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SOME ESTIMATIONS ON CONTINUOUS RANDOM VARIABLES INVOLVING FRACTIONAL CALCULUS

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Abstract. Using fractional calculus, new fractional bounds estimating the $w-$ weighted expectation, the $w-$ weighted variance and the $w-$weighted moment of continuous random variables are obtained. Some recent results on classical bounds estimations are generalized.

1. Introduction

It is known that the integral inequalities play an important role in the theory of differential equations, probability theory and in applied sciences. For more details, we refer to [2, 3, 11–13, 16] and the references therein. Moreover, the study of the integral inequalities using fractional calculus is also of great importance, we refer the reader to [1, 4–6, 8, 14, 15] for further information and applications.
In this sense, in a recent work [4], by introducing new concepts on probability theory using fractional calculus, the author extended some classical results of the papers [3, 11].
Then, based on [4], the authors in [9] introduced other classes of weighted concepts and generalized some classical results of [3, 12].
Very recently, in [7], the author presented some fractional applications for continuous random variables having probability density functions (p.d.f.) defined on finite real lines.
Motivated by the papers in [4, 7, 9, 11], in this work, we focus our attention on the applications of fractional calculus on probability theory. We establish new fractional bounds that estimate the $w-$ weighted expectation, the $w-$ weighted variance and the $w-$weighted moment of continuous random variables. Some recent results on classical random variable bound estimations are also generalized.

2. Preliminaries

In this section, we recall some preliminaries that will be used in this work. We begin by the following definition.

Definition 2.1. *[10] The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a continuous function f on $[a,b]$ is defined as*

$$J_a^{\alpha}[f(t)] = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1} f(\tau)\,d\tau,\ \alpha>0,\ a<t\leq b, \tag{2.1}$$
$$J_a^0[f(t)] = f(t),$$

where $\Gamma(\alpha) := \int_0^{\infty} e^{-u}u^{\alpha-1}du$.

For $\alpha>0$, $\beta>0$, we have:

$$J_a^{\alpha}J_a^{\beta}[f(t)] = J_a^{\alpha+\beta}[f(t)] \tag{2.2}$$

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and

$$J_a^\alpha J_a^\beta [f(t)] = J_a^\beta J_a^\alpha [f(t)]. \tag{2.3}$$

Let us now consider a positive continuous function w defined on $[a,b]$. We recall the $w-$concepts [9] :

Definition 2.2. *The fractional $w-$weighted expectation function of order $\alpha > 0$, for a random variable X with a positive p.d.f. f defined on $[a,b]$ is defined as*

$$E_{X,\alpha,w}(t) := J_a^\alpha [twf(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \tau w(\tau) f(\tau) d\tau, \ a \le t < b, \ \alpha > 0, \tag{2.4}$$

where $w : [a,b] \to \mathbb{R}^+$ is a positive continuous function.

Definition 2.3. *The fractional $w-$weighted expectation function of order $\alpha > 0$ for the random variable $X - E(X)$ is given by*

$$E_{X-E(X),\alpha,w}(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - E(X)) w(\tau) f(\tau) d\tau, \ a \le t < b, \ \alpha > 0. \tag{2.5}$$

where $f : [a,b] \to \mathbb{R}^+$ is the $(p.d.f)$ of X.

Definition 2.4. *The fractional $w-$weighted expectation of order $\alpha > 0$ for a random variable X with a positive p.d.f. f defined on $[a,b]$ is defined as*

$$E_{X,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau w(\tau) f(\tau) d\tau, \ \alpha > 0. \tag{2.6}$$

For the $w-$weighted fractional variance of X, we recall the definitions [9]:

Definition 2.5. *The fractional $w-$weighted variance function of order $\alpha > 0$ for a random variable X having a positive p.d.f. f on $[a,b]$ is defined as*

$$\begin{aligned} \sigma^2_{X,\alpha,w}(t) \quad &: \quad = J_a^\alpha \left[(t - E(X))^2 wf(t) \right] \qquad (2.7) \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} (\tau - E(X))^2 w(\tau) f(\tau) d\tau, \ a \le t < b, \ \alpha > 0. \end{aligned}$$

Definition 2.6. *The fractional $w-$weighted variance of order $\alpha > 0$ for a random variable X having a positive p.d.f. f on $[a,b]$ is given by*

$$\sigma^2_{X,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} (\tau - E(X))^2 w(\tau) f(\tau) d\tau, \ \alpha > 0. \tag{2.8}$$

For the fractional $w-$weighted moments, we recall the following definitions [9]:

Definition 2.7. *The fractional $w-$weighted moment function of orders $r > 0$, $\alpha > 0$ for a continuous random variable X having a p.d.f. f defined on $[a,b]$ is defined as*

$$M_{X^r,\alpha,w}(t) := J_a^\alpha [t^r wf(t)] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} \tau^r w(\tau) f(\tau) d\tau, \ a \le t < b, \ \alpha > 0. \tag{2.9}$$

Definition 2.8. *The fractional $w-$weighted moment of orders $r > 0$, $\alpha > 0$ for a continuous random variable X having a p.d.f. f defined on $[a,b]$ is defined by*

$$M_{X^r,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} \tau^r w(\tau) f(\tau) d\tau, \ \alpha > 0. \tag{2.10}$$

Based on the above definitions, we give the following remark:

Remark 2.1. *(1:) If we take $\alpha = 1, w(t) = 1, t \in [a,b]$ in Definition 2, we obtain the classical expectation: $E_{X,1,1} = E(X)$.*
(2:) If we take $\alpha = 1, w(t) = 1, t \in [a,b]$ in Definition 5, we obtain the classical variance: $\sigma^2_{X,1,1} = \sigma^2(X) = \int\limits_a^b (\tau - E(X))^2 f(\tau)\, d\tau$.
(3:) If we take $\alpha = 1, w(t) = 1, t \in [a,b]$ in Definition 7, we obtain the classical moment of order $r > 0, M_r := \int\limits_a^b \tau^r f(\tau)\, d\tau$.

3. Main Results

In this section, based on [7], we establish new $w-$weighted integral inequalities (with new fractional bounds) for random variables with p.d.f. that are defined on finite real intervals. We begin by proving the following property that generalizes an important property of the classical variance:

Theorem 3.1. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$, and let $w : [a,b] \to \mathbb{R}^+$ be a positive continuous function. Then for all $\alpha > 0, n = [\alpha - 1]$ we have :*

$$\sigma^2_{X,\alpha,w} = E_{X^2,\alpha,w} - 2E(X)\, E_{X,\alpha,w} + E^2(X)\, \frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w} \right] \right. \tag{3.1}$$

Proof. By Definition 6, we can write :

$$\sigma^2_{X,\alpha,w} := \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} (\tau - E(X))^2\, w(\tau)\, f(\tau)\, d\tau, \ \ \alpha > 0. \tag{3.2}$$

Hence,

$$\sigma^2_{X,\alpha,w} = E_{X^2,\alpha,w} - 2E(X)\, E_{X,\alpha,w} + E^2(X)\, J^{\alpha} w f(b). \tag{3.3}$$

On the other hand, we have

$$J^{\alpha} w f(b) = \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[\left[(-1)^i C_n^i b^{n-i} \int_a^b (b-\tau)^s\, \tau^i w f(\tau) d\tau \right. \right], \tag{3.4}$$

where $\alpha = n + s; n = [\alpha]; s \in (0;1)$.
Definition 8 allows us to write

$$J^{\alpha} w f(b) = \frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w} \right] \right.. \tag{3.5}$$

Then, using (3.3) and (3.5), we get (3.1). □

Remark 3.1. *a*: Taking $w(t) = 1$ on $[a,b]$ in the above theorem, we obtain Theorem 3.3 of [7].*
b: Taking $\alpha = 1$ and $w(t) = 1, t \in [a,b]$, we obtain $\sigma^2_{X,1,1} = E(X^2) - E^2(X)$.*

Another result is the following:

Theorem 3.2. *Let X be a continuous random variable with a p.d.f. $f : [a,b] \to \mathbb{R}^+$, and let $w : [a,b] \to \mathbb{R}^+$ be a positive continuous function. Then for all $\alpha > 0, n = [\alpha - 1]$ the following estimations are valid.*

$$\begin{aligned} & \left(E_{X^2,\alpha,w} - 2E(X)\, E_{X,\alpha,w} + E^2(X)\, \frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w} \right] \right. \right) \\ & \times \left(\frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w} \right] \right. \right) - \left(E_{X-E(X),\alpha,w} \right)^2 \\ \leq \quad & \|f\|^2_{\infty}\, J_a^{\alpha} w(b) \left[J_a^{\alpha} \left[w(b) b^2 \right] - \left(J_a^{\alpha} \left[w(b) b \right] \right)^2 \right] \ , \ f \in L_{\infty}[a,b] \end{aligned} \tag{3.6}$$

and

$$\left(E_{X^2,\alpha,w}-2E\left(X\right)E_{X,\alpha,w}+E^2\left(X\right)\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^iC_n^ib^{n-i}M_{X^i,\alpha-n,w}\right]\right)\right.$$
$$\times\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^iC_n^ib^{n-i}M_{X^i,\alpha-n,w}\right]\right)-\left(E_{X-E(X),\alpha,w}\right)^2 \tag{3.7}$$
$$\leq\frac{1}{2}\left(b-a\right)^2\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^iC_n^ib^{n-i}M_{X^i,\alpha-n,w}\right]\right)^2.$$

Proof. To prove the above theorem, we use Theorem 3.1 of [4]. We find that:

$$\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-x\right)^{\alpha-1}\left(b-y\right)^{\alpha-1}\left(x-y\right)^2p\left(x\right)p\left(y\right)dxdy \tag{3.8}$$
$$=2J_a^\alpha\left[p(b)\right]J_a^\alpha\left[p(b)(b-E(X))^2\right]-2(J_a^\alpha\left[p(b)(b-E(X)\right])^2$$

Then, taking $p\left(t\right)=w(t)f\left(t\right),t\in[a,b]$ in (3.8), it yields that

$$\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-x\right)^{\alpha-1}\left(b-y\right)^{\alpha-1}\left(x-y\right)^2w(x)f\left(x\right)w(y)f\left(y\right)dxdy$$
$$=2J_a^\alpha\left[wf(b)\right]\sigma^2_{X,\alpha,w}-2\left(E_{X-E(X),\alpha,w}\right)^2. \tag{3.9}$$

By the hypothesis $f\in L_\infty\left([a,b]\right)$, we obtain

$$\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-x\right)^{\alpha-1}\left(b-y\right)^{\alpha-1}\left(x-y\right)^2w(x)w(y)f\left(x\right)f\left(y\right)dxdy$$
$$\leq2\left\|f\right\|^2_\infty\left[J_a^\alpha\left[w\left(b\right)\right]J_a^\alpha\left[w(b)b^2\right]-\left(J_a^\alpha\left[w(b)b\right]\right)^2\right]. \tag{3.10}$$

Thanks to (3.9),(3.10),(3.5) and applying Theorem 1, we obtain (3.6).

On the other hand,

$$\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-x\right)^{\alpha-1}\left(b-y\right)^{\alpha-1}w(x)w(y)\left(x-y\right)^2f\left(x\right)f\left(y\right)dxdy \tag{3.11}$$
$$\leq\sup_{x,y\in[a,b]}\left|(x-y)\right|^2\frac{1}{\Gamma^2\left(\alpha\right)}\int_a^b\int_a^b\left(b-x\right)^{\alpha-1}\left(b-y\right)^{\alpha-1}w(x)w(y)f\left(x\right)f\left(y\right)dxdy$$
$$=\left(b-a\right)^2\left(J_a^\alpha\left[wf(b)\right]\right)^2.$$

So, by (3.9),(3.11),(3.1) and (3.5), we obtain (3.7). □

Remark 3.2. (1) *If we take $w=1$ on $[a,b]$ in Theorem 2, we obtain the first part of Theorem 3.5 of [7],*
(2) *and taking $\alpha=1,w=1$ on $[a,b]$, we obtain the first part of Theorem 1 in [3].*

In what follows, we prove a more general theorem.

Theorem 3.3. *Suppose that X is a continuous random variable with a p.d.f. $f:[a,b]\to\mathbb{R}^+$ and let $w:[a,b]\to\mathbb{R}^+$ b a continuous function. Then,*

(I): *For all* $\alpha > 0, \beta > 0; n = [\alpha - 1], m = [\alpha - 1]$

$$\left(E_{X^2,\beta,w} - 2E(X)E_{X,\beta,w} + E^2(X)\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right)\right. \tag{3.12}$$
$$\times\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)\right. + \frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{n}\left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right.$$
$$\times\left(E_{X^2,\alpha,w} - 2E(X)E_{X,\alpha,w} + E^2(X)\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)\right.$$
$$-2E_{X-E(X),\alpha,w}E_{X-E(X),\beta,w}$$
$$\leq \|f\|_\infty^2\left[\begin{array}{c} J_a^\alpha[w(b)]J_a^\beta\left[w(b)b^2\right] + J_a^\beta[w(b)]J_a^\alpha\left[w(b)b^2\right] \\ -2J_a^\alpha[w(b)b]J_a^\beta[w(b)b]\end{array}\right],\ f \in L_\infty([a,b]).$$

(II) *Also, the following estimation*

$$\left(E_{X^2,\beta,w} - 2E(X)E_{X,\beta,w} + E^2(X)\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\beta-n,w}\right]\right)\right.$$
$$\times\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)\right. + \left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right)\right.$$
$$\times\left(E_{X^2,\alpha,w} - 2E(X)E_{X,\alpha,w} + E^2(X)\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)\right.$$
$$-2E_{X-E(X),\alpha,w}E_{X-E(X),\beta,w} \tag{3.13}$$
$$\leq (b-a)^2\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)\right.\left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right)\right.$$

is also valid for any $\alpha > 0, \beta > 0$.

Proof. We have (see [4]):

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}(x-y)^2 p(x)p(y)dxdy$$
$$= J_a^\alpha[wf(b)]J_a^\beta\left[wf(b)(b-E(X))^2\right] + J_a^\beta[wf(b)]J_a^\alpha\left[wf(b)(b-E(X))^2\right]$$
$$-2J_a^\alpha[wf(b)(b-E(X))]J_a^\beta[wf(b)(b-E(X))]. \tag{3.14}$$

In (3.14), if we take $p = wf$, we obtain

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}(x-y)^2 w(x)w(y)f(x)f(y)dxdy$$
$$= J_a^\alpha[wf(b)]\sigma^2_{X,\beta,w} + J_a^\beta[wf(b)]\sigma^2_{X,\alpha,w} - 2E_{X-E(X),\alpha,w}E_{X-E(X),\beta,w}. \tag{3.15}$$

On the other hand, it is clear that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}(x-y)^2 w(x)w(y)f(x)f(y)dxdy \tag{3.16}$$
$$\leq \|f\|_\infty^2\left[J_a^\alpha[w(b)]J_a^\beta\left[w(b)b^2\right] + J_a^\beta[w(b)]J_a^\alpha\left[w(b)b^2\right] - 2J_a^\alpha[w(b)b]J_a^\beta[w(b)b]\right].$$

Consequently, by (3.15), (3.16) and (3.1), we obtain (3.12).

For the second inequality of Theorem 3, we observe that

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}(x-y)^2 w(x)w(y)f(x)f(y)\,dxdy\\
\leq\ & \sup_{x,y\in[a,b]}|(x-y)|^2\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1} w(x)w(y)f(x)f(y)dxdy\\
\leq\ & (b-a)^2 J_a^\alpha[wf(b)]\,J_a^\beta[wf(b)]\,. \qquad (3.17)
\end{aligned}$$

So, applying Theorem 1 and thanks to (3.15) and (3.17), we get (3.13). □

Remark 3.3. (i) : *Applying Theorem 14 for* $\alpha=\beta$, *we obtain Theorem 12.*
(ii) : *Taking* w *equal to* 1 *on* $[a,b]$ *in theorem 14, we obtain the last part of Theorem 3.7 of [7].*

Also, we present to reader the following estimation:

Theorem 3.4. *Let* f *be the p.d.f of* X *on* $[a,b]$ *and* $w:[a,b]\to\mathbb{R}^+$*.Then for all* $\alpha>0, n=[\alpha-1]$ *the following fractional inequality holds:*

$$\begin{aligned}
&\left(E_{X^2,\alpha,w}-2E(X)E_{X,\alpha,w}+E^2(X)\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)\right.\\
&\times\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)-\left(E_{X-E(X),\alpha,w}\right)^2 \qquad (3.18)\\
\leq\ & \frac{1}{4}(b-a)^2\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right)^2.
\end{aligned}$$

Proof. In [4], it has been proved that

$$0\leq J_a^\alpha[p(b)]\,J_a^\alpha\left[p(b)(b-E(X))^2\right]-\left(J_a^\alpha[p(b)(b-E(X))]\right)^2\leq\frac{1}{4}(b-a)^2\left(J_a^\alpha[p(b)]\right)^2. \qquad (3.19)$$

Hence, taking $p(b)=wf(b)$ in (3.19), we observe that

$$J_a^\alpha[wf(b)]\,\sigma^2_{X,\alpha,w}-\left(E_{X-E(X),\alpha,w}\right)^2\leq\frac{1}{4}(b-a)^2\left(J_a^\alpha[wf(b)]\right)^2. \qquad (3.20)$$

Thanks to Theorem 1 and by the relation (3.5), we obtain (3.18). □

Remark 3.4. *Taking* $w(t)=1, t\in[a,b]$ *in Theorem 4, we obtain Theorem 3.8 of [7] .*

Another result related to the moments is the following theorem.

Theorem 3.5. *Let f be the p.d.f of the random variable X on $[a,b]$ and $w:[a,b]\to\mathbb{R}^{+}$. Then for all $\alpha>0,\beta>0;n=[\alpha-1],m=[\beta-1]$, the inequality*

$$\begin{aligned}
&\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]\right.\\
&\times\left(E_{X^{2},\beta,w}-2E\left(X\right)E_{X,\beta,w}+E^{2}\left(X\right)\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^{i}C_{m}^{i}b^{m-i}M_{X^{i},\beta-m,w}\right]\right)\right.\\
&+\frac{\Gamma(\beta-1+m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^{i}C_{m}^{i}b^{m-i}M_{X^{i},\beta-m,w}\right]\right. \qquad (3.21)\\
&\times\left(E_{X^{2},\alpha,w}-2E\left(X\right)E_{X,\alpha,w}+E^{2}\left(X\right)\frac{\Gamma(\beta-1+m)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]\right)\right.\\
&+2\left(a-E\left(X\right)\right)\left(b-E\left(X\right)\right)\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]\right)\right.\\
&\times\left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^{i}C_{m}^{i}b^{n-i}M_{X^{i},\beta-m,w}\right]\right)\right..\\
\leq\ &(a+b-2E(X))\left[\begin{array}{c}\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]\right)E_{X-E(X),\beta,w}\right.\\+\left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^{i}C_{m}^{i}b^{m-i}M_{X^{i},\beta-m,w}\right]\right)E_{X-E(X),\alpha,w}\right.\end{array}\right]
\end{aligned}$$

is valid.

Proof. We have

$$\begin{aligned}
&\left[J_{a}^{\alpha}\left[p(b)\right]J_{a}^{\beta}\left[p(b)(b-E(X))^{2}\right]+J_{a}^{\beta}\left[p(b)\right]J_{a}^{\alpha}\left[p(b)(b-E(X))^{2}\right]\right.\\
&\left.-2J_{a}^{\alpha}\left[p(b)(b-E(X))\right]J_{a}^{\beta}\left[p(b)(b-E(X))\right]\right]^{2} \qquad (3.22)\\
\leq\ &\left[\left(MJ_{a}^{\alpha}\left[p(b)\right]-J_{a}^{\alpha}\left[p(b)(b-E(X))\right]\right)\left(J_{a}^{\beta}\left[p(b)(b-E(X))\right]-\tilde{m}J_{a}^{\beta}\left[p(b)\right]\right)\right.\\
&\left.+\left(J_{a}^{\alpha}\left[p(b)(b-E(X))\right]-\tilde{m}J_{a}^{\alpha}\left[p(b)\right]\right)\left(MJ_{a}^{\beta}\left[p(b)\right]-J_{a}^{\beta}\left[p(b)(b-E(X))\right]\right)\right]^{2}.
\end{aligned}$$

Taking : $p=wf, M=b-E(X)$, $\tilde{m}=a-E(X)$ in (3.22), we can write

$$\begin{aligned}
&J_{a}^{\alpha}\left[wf(b)\right]\sigma_{X,\beta,w}^{2}+J_{a}^{\beta}\left[wf(b)\right]\sigma_{X,\alpha,w}^{2}+2\left(a-E\left(X\right)\right)\left(b-E\left(X\right)\right)J_{a}^{\alpha}\left[wf(b)\right]J_{a}^{\beta}\left[wf(b)\right]\\
\leq\ &(a+b-2E(X))\left[J_{a}^{\alpha}\left[wf(b)\right]E_{X-E(X),\beta,w}+J_{a}^{\beta}\left[wf(b)\right]E_{X-E(X),\alpha,w}\right]. \qquad (3.23)
\end{aligned}$$

By Theorem 1 and using (3.5), we get (3.21). □

Remark 3.5. *If we take $w=1$ in Theorem 5, we obtain Theorem 3.10 of [7].*

We prove also:

Theorem 3.6. *Let X be a continuous random variable having a p.d.f. $f:[a,b]\to\mathbb{R}^{+}$, $w:[a,b]\to\mathbb{R}^{+}$. Then, for all $\alpha>0$, the following two inequalities hold:*

$$\begin{aligned}
&\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]E_{X^{r-1}(X-E(X)),\alpha,w}-\left(E_{X-E(X),\alpha,w}\right)M_{X^{r-1},\alpha,w}\right.\\
\leq\ &\|f\|_{\infty}^{2}\left[J_{a}^{\alpha}\left[w(b)\right]J_{a}^{\alpha}\left[b^{r}w(b)\right]-J_{a}^{\alpha}\left[bw(b)\right]J_{a}^{\alpha}\left[b^{r-1}w(b)\right]\right] \qquad (3.24)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]\right)E_{X^{r-1}(X-E(X)),\alpha,w}-\left(E_{X-E(X),\alpha,w}\right)M_{X^{r-1},\alpha,w}\right.\\
\leq\ &\frac{(b-a)}{2}\left(b^{r-1}-a^{r-1}\right)\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^{i}C_{n}^{i}b^{n-i}M_{X^{i},\alpha-n,w}\right]\right)^{2}\right.. \qquad (3.25)
\end{aligned}$$

Proof. We have

$$\begin{aligned} & \frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-x)^{\alpha-1}(b-y)^{\alpha-1} p(x)p(y)(g(x)-g(y))(h(x)-h(y)) \\ = \quad & 2J_a^{\alpha}\left[p(b)\right] J_a^{\alpha}\left[pgh(b)\right] - 2(J_a^{\alpha}\left[pg(b)\right] J_a^{\alpha}\left[ph(b)\right]) \end{aligned} \tag{3.26}$$

Taking $p = wf$, $g(b) = b - E(X)$ and $h(b) = b^{r-1}$, we obtain

$$\begin{aligned} & \frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-x)^{\alpha-1}(b-y)^{\alpha-1}(x-y)\left(x^{r-1}-y^{r-1}\right) w(x)w(y)f(x)f(y)dxdy \\ = \quad & 2J_a^{\alpha}\left[wf(b)\right] E_{X^{r-1}(X-E(X)),\alpha,w} - 2\left(E_{X-E(X),\alpha,w}\right) M_{X^{r-1},\alpha,w}. \end{aligned} \tag{3.27}$$

Therefore,

$$\begin{aligned} & \frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-x)^{\alpha-1}(b-y)^{\alpha-1}(x-y)\left(x^{r-1}-y^{r-1}\right) w(x)w(y)f(x)f(y)dxdy \\ \leq \quad & \|f\|_{\infty}^2 \left[2J_a^{\alpha}\left[w(b)\right] J_a^{\alpha}\left[b^r w(b)\right] - 2J_a^{\alpha}\left[bw(b)\right] J_a^{\alpha}\left[b^{r-1}w(b)\right]\right]. \end{aligned} \tag{3.28}$$

Combining (3.27), (3.28) and (3.5), we obtain (3.24).

To obtain (3.25), it suffices to see that

$$\begin{aligned} & \frac{1}{\Gamma^2(\alpha)} \int_a^b \int_a^b (b-x)^{\alpha-1}(b-y)^{\alpha-1}(x-y)\left(x^{r-1}-y^{r-1}\right) w(x)w(y)f(x)f(y)dxdy \\ \leq \quad & (b-a)\left(b^{r-1}-a^{r-1}\right)\left(J_a^{\alpha}\left[wf(b)\right]\right)^2 \end{aligned} \tag{3.29}$$

and to combine (3.28), (3.29) and (3.5). □

Remark 3.6. *Taking $\alpha = 1$, we obtain Theorem 3.1 of [11].*

Theorem 3.7. *Let X be a continuous random variable having a p.d.f. $f : [a,b] \to \mathbb{R}^+$, $w : [a,b] \to \mathbb{R}^+$. Then we have:*

(I^*): For any $\alpha > 0, \beta > 0; n = [\alpha - 1], m = [\beta - 1]$

$$\begin{aligned} & \left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} \sum_{i=0}^{n} \left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right) E_{X^{r-1}(X-E(X)),\beta,w} \\ & + \left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)} \sum_{i=0}^{m} \left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right) E_{X^{r-1}(X-E(X)),\alpha,w} \\ & - E_{X,\alpha,w} M_{X^{r-1},\beta,w} - E_{X,\beta,w} M_{X^{r-1},\alpha,w} \\ \leq \quad & \|f\|_{\infty}^2 \left[J_a^{\alpha}\left[w(b)\right] J_a^{\beta}\left[b^r w(b)\right] + J_a^{\beta}\left[w(b)\right] J_a^{\alpha}\left[b^r w(b)\right] \right. \\ & \left. - J_a^{\alpha}\left[bw(b)\right] J_a^{\beta}\left[b^{r-1}w(b)\right] - J_a^{\beta}\left[bw(b)\right] J_a^{\alpha}\left[b^{r-1}w(b)\right]\right] \end{aligned} \tag{3.30}$$

where $f \in L_{\infty}[a,b]$.

(II^*): *The inequality*

$$\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right) E_{X^{r-1}(X-E(X)),\beta,w}$$
$$+\left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right) E_{X^{r-1}(X-E(X)),\alpha,w}$$
$$-E_{X,\alpha,w}M_{X^{r-1},\beta,w}-E_{X,\beta,w}M_{X^{r-1},\alpha,w}$$
$$\leq \frac{(b-a)}{2}\left(b^{r-1}-a^{r-1}\right)\left(\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)}\sum_{i=0}^{n}\left[\left[(-1)^i C_n^i b^{n-i} M_{X^i,\alpha-n,w}\right]\right) \tag{3.31}$$
$$\times\left(\frac{\Gamma(\beta-m)}{\Gamma(\beta)}\sum_{i=0}^{m}\left[\left[(-1)^i C_m^i b^{m-i} M_{X^i,\beta-m,w}\right]\right)$$

holds for all $\alpha>0,\ \beta>0; n=[\alpha-1], m=[\beta-1]$.

Proof. In [4], it has been proved that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}p(x)p(y)(g(x)-g(y))(h(x)-h(y))$$
$$= J_a^\alpha[p(b)]\,J_a^\alpha[pgh(b)]+J_a^\beta[p(b)]\,J_a^\beta[pgh(b)] \tag{3.32}$$
$$-(J_a^\alpha[pg(b)]\,J_a^\alpha[ph(b)])-(J_a^\beta[pg(b)]\,J_a^\beta[ph(b)])$$

In (3.32), we take $p=wf$, $g(b)=b-E(X)$, $h(b)=b^{r-1}$. We obtain

$$\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1})(x-y)\left(x^{r-1}-y^{r-1}\right)w(x)w(y)f(x)f(y)\,dxdy$$
$$= J_a^\alpha[wf(b)]\,E_{X^{r-1}(X-E(X)),\beta,w}+J_a^\beta[wf(b)]\,E_{X^{r-1}(X-E(X)),\alpha,w} \tag{3.33}$$
$$-E_{X,\alpha,w}M_{X^{r-1},\beta,w}-E_{X,\beta,w}M_{X^{r-1},\alpha,w}.$$

On the other hand, it is clear that

$$\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}(x-y)\left(x^{r-1}-y^{r-1}\right)w(x)w(y)f(x)f(y)dxdy$$
$$\leq \|f\|_\infty^2\left[J_a^\alpha[w(b)]\,J_a^\beta[b^r w(b)]+J_a^\beta[w(b)]\,J_a^\alpha[b^r w(b)]\right. \tag{3.34}$$
$$\left.-J_a^\alpha[bw(b)]\,J_a^\beta\left[b^{r-1}w(b)\right]-J_a^\beta[bw(b)]\,J_a^\alpha\left[b^{r-1}w(b)\right]\right].$$

Consequently, by (3.33), (3.34) and (3.5), we deduce (3.30).

To prove the second part, we observe that

$$\frac{1}{\Gamma(\alpha)}\frac{1}{\Gamma(\beta)}\int_a^b\int_a^b (b-x)^{\alpha-1}(b-y)^{\beta-1}w(x)w(y)(x-y)\left(x^{r-1}-y^{r-1}\right)f(x)f(y)dxdy$$
$$= (b-a)\left(b^{r-1}-a^{r-1}\right)J_a^\alpha[wf(b)]\,J_a^\beta[wf(b)]. \tag{3.35}$$

Then, we take into account (3.33) and (3.35). We obtain (3.31). □

Remark 3.7. *Taking* $\alpha=\beta$ *in the above theorem, we obtain Theorem 5.*

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