.edu/Volumes/2011/80/gavruta.pdf.

[20] Qarawani M. N., (2012), Hyers-Ulam Stability of a Generalized Second- Order Nonlinear Differential Equation, Applied Mathematics,Vol. 3, No. 12, pp. 1857-1861. doi: 10.4236/am.2012.312252.

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