International Journal of Analysis and Applications
ISSN 2291-8639
Volume 15, Number 1 (2017), 1-7
http://www.etamaths.com

THE ESSENTIAL SPECTRUM OF A SEQUENCE OF LINEAR OPERATORS IN BANACH SPACES

AYMEN AMMAR[1,*], NOUI DJAIDJA[2] AND AREF JERIBI[1]

Abstract. In this work we introduce some essential spectra ($\sigma_{ei},\ i=1,...,5$) of a sequence of closed linear operators $(T_n)_{n\in\mathbb{N}}$ on Banach space, we prove that if $(T_n)_{n\in\mathbb{N}}$ converges in the generalized sense to a closed linear operator T, then there exists $n_0\in\mathbb{N}$ such that, for every $n\geq n_0$, we have $\sigma_{ei}(\lambda_0-(T_n+B))\subseteq\sigma_{ei}(\lambda_0-(T+B)),\ i=1,...,5$, where B is a bounded linear operator, and $\lambda_0\in\mathbb{C}$. The same treatment is made when (T_n-T) converges to zero compactly.

1. Introduction

Let X and Y be two Banach spaces. We denote by $\mathcal{L}(X,Y)$ (resp., $\mathcal{C}(X,Y)$) the set of all bounded (resp., closed, densely defined) linear operators from X into Y while $\mathcal{K}(X,Y)$ designates the subspace of compact operators from X into Y. If $T\in\mathcal{C}(X,Y)$, we write $\mathcal{N}(T)$ and $\mathcal{R}(T)$ for the null space and range of T, we set $\alpha(T)$=dim$\mathcal{N}(T)$, $\beta(T)=codim\mathcal{R}(T)$. The classes of Fredholm, upper semi-Fredholm and lower semi-Fredholm operators from X into Y are, respectively, the following:

$$\Phi(X,Y):=\big\{T\in\mathcal{C}(X,Y):\ \alpha(T)<\infty \text{ and } \ \beta(T)<\infty\ , R(T) \text{ is closed in} Y\ \big\}.$$

$$\Phi_+(X,Y):=\big\{T\in\mathcal{C}(X,Y):\ \alpha(T)<\infty \text{ and } \ R(T) \text{ is closed in } Y\big\}.$$

$$\Phi_-(X,Y):=\big\{T\in\mathcal{C}(X,Y):\ \beta(T)<\infty \text{ and } R(T) \text{ is closed in Y }\big\}.$$

The set of semi-Fredholm operators from X into Y is defined by

$$\Phi_\pm(X,Y):=\Phi_+(X,Y)\cup\Phi_-(X,Y).$$

The set of Fredholm operators from X into Y is defined by

$$\Phi(X,Y):=\Phi_+(X,Y)\cap\Phi_-(X,Y).$$

For $T\in\Phi_\pm(X,Y)$, the number $i(T)=\alpha(T)-\beta(T)$ is called the index of T.

Definition 1.1. *An operator $F\in\mathcal{L}(X,Y)$ is called a Fredholm perturbation if $T+F\in\Phi(X,Y)$ whenever $T\in\Phi(X,Y)$. F is called an upper (respectively, lower) Fredholm perturbation if $T+F\in\Phi_+(X,Y)$ $\big($respectively, $\Phi_-(X,Y)\big)$ whenever $T\in\Phi_+(X,Y)$ $\big($respectively, $\Phi_-(X,Y)\big)$. The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y)$, $\mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.*

Let $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the set $\Phi(X,Y)\cap\mathcal{L}(X,Y)$, $\Phi_+(X,Y)\cap\mathcal{L}(X,Y)$ and $\Phi_-(X,Y)\cap\mathcal{L}(X,Y)$, respectively.

Definition 1.2. *Let A be a closable linear operator in a Banach space X. The resolvent set and the spectrum of A are, respectively, defined as*
$\rho(A):=\big\{\lambda\in\mathbb{C},$ *such that* $(\lambda-A)$ *is injective and* $(\lambda-A)^{-1}\in\mathcal{L}(X)\big\}$,
$\sigma(A):=\mathbb{C}\backslash\rho(A)$.

Received 26th April, 2017; accepted 6th July, 2017; published 1st September, 2017.
2010 *Mathematics Subject Classification.* 47A10.
Key words and phrases. essential spectra; convergence in the generalized sense; convergence to zero compactly.

Definition 1.3. *Let A be a closed linear operator in a Banach space X. We define the sets*

$\sigma_{e1}(A) := \{\lambda \in \mathbb{C},\ \text{such that } \lambda - A \notin \Phi_+(X)\}$,
$\sigma_{e2}(A) := \{\lambda \in \mathbb{C},\ \text{such that } \lambda - A \notin \Phi_-(X)\}$,
$\sigma_{e3}(A) := \{\lambda \in \mathbb{C},\ \text{such that } \lambda - A \notin \Phi_-(X) \cup \Phi_+(X)\}$,
$\sigma_{e4}(A) := \{\lambda \in \mathbb{C},\ \text{such that } \lambda - A \notin \Phi(X)\}$,
$\sigma_{e5}(A) := \bigcap_{k \in \mathcal{K}(X)} \sigma(T+K)$.

$\sigma_{e1}(.)$ and $\sigma_{e2}(.)$ are the Gustafson and Weidman's essential spectra. $\sigma_{e3}(.)$ is the Kato's essential spectrum. $\sigma_{e4}(.)$ is the Wolf's essential spectrum, and $\sigma_{e5}(.)$ is the Schechter's essential spectrum.

Proposition 1.1. [8, Theorem 7.27, p.172] *Let $T \in \mathcal{C}(X)$. Then $\lambda \notin \sigma_{e5}(T)$ if, and only if, $(\lambda - T) \in \Phi(X)$ and $i(\lambda - T) = 0$.*

Definition 1.4. *Let X be a Banach space and E, F be closed subspaces of X. Let B_E be the unit sphere of E. Let us define*

$$\delta(E,F) := \begin{cases} \sup\limits_{x \in \mathrm{B}_E} dist(x,F), & \text{if } E \neq \{0\}, \\ 0, & \text{otherwise}, \end{cases}$$

and $\widehat{\delta}(E,F) := \max\{\delta(E,F), \delta(F,E)\}$. The quantity $\widehat{\delta}(E,F)$ is called the gap between the subspaces E and F.

Remark 1.1. (i) *The gap measures the distance between two subspaces and it easily follows, from the definitions,*

- (i_1) $\delta(E,F) = \delta(\overline{E}, \overline{F})$ *and* $\widehat{\delta}(E,F) = \widehat{\delta}(\overline{E}, \overline{F})$.
- (i_2) $\delta(E,F) = 0$ *if, and only if,* $\overline{E} \subset \overline{F}$.
- (i_3) $\widehat{\delta}(E,F) = 0$ *if, and only if,* $\overline{E} = \overline{F}$.

(ii) *$\widehat{\delta}(\cdot,\cdot)$ is a metric on the set $\mathcal{V}(X)$ of all linear closed subspaces of X and the convergence $E_n \to F$ in $\mathcal{V}(X)$ is obviously defined by $\widehat{\delta}(E_n, F) \to 0$. Moreover, $(\mathcal{V}(X), \widehat{\delta})$ is a complete metric space.*

Definition 1.5. (i) *Let X and Y be two Banach spaces, and let T, S be two closed linear operators acting from X to Y. Let us define*

$$\delta\big(G(T), G(S)\big) = \sup_{\substack{x \in \mathcal{D}(T) \\ \|x\|^2 + \|Tx\|^2 = 1}} \left[\inf_{y \in \mathcal{D}(S)} \left(\|x-y\|^2 + \|Tx - Sy\|^2 \right)^{\frac{1}{2}} \right].$$

$\widehat{\delta}\big(T,S\big)$ is called the gap between S and T.

(ii) *Let T and S be two closable operators. We define the gap between T and S by $\delta(T,S) = \delta(\overline{T}, \overline{S})$ and $\widehat{\delta}(T,S) = \widehat{\delta}(\overline{T}, \overline{S})$.*

Definition 1.6. *A sequence $(T_n)_{n\in\mathbb{N}}$ of bounded linear operators mapping on X is said to converge to zero compactly if for all $x \in X$, $T_n x \to 0$ and $(T_n x_n)_n$ is relatively compact for every bounded sequence $(x_n)_n \subset X$.*

Remark 1.2. *Clearly, T_n converges to 0 implies that T_n converges to zero compactly.*

Definition 1.7. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closable linear operators from X into Y and let T be a closable linear operator from X into Y. $(T_n)_{n\in\mathbb{N}}$ is said to converge in the generalized sense to T if $\widehat{\delta}(T_n, T)$ converges to 0 as, $n \to \infty$.*

2. Preliminaries

Theorem 2.1. [2, Theorem 4] *Let A_n be a sequence of bounded linear operators converging to zero compactly and let T be a closed linear operator. If T is a semi-Fredholm operator, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,*

(i) $(T + A_n)$ *is semi-Fredholm,*
(ii) $\alpha(T + A_n) < \alpha(T)$*,*
(iii) $\beta(T + A_n) < \beta(T)$*, and*
(iv) $i(T + A_n) = i(T)$*.*

Proposition 2.1. [3, Proposition 7.8.1]. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of bounded linear operators and let* $T \in \mathcal{L}(X)$ *such that* $T_n - T$ *converges to zero compactly. Then,*

(i) If $T_n \in \mathcal{F}^b(X)$*, then* $T \in \mathcal{F}^b(X)$*,*

(ii) If $T_n \in \mathcal{F}^b_+(X)$*, then* $T \in \mathcal{F}^b_+(X)$*, and*

(iii) If $T_n \in \mathcal{F}^b_-(X)$*, then* $T \in \mathcal{F}^b_-(X)$*.*

Theorem 2.2. [1, theorem 2.1] *Let* T *and* S *be two closed densely defined linear operators. Then, we have*:
(i) $\delta(T, S) = \delta(S^*, T^*)$ *and* $\widehat{\delta}(T, S) = \widehat{\delta}(S^*, T^*)$*.*

(ii) If S *and* T *are one-to-one, then* $\delta(S, T) = \delta(S^{-1}, T^{-1})$ *and* $\widehat{\delta}(S, T) = \widehat{\delta}(S^{-1}, T^{-1})$*.*

(iii) Let $A \in \mathcal{L}(X, Y)$*. Then* $\widehat{\delta}(A + S, A + T) \leq 2(1 + \|A\|^2)\widehat{\delta}(S, T)$*.*

(iv) Let T *be Fredholm operator (respectively semi-Fredholm operator). If* $\widehat{\delta}(T, S) < \gamma(T)(1+[\gamma(T)]^2)^{\frac{-1}{2}}$*, then* S *is Fredholm operator (respectively semi-Fredholm operator),* $\alpha(S) \leq \alpha(T)$ *and* $\beta(S) \leq \beta(T)$*. Furthermore, there exists* $b > 0$ *such that* $\widehat{\delta}(T, S) < b$*, which implies* $i(S) = i(T)$*.*

(v) Let $T \in \mathcal{L}(X, Y)$*. If* $S \in \mathcal{C}(X, Y)$ *and* $\widehat{\delta}(T, S) \leq \left[1 + \|T\|^2\right]^{-\frac{1}{2}}$*, then* S *is bounded operator (so that* $\mathcal{D}(S)$ *is closed).*

Theorem 2.3. [1, theorem 2.3] *Let* $(T_n)_{n\in\mathbb{N}}$ *be a sequence of closable linear operators from* X *into* Y *and let* T *be a closable linear operator from* X *into* Y*.*

(i) The sequence T_n *converges in the generalized sense to* T *if, and only if,* $T_n + S$ *converges in the generalized sense to* $T + S$*, for all* $S \in \mathcal{L}(X, Y)$*.*

(ii) Let $T \in \mathcal{L}(X, Y)$*.* T_n *converges in the generalized sense to* T *if, and only if,* $T_n \in \mathcal{L}(X, Y)$ *for sufficiently larger* n *and* T_n *converges to* T*.*

(iii) Let T_n *converges in the generalized sense to* T*. Then,* T^{-1} *exists and* $T^{-1} \in \mathcal{L}(Y, X)$*, if, and only if,* T_n^{-1} *exists and* $T_n^{-1} \in \mathcal{L}(Y, X)$ *for sufficiently larger* n *and* T_n^{-1} *converges to* T^{-1}*.*

3. The main result

In this section we investigate the essential spectra ($\sigma_{ei},\ i = 1, \ldots, 5$) of the sequence of linear operators in a Banach space X.

Theorem 3.1. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a bounded linear operators mapping on* X*, and let* T *and* B *be two operators in* $\mathcal{L}(X)$*,* $\lambda_0 \in \mathbb{C}$*, and* $\mathcal{U} \subseteq \mathbb{C}$ *is open .*

(a) If $((\lambda_0 - T_n - B) - (\lambda_0 - T - B))$ *converges to zero compactly, and* $0 \in \mathcal{U}$*, then there exists* $n_0 \in \mathbb{N}$ *such that, for all* $n \geq n_0$*.*

$$\sigma_{ei}(\lambda_0 - T_n - B) \subseteq \sigma_{ei}(\lambda_0 - T - B) + \mathcal{U}.$$

And, $\delta\big(\sigma_{ei}(\lambda_0 - T_n - B), \sigma_{ei}(\lambda_0 - T - B)\big) = 0$*, i=1,...,5*

(b) If $(\lambda_0 - T_n - B)$ *converges to zero compactly then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$

$$\sigma_{ei}((\lambda_0 - T - B) + (\lambda_0 - T_n - B)) \subseteq \sigma_{ei}(\lambda_0 - T - B).$$

And, $\delta\big(\sigma_{ei}((\lambda_0 - T - B) + (\lambda_0 - T_n - B)), \sigma_{ei}(\lambda_0 - T - B)\big) = 0,\ i = 1, \ldots, 5.$

Proof. (a) For $i = 1$. Assume that the assertion fails. Then by passing to a subsequence, it may be deduced that, for each n, there exists $\lambda_n \in \sigma_{e1}(\lambda_0 - T_n - B)$ such that $\lambda_n \notin \sigma_{e1}(\lambda_0 - T - B) + \mathcal{U}$. It is clear that $\lim\limits_{n\to+\infty} \lambda_n = \lambda$ since (λ_n) is bounded, this implies that $\lambda \notin \sigma_{e1}(\lambda_0 - T - B) + \mathcal{U}$. Using the fact that $0 \in \mathcal{U}$, hence we have $\lambda \notin \sigma_{e1}(\lambda_0 - T - B)$, and therefore, $(\lambda - (\lambda_0 - T - B)) \in \Phi_+^b(X)$. Let $A_n = \lambda_n - \lambda + (\lambda_0 - T - B) - (\lambda_0 - T_n - B)$. Since A_n converges to zero compactly, writing $\lambda_n - (\lambda_0 - T_n - B) = \lambda - (\lambda_0 - T - B) + A_n$ and according to Theorem 2.1, we infer that, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $(\lambda_n - (\lambda_0 - T_n - B)) \in \Phi_+(X)$ and $i(\lambda_n - (\lambda_0 - T_n - B)) = i(\lambda - (\lambda_0 - T - B) + A_n) = i(\lambda - (\lambda_0 - T - B))$. So, $\lambda_n \notin \sigma_{e1}(\lambda_0 - T_n - B)$, which is a contradiction. Then

$$\sigma_{e1}(\lambda_0 - T_n - B) \subseteq \sigma_{e1}(\lambda_0 - T - B) + \mathcal{U}, \text{ for all } n \geq n_0.$$

Since $0 \in \mathcal{U}$, we obtain $\sigma_{e1}(\lambda_0 - T_n - B) \subseteq \sigma_{e1}(\lambda_0 - T - B)$. Hence by Remark 1.1 (i_2), we get $\delta\big(\sigma_{e1}(\lambda_0 - T_n - B), \sigma_{e1}(\lambda_0 - T - B)\big) = 0$, for all $n \geq n_0$.

For $i = 2, 3, 4$, by using a similar proof as in $(i = 1)$, by replacing $\sigma_{e1}(.)$, and $\Phi_+(X)$ by $\sigma_{e2}(.)$, $\sigma_{e3}(.)$, $\sigma_{e4}(.)$, and $\Phi_-(X)$, $\Phi_-(X) \cup \Phi_+(X)$, $\Phi(X)$, respectively, we get
If $((\lambda_0 - T_n - B) - (\lambda_0 - T - B))$ converges to zero compactly, and $0 \in \mathcal{U}$, then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$.

$$\sigma_{ei}(\lambda_0 - T_n - B) \subseteq \sigma_{ei}(\lambda_0 - T - B) + \mathcal{U}.$$

And

$$\delta\big(\sigma_{ei}(\lambda_0 - T_n - B), \sigma_{ei}(\lambda_0 - T - B)\big) = 0.$$

For $i = 5$. Assume that the assertion fails. Then by passing to a subsequence, it may be deduced that, for each n, there exists $\lambda_n \in \sigma_{e5}(\lambda_0 - T_n - B)$ such that $\lambda_n \notin \sigma_{e5}(\lambda_0 - T - B) + \mathcal{U}$. It is clear that $\lim\limits_{n \to +\infty} \lambda_n = \lambda$ since (λ_n) is bounded, this implies that $\lambda \notin \sigma_{e5}(\lambda_0 - T - B) + \mathcal{U}$. Using the fact that $0 \in \mathcal{U}$, we have $\lambda \notin \sigma_{e5}(\lambda_0 - T - B)$ and therefore, $\lambda - (\lambda_0 - T - B) \in \Phi^b(X)$ and $i(\lambda - (\lambda_0 - T - B)) = 0$. Let $A_n = \lambda_n - \lambda + (\lambda_0 - T - B) - (\lambda_0 - T_n - B)$. Since A_n converges to zero compactly, writing $\lambda_n - (\lambda_0 - T_n - B) = \lambda - (\lambda_0 - T - B) + A_n$ and according to Theorem 2.1, we infer that, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\lambda_n - (\lambda_0 - T_n - B) \in \Phi(X)$ and $i(\lambda_n - (\lambda_0 - T_n - B)) = i(\lambda - (\lambda_0 - T - B) + A_n) = i(\lambda - (\lambda_0 - T - B)) = 0$. So, $\lambda_n \notin \sigma_{e5}(\lambda_0 - T_n - B)$, which is a contradiction. Then

$$\sigma_{e5}(\lambda_0 - T_n - B) \subseteq \sigma_{e5}(\lambda_0 - T - B) + \mathcal{U}, \quad \text{for all } n \geq n_0.$$

Since $0 \in \mathcal{U}$, we have $\sigma_{e5}(\lambda_0 - T_n - B) \subseteq \sigma_{e5}(\lambda_0 - T - B)$. Hence by Remark 1.1 (i_2), we have

$$\delta\big(\sigma_{e5}(\lambda_0 - T_n - B), \sigma_{e5}(\lambda_0 - T - B)\big) = 0, \text{ for all } n \geq n_0.$$

(b) For $i = 1$. Let $\lambda \notin \sigma_{e1}(\lambda_0 - T - B)$. Then, $(\lambda - (\lambda_0 - T - B)) \in \Phi_+^b(X)$. Since $(\lambda_0 - T_n - B)$ converges to zero compactly and applying [2, Theorem 4] to the operators $(\lambda_0 - T - B)$ and $(\lambda_0 - T_n - B)$, we prove that, there exists $n_0 \in \mathbb{N}$ such that $(\lambda - (\lambda_0 - T - B) + (\lambda_0 - T_n - B)) \in \Phi_+(X)$ for all $n \geq n_0$. Hence $\lambda \notin \sigma_{e1}((\lambda_0 - T - B) + (\lambda_0 - T_n - B))$. We conclude that

$$\sigma_{e1}(\lambda_0 - T_n - B) \subseteq \sigma_{e1}(\lambda_0 - T - B).$$

Now applying Remark 1.1 (i_2) we obtain
$\delta\big(\sigma_{e1}((\lambda_0 - T - B) + (\lambda_0 - T_n - B)), \sigma_{e1}(\lambda_0 - T - B)\big) = 0$, for all $n \geq n_0$.

For $i = 2, 3, 4$, by using a similar proof as in $(i = 1)$, by replacing $\sigma_{e1}(.)$, and $\Phi_+(X)$ by $\sigma_{e2}(.)$, $\sigma_{e3}(.)$, $\sigma_{e4}(.)$, and $\Phi_-(X)$, $\Phi_-(X) \cup \Phi_+(X)$, $\Phi(X)$, respectively, we get

If $(\lambda_0 - T_n - B)$ converges to zero compactly then there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$.

$$\sigma_{ei}((\lambda_0 - T + B) + (\lambda_0 - T_n - B)) \subseteq \sigma_{ei}(\lambda_0 - T - B).$$

And,

$$\delta\big(\sigma_{ei}((\lambda_0 - T - B) + (\lambda_0 - T_n - B)), \sigma_{ei}(\lambda_0 - T - B)\big) = 0, \text{ for all } n \geq n_0.$$

For $i = 5$. Let $\lambda \notin \sigma_{e5}(\lambda_0 - T - B)$. Then, $(\lambda - (\lambda_0 - T - B)) \in \Phi^b(X)$ and
$i(\lambda - (\lambda_0 - T - B)) = 0$. Since $(\lambda_0 - T_n - B)$ converges to zero compactly and by applying the [2, Theorem 4] to the operators $(\lambda_0 - T - B)$ and $(\lambda_0 - T_n - B)$, we prove that, there exists $n_0 \in \mathbb{N}$ such that $(\lambda - (\lambda_0 - T - B) + (\lambda_0 - T_n - B)) \in \Phi(X)$ for all $n \geq n_0$. Hence $\lambda \notin \sigma_{e5}((\lambda_0 - T - B) + (\lambda_0 - T_n - B))$. We conclude that

$$\sigma_{e5}(\lambda_0 - T_n - B) \subseteq \sigma_{e5}(\lambda_0 - T - B)$$

Now applying Remark 1.1 (i_2) we have
$\delta\big(\sigma_{e5}((\lambda_0-T-B)+(\lambda_0-T_n-B)),\sigma_{e5}(\lambda_0-T-B)\big)=0,$ for all $n\geq n_0$. $\square$

Theorem 3.2. *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of closed linear operators mapping on Banach spaces X and let $T\in\mathcal{C}(X)$,and let B and L be two operators in $\mathcal{L}(X)$, $\lambda_0\in\mathbb{C}$ such that T_n converges in the generalized sense to T, and $\lambda_0\in\rho(T+B)$, $\mathcal{U}\subseteq\mathbb{C}$ is open.*
(a) If $0\in\mathcal{U}$, then there exists $n_0\in\mathbb{N}$ such that, for every $n\geq n_0$, we have

$$\sigma_{ei}(\lambda_0-T_n-B)\subseteq\sigma_{ei}(\lambda_0-T-B)+\mathcal{U}. \tag{3.1}$$

And, $\delta\Big(\sigma_{ei}(\lambda_0-T_n-B),\sigma_{ei}(\lambda_0-T-B)\Big)=0,\ i=1,\ldots,5.$

(b) There exist $\varepsilon>0$ and $n\in\mathbb{N}$ such that, for all $\|L\|<\varepsilon$, we have

$$\sigma_{ei}(\lambda_0-T_n-B+L)\subseteq\sigma_{ei}(\lambda_0-T-B)+\mathcal{U},\ \textit{for all } n\geq n_0.$$

And, $\delta\Big(\sigma_{ei}(\lambda_0-T_n-B+L),\sigma_{ei}(\lambda_0-T-B)\Big)=\delta\Big(\sigma_{ei}(\lambda_0-T-B+L),\sigma_{ei}(\lambda_0-T-B)\Big),\ i=1,\ldots,5.$
$\diamondsuit$

Proof. (a) For $i=1$, since $(B-\lambda_0)$ be a bounded operator and $\lambda_0\in\rho(T+B)$. According to Theorem 2.3 (i) and $(iii)$the sequence (λ_0-T_n-B) converges in the generalized sense to (λ_0-T-B), and $\lambda_0\in\rho(T_n+B)$ for a sufficiently large n and $(\lambda_0-T_n-B)^{-1}$ converges to $(\lambda_0-T-B)^{-1}$. Now to prove such that the inclusion (3.1)holds it suffices to prove there exist $n_0\in\mathbb{N}$, such that for all $n\geq n_0$, we have

$$\sigma_{e1}(\lambda_0-T_n-B)^{-1}\subseteq\sigma_{e1}(\lambda_0-T-B)^{-1}+\mathcal{U}. \tag{3.2}$$

In first step by an indirect proof, we suppose that the (3.2) does not hold, and for each $n\in\mathbb{N}$ there exists $\lambda_n\in\sigma_{e1}(\lambda_0-T_n-B)^{-1}$ such that $\lambda_n\notin\sigma_{e1}(\lambda_0-T-B)^{-1}+\mathcal{U}$. It is clear that $\lim\limits_{n\to+\infty}\lambda_n=\lambda$ since (λ_n) is bounded, this implies that $\lambda\notin\sigma_{e1}(\lambda_0-T-B)^{-1}+\mathcal{U}$. Using the fact that $0\in\mathcal{U}$ hence we have $\lambda\notin\sigma_{e1}(\lambda_0-T-B)^{-1}$. Therefore $(\lambda-(\lambda_0-T-B)^{-1})\in\Phi_+^b(X)$ and applying Theorem 2.3 (ii), we conclude that

$$\widehat{\delta}(\lambda_n-(\lambda_0-T_n-B)^{-1},\lambda-(\lambda_0-T-B)^{-1})\to 0,\ \textit{as } n\to\infty.$$

Let $\gamma(\lambda-(\lambda_0-T-B)^{-1})=\delta>0$. Then there exists $N\in\mathbb{N}$ such that, for all $n\geq N$ we have $\widehat{\delta}(\lambda_n-(\lambda_0-T_n-B)^{-1},\lambda-(\lambda_0-T-B)^{-1})\leq\frac{\delta}{\sqrt{1+\delta^2}}$. According Theorem 2.2 (iv) we infer $(\lambda_n-(\lambda_0-T_n-B)^{-1})\in\Phi_+^b(X)$. Then we obtain $\lambda_n\notin\sigma_{e1}((\lambda_0-T_n-B)^{-1})$, which this is a contradicts our assumption. Hence (3.2) holds. Now, if $\lambda\in\sigma_{e1}(\lambda_0-T_n-B)$ then $\frac{1}{\lambda}\in\sigma_{e1}((\lambda_0-T_n-B)^{-1})$. According then (3.1) we conclude that

$$\frac{1}{\lambda}\in\sigma_{e1}((\lambda_0-T-B)^{-1})+\mathcal{U}. \tag{3.3}$$

Since $0\in\mathcal{U}$, then (3.3) implies that $\frac{1}{\lambda}\in\sigma_{e1}((\lambda_0-T-B)^{-1})$. We have to prove

$$\lambda\in\sigma_{e1}(\lambda_0-T-B)+\mathcal{U}. \tag{3.4}$$

We will proceed by contradiction, we suppose that $\lambda\notin\sigma_{e1}(\lambda_0-T-B)+\mathcal{U}$. The fact that $0\in\mathcal{U}$ implies that $\lambda\notin\sigma_{e1}(\lambda_0-T-B)$ and so, $\frac{1}{\lambda}\notin\sigma_{e1}((\lambda_0-T-B)^{-1})$ which this is a contradicts our assumption. So $\lambda\in\sigma_{e1}(\lambda_0-T-B)+\mathcal{U}$. Therefore (3.1) holds. Since $\mathcal{U}$ is an arbitrary neighborhood of 0 and by using the relation (3.1) we have $\sigma_{e1}(\lambda_0-T_n-B)\subseteq\sigma_{e1}(T+B-\lambda_0)$, for all $n\geq n_0$. Hence by Remark 1.1 (i_2)

$$\delta\Big(\sigma_{e1}(\lambda_0-T_n-B),\sigma_{e1}(\lambda_0-T-B)\Big)=\delta\Big(\overline{\sigma_{e1}(\lambda_0-T_n-B)},\overline{\sigma_{e1}(\lambda_0-T+B)}\Big)=0$$

for all $n\geq n_0$.This ends the proof (i=1).

For $i=2,3,4$, by using a similar proof as in ((a) *for* $i=1$), by replacing $\sigma_{e1}(.)$, and $\Phi_+(X)$ by $\sigma_{e2}(.)$, $\sigma_{e3}(.)$, $\sigma_{e4}(.)$, and $\Phi_-(X)$, $\Phi_-(X)\cup\Phi_+(X)$, $\Phi(X)$, respectively, we get

$$\sigma_{ei}(\lambda_0-T_n-B)\subseteq\sigma_{ei}(\lambda_0-T-B)+\mathcal{U}.$$

And, $\delta\Big(\sigma_{ei}(\lambda_0-T_n-B),\sigma_{ei}(\lambda_0-T-B)\Big)=0,$ for all $n\geq n_0$.

For $i=5$, since (λ_0-B) be a bounded operator and $\lambda_0\in\rho(T+B)$, according to Theorem 2.3 (i) and (iii) the sequence (λ_0-T_n-B) converges in the generalized sense to (λ_0-T-B), and $\lambda_0\in\rho(T_n+B)$ for a sufficiently large n and $(\lambda_0-T_n-B)^{-1}$ converges to $(\lambda_0-T-B)^{-1}$. Now to prove that (3.1)holds it suffices to prove there exist $n_0\in\mathbb{N}$, such that for all $n\geq n_0$, we have

$$\sigma_{e5}(\lambda_0-T_n-B)^{-1}\subseteq\sigma_{e5}(\lambda_0-T-B)^{-1}+\mathcal{U}. \tag{3.5}$$

In first step by an indirect proof, we suppose that the inclusion (3.5) does not hold, and for each $n\in\mathbb{N}$ there exists $\lambda_n\in\sigma_{e5}(\lambda_0-T_n-B)^{-1}$ such that $\lambda_n\notin\sigma_{e5}(\lambda_0-T-B)^{-1}+\mathcal{U}$. It is clear that $\lim\limits_{n\to+\infty}\lambda_n=\lambda$ since (λ_n) is bounded, this implies that $\lambda\notin\sigma_{e5}(\lambda_0-T-B)^{-1}+\mathcal{U}$. Using the fact that $0\in\mathcal{U}$, hence we have $\lambda\notin\sigma_{e5}(\lambda_0-T-B)^{-1}$. Therefore $(\lambda-(\lambda_0-T-B)^{-1})\in\Phi^b(X)$ and $i(\lambda-(\lambda_0-T-B)^{-1})$=0, and applying Theorem 2.3 (ii), we conclude that

$$\widehat{\delta}(\lambda_n-(\lambda_0-T_n-B)^{-1},\lambda-(\lambda_0-T-B)^{-1})\to 0\ as\ n\to\infty.$$

Let $\gamma(\lambda-(\lambda_0-T-B)^{-1})=\delta>0$. Then there exists $N\in\mathbb{N}$ such that, for all $n\geq N$ we have

$$\widehat{\delta}(\lambda_n-(\lambda_0-T_n-B)^{-1},\lambda-(\lambda_0-T-B)^{-1})\leq\frac{\delta}{\sqrt{1+\delta^2}}.$$

According to Theorem 2.2 (iv) we infer $(\lambda_n-(\lambda_0-T_n-B)^{-1})\in\Phi^b(X)$ and

$i(\lambda_n-(\lambda_0-T_n-B)^{-1})=i(\lambda-(\lambda_0-T-B)^{-1})$=0. Then we obtain $\lambda_n\notin\sigma_{e5}((\lambda_0-T_n-B)^{-1})$, which this is a contradicts our assumption. Hence (3.1) holds. Now, if $\lambda\in\sigma_{e5}(\lambda_0-T_n-B)$ then $\frac{1}{\lambda}\in\sigma_{e5}((\lambda_0-T_n-B)^{-1})$. According then (3.1) we conclude that

$$\frac{1}{\lambda}\in\sigma_{e5}((\lambda_0-T-B)^{-1})+\mathcal{U}. \tag{3.6}$$

Since $0\in\mathcal{U}$, then (3.6) implies that $\frac{1}{\lambda}\in\sigma_{e5}(\lambda_0-T-B)^{-1}$. We have to prove

$$\lambda\in\sigma_{e5}(\lambda_0-T-B)+\mathcal{U}. \tag{3.7}$$

We will proceed by contradiction , we suppose that $\lambda\notin\sigma_{e5}(\lambda_0-T-B)+\mathcal{U}$. The fact that $0\in\mathcal{U}$ implies that $\lambda\notin\sigma_{e5}(\lambda_0-T-B)$ and so, $\frac{1}{\lambda}\notin\sigma_{e5}((\lambda_0-T-B)^{-1})$ which this is a contradicts our assumption. So $\lambda\in\sigma_{e5}(\lambda_0-T-B)+\mathcal{U}$. Therefore (3.1)holds. Since $\mathcal{U}$ is an arbitrary neighborhood of 0 and by using (3.1) we have $\sigma_{e5}(\lambda_0-T_n-B)\subseteq\sigma_{e5}(\lambda_0-T-B)$ for all $n\geq n_0$. Hence by Remark 1.1 (i_2)

$$\delta\Big(\sigma_{e5}(\lambda_0-T_n-B),\sigma_{e5}(\lambda_0-T-B)\Big)=\delta\Big(\overline{\sigma_{e5}(\lambda_0-T_n-B)},\overline{\sigma_{e5}(\lambda_0-T-B)}\Big)=0$$

for all $n\geq n_0$.This ends the proof of, (a) .

(b) For $i=1$, since $\lambda_0\in\rho(T+B)$, then $(T+B-\lambda_0)^{-1}$ exists and bounded. We put $\frac{1}{\|(\lambda_0-T-B)^{-1}\|}=\varepsilon_1$. Let $L\in\mathcal{L}(X)$ such that $\|L\|<\varepsilon_1$ this implies

$$\|L\,(\lambda_0-T-B)^{-1}\,\|<1.$$

By according Theorem 2.3 (i) the squence (λ_0-T_n-B+L) converges in the generalized sense to $(\lambda_0-T-B+L)$, and the Neumann series $\sum_{k=0}^{\infty}(-L\,(\lambda_0-T-B)^{-1})^k$ converges to $(I+L\,(\lambda_0-T-B)^{-1})^{-1}$ and

$$\|(I+L\,(\lambda_0-T-B)^{-1})^{-1}\|<\frac{1}{1-\|L\|\|\,(\lambda_0-T-B)^{-1}\,\|}$$

Since $(\lambda_0-T-B+L)^{-1}=(\lambda_0-T-B)^{-1})(I+L\,(\lambda_0-T-B)^{-1}))^{-1}$, then $\lambda_0\in\rho(T+B+L)$. Now applying $((a)for\ i=1)$, we deduce that there exists $n_0\in\mathbb{N}$ such that $\sigma_{e1}(\lambda_0-T_n-B+L)\subseteq\sigma_{e1}(\lambda_0-T-B+L)+\mathcal{U}$, for all $n\geq n_0$. Let $\lambda\notin\sigma_{e1}(\lambda_0-T-B)$. Then $(\lambda-(\lambda_0-T-B))\in\Phi_+(X)$. By applying [8, Theorem 7.9] there exists $\varepsilon_2>0$ such that for $\|L\|<\varepsilon_2$, one has $(\lambda-(\lambda_0-T-B)-L)\in\Phi_+(X)$ and, this implies that $\lambda\notin\sigma_{e1}(\lambda_0-T-B+L)$. From what has been mentioned and if we take $\varepsilon=\min(\varepsilon_1,\varepsilon_2)$ then for all $\|L\|<\varepsilon$, there exists $n_0\in\mathbb{N}$ such that $\sigma_{e1}(\lambda_0-T_n-B+L)\subseteq\sigma_{e1}(\lambda_0-T-B)+\mathcal{U}$, for all $n\geq n_0$. Since $0\in\mathcal{U}$ then we have

$$\delta\big(\sigma_{e1}(\lambda_0-T_n-B+L),\sigma_{e1}(\lambda_0-T-B)\big)=0$$

and

$$\delta\big(\sigma_{e1}(\lambda_0-T-B+L),\sigma_{e1}(\lambda_0-T-B)\big)=0.$$

Therefore, $(i = 1)$ holds.

For $i = 2, 3, 4$. By using a similar proof as in $((b) for\ i = 1)$, by replacing $\sigma_{e1}(.)$, and $\Phi_+(X)$ by $\sigma_{e2}(.)$, $\sigma_{e3}(.)$, $\sigma_{e4}(.)$, and $\Phi_-(X)$, $\Phi_-(X) \cup \Phi_+(X)$, $\Phi(X)$, respectively, we get
There exist $\varepsilon > 0$ and $n \in \mathbb{N}$ such that, for all $\|L\| < \varepsilon$, we have

$$\sigma_{ei}(\lambda_0 - T_n - B + L) \subseteq \sigma_{ei}(\lambda_0 - T - B) + \mathcal{U}, \text{ for all } n \geq n_0.$$

And,

$$\delta\Big(\sigma_{ei}(\lambda_0 - T_n - B + L), \sigma_{ei}(\lambda_0 - T - B)\Big) = \delta\Big(\sigma_{ei}(\lambda_0 - T - B + L), \sigma_{ei}(\lambda_0 - T - B)\Big).$$

For $i = 5$, since $\lambda_0 \in \rho(T + B)$, then $(\lambda_0 - T - B)^{-1}$ exists and bounded. We put $\frac{1}{\|(\lambda_0 - T - B)^{-1}\|} = \varepsilon_1$. Let $L \in \mathcal{L}(X)$ such that $\|L\| < \varepsilon_1$ this implies

$$\|L\,(\lambda_0 - T - B)^{-1}\| < 1.$$

By according theorem 2.3 (i) we have $(\lambda_0 - T_n - B + L)$ converges in the generalized sense to $(\lambda_0 - T - B + L)$, and the Neumann series $\sum_{k=0}^{\infty}(-L\,(\lambda_0 - T - B)^{-1})^k$ converges to $(I + L\,(\lambda_0 - T - B)^{-1})^{-1}$ and

$$\|(I + L\,(\lambda_0 - T - B)^{-1})^{-1}\| < \frac{1}{1 - \|L\|\|\,(\lambda_0 - T - B)^{-1}\,\|}.$$

Since $(\lambda_0 - T - B + L)^{-1} = (\lambda_0 - T - B)^{-1})(I + L\,(\lambda_0 - T - B)^{-1}))^{-1}$, then $\lambda_0 \in \rho(T + B + L)$. Now applying $((a) for\ i = 5)$, we deduce that there exists $n_0 \in \mathbb{N}$ such that $\sigma_{e5}(\lambda_0 - T_n - B + L) \subseteq \sigma_{e5}(\lambda_0 - T - B + L) + \mathcal{U}$, for all $n \geq n_0$. Let $\lambda \notin \sigma_{e5}(\lambda_0 - T - B)$. Then $(\lambda - (\lambda_0 - T - B)) \in \Phi(X)$. By applying [8, Theorem 7.9] there exists $\varepsilon_2 > 0$ such that for $\|L\| < \varepsilon_2$, one has $(\lambda - (\lambda_0 - T - B) - L) \in \Phi(X)$ and $i(\lambda - (\lambda_0 - T - B - L)) = i(\lambda - (\lambda_0 - T - B) = 0$. This implies that $\lambda \notin \sigma_{e5}(\lambda_0 - T - B + L)$. From what has been mentioned and if we take $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$ then for all $\|L\| < \varepsilon$, there exists $n_0 \in \mathbb{N}$ such that $\sigma_{e5}(\lambda_0 - T_n - B + L) \subseteq \sigma_{e5}(\lambda_0 - T - B) + \mathcal{U}$, for all $n \geq n_0$. Since $0 \in \mathcal{U}$ then we have

$$\delta\big(\sigma_{e5}(\lambda_0 - T_n - B + L), \sigma_{e5}(\lambda_0 - T - B)\big) = 0 = \delta\big(\sigma_{e5}(\lambda_0 - T - B + L), \sigma_{e5}(\lambda_0 - T - B)\big).$$

Therefore, $(i = 5)$ holds. $\square$

References

[1] A. Ammar and A. Jeribi, The weyl essential spectrum of a sequence of linear operators in Banach spaces, Indag. Math., New Ser. 27 (1) (2016), 282-295.

[2] S. Goldberg, Perturbations of semi-Fredholm operators by operators converging to zero compactly, Proc. Amer. Math. Soc. 45 (1974), 93-98 .

[3] A. Jeribi, Spectral Theory and Applications of Linear Operators and Block Operator Matrices. Springer-Verlag, New York, 2015.

[4] A. Jeribi and N. Moalla, A characterisation of some subsets of Schechter's essential spectrum and applications to singular transport equation, J. Math. Anal. Appl. 358 (2) (2009), 434-444.

[5] A.Jeribi, Une nouvelle caractrisation du spectre essentiel et application, Comp Rend.Acad.Paris serie I, 331 (2000),525-530.

[6] T. Kato, Perturbation theory for linear operators, Second edition. Grundlehren der Mathematischen Wissenschaften, Band 132. Springer-Verlag, Berlin-New York, 1966.

[7] K. Latrach, A. Jeribi, Some results on Fredholm operators, essential spectra, and application, J. Math. Anal. Appl. 225 (1998), 461-485.

[8] M. Schechter, Principles of Functional Analysis, Grad. Stud. Math. vol. 36, Amer. Math. Soc., Providence, 2002.

[9] M. Schechter, Riesz operators and Fredholm perturbations, Bull. Amer. Math. Soc. 74 (1968), 1139-1144.

[1]Department of Mathematics, University of Sfax, Faculty of Sciences of Sfax, Tunisia

[2]Department of Mathematics, University of Mohamed Boudiaf, M'sila, Algeria

[*]Corresponding author: ammar_aymen84@yahoo.fr