International Journal of Analysis and Applications
ISSN 2291-8639
Volume 15, Number 1 (2017), 46-56
http://www.etamaths.com

SOME CHARACTERIZATIONS OF GENERAL PREINVEX FUNCTIONS

MUHAMMAD UZAIR AWAN[1], MUHAMMAD ASLAM NOOR[2,3], VISHNU NARAYAN MISHRA[4,5,*] AND KHALIDA INAYAT NOOR[3]

Abstract. In this paper, we consider a new class of general preinvex functions involving an arbitrary function. We show that the optimality condition for general preinvex functions on general invex set can be characterized by a class of variational-like inequality. We also derive some integral inequalities of Hermite-Hadamard type via general preinvex functions. Some special cases are also discussed. Our results represent a significant refinement of the previously known results. These results may stimulate further research in this area.

1. Introduction

In recent years, several extensions and generalizations have been introduced and considered for classical convexity using novel and innovative techniques, see [1, 13]. A significant generalization of convex functions was that of invex functions which was introduced by Hanson [7]. Ben-Israel and Mond [8] introduced another class of convex functions, which is called preinvex functions. We remark that the differentiable preinvex functions are invex functions, but the converse may be true. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. Many researchers have investigated different properties of the preinvex functions and their role in different fields of sciences such as optimization, variational inequalities, equilibrium problems and integral inequalities, see [14, 15, 17, 18, 22–24]. Another significant generalization of classical convex sets and functions was the introduction of general nonconvex (φ-convex) sets and general nonconvex (φ-convex) functions with respect to an arbitrary function, respectively by Youness [25]. These general convex set may not be a classical convex set, see [5]. Noor [16] has investigated the applications of general nonconvex functions in variational inequalities and optimization theory.

It is obvious that preinvex functions and general convex functions are distinctly two different classes of convex functions. This motivated Fulga et al. [6], to consider another class of convex functions by combining these two classes. This new class of convex function is called the general preinvex function. In this paper, we discuss some properties of the general preinvex functions. We show that the optimization of the differentiable general preinvex functions can be characterized by a class of variational-like inequality, which is called general variational-like inequality. In the last section, we derive some Hermite-Hadamard type inequalities via general preinvex functions. This may be starting point for some new research in this field.

2. Preliminaries

In this section, we define some basic results and also discuss several special cases. Before proceeding further, we suppose $K_{\eta\varphi}$ be a nonempty closed set in a Hilbert space H. We denote $\langle .,. \rangle$ by norm and $\|.\|$ by inner product, respectively. Also suppose that $\eta(.,.) : K_{\eta\varphi} \times K_{\eta\varphi} \to H$ and $\varphi : H \to H$ be arbitrary functions.

Definition 2.1 ([25]). *A set $K_\varphi \subseteq \mathbb{R}^n$ is said to be a general convex (φ-convex) set, if and only if, there exists an arbitrary function φ such that,*

$$(1-t)\varphi(u) + t\varphi(v) \in K_\varphi, \quad \forall u, v \in H : \varphi(u), \varphi(v) \in K_\varphi, t \in [0,1].$$

Received 8th May, 2017; accepted 14th July, 2017; published 1st September, 2017.

2010 *Mathematics Subject Classification.* 26A51, 26D15, 49J40, 47H10, 90C33.

Key words and phrases. convex functions; general preinvex functions; differentiability; Hermite-Hadamard inequality.

Definition 2.2 ([6]). *A set $K_{\eta\varphi}$ is said to be general invex set with respect to $\eta(.,.)$ and φ, if*

$$\varphi(u)+t\eta(\varphi(v),\varphi(u))\in K_{\eta\varphi},\quad \forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1]. \tag{2.1}$$

It is known that general invex set may not be a general convex set see [6].
Note that, if $\eta(\varphi(v),\varphi(u))=\varphi(v)-\varphi(u)$, then our definition reduces to the definition of general convex set, which is mainly due to Youness [25]. If along with $\eta(\varphi(v),\varphi(u))=\varphi(v)-\varphi(u)$, we have $\varphi=I$, where I is identity function, then we have the definition of classical convex set.

Definition 2.3 ([6]). *A function F on $K_{\eta\varphi}$ is said to be general preinvex with respect to arbitrary functions η and φ, if*

$$\begin{aligned}F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))&\le(1-t)F(\varphi(u))+tF(\varphi(v)),\\ &\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1].\end{aligned} \tag{2.2}$$

If $\eta(\varphi(v),\varphi(u))=\varphi(v)-\varphi(u)$, then our definition reduces to the definition of general convex function [25]. If $\varphi=I$, where I is identity function, then we have the definition of preinvex functions [24].

Definition 2.4. *A function F is said to be general mid-preinvex with respect to arbitrary functions η and φ, if*

$$F\left(\frac{2\varphi(u)+\eta(\varphi(v),\varphi(u))}{2}\right)\le\frac{F(\varphi(u))+F(\varphi(v))}{2},\quad \forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi}.$$

Definition 2.5. *A function F on $K_{\eta\varphi}$ is said to be general semistrictly preinvex with respect to arbitrary functions η and φ, if and only if $\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi}$, $F(\varphi(u))\neq F(\varphi(v))$, $t\in(0,1)$, we have*

$$\begin{aligned}F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))&\le(1-t)F(\varphi(u))+tF(\varphi(v)),\\ &\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1].\end{aligned} \tag{2.3}$$

Definition 2.6. *A function F on $K_{\eta\varphi}$ is said to be general strictly preinvex with respect to arbitrary functions η and φ, if and only if $\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi}$, $\varphi(u)\neq\varphi(v)$, $t\in(0,1)$, we have*

$$\begin{aligned}F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))&<(1-t)F(\varphi(u))+tF(\varphi(v)),\\ &\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1].\end{aligned} \tag{2.4}$$

Definition 2.7 ([6]). *A function F on general invex set $K_{\eta\varphi}$ is said to be quasi general preinvex function, if*

$$\begin{aligned}F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))&\le\max\{F(\varphi(u)),F(\varphi(v))\},\\ &\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1].\end{aligned}$$

Definition 2.8. *A function F on general invex set $K_{\eta\varphi}$ is said to be general logarithmic preinvex function, if*

$$\begin{aligned}F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))&\le(F(\varphi(u)))^{1-t}(F(\varphi(v)))^{t},\\ &\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1],\end{aligned}$$

where $F(.)>0$.

From above definition, we have

$$\begin{aligned}F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))&\le(F(\varphi(u)))^{1-t}(F(\varphi(v)))^{t}\\ &\le(1-t)F(\varphi(u))+tf(\varphi(v))\\ &\le\max\{F(\varphi(u)),F(\varphi(v))\}.\end{aligned}$$

Following conditions are useful in studying various properties of our proposed results.

Condition C. Let $\eta(.,.):K_{\eta\varphi}\times K_{\eta\varphi}\to H$ satisfies the following assumptions

$$\begin{aligned}&\eta(\varphi(u),\varphi(u)+t\eta(\varphi(v),\varphi(u)))=-t\eta(\varphi(v),\varphi(u)),\\ &\eta(\varphi(v),\varphi(u)+t\eta(\varphi(v),\varphi(u)))=(1-t)\eta(\varphi(v),\varphi(u)),\\ &\qquad\qquad\forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi},t\in[0,1].\end{aligned}$$

For more information, see [9]. For $t = 1$ in Definition 2.3, we have following condition.
Condition A. Let F be general preinvex function, then
$F(\varphi(u) + \eta(\varphi(v), \varphi(u))) \leq F(\varphi(v))$.

Let $K_{\eta\varphi} = I_{\eta\varphi} = [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))]$ be the interval. We now define general preinvex functions on I.

Definition 2.9. *Let $I_{\eta\varphi} = [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))]$. Then F is a general preinvex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ \varphi(a) & \varphi(x) & \varphi(a) + \eta(\varphi(b), \varphi(a)) \\ F(\varphi(a)) & F(\varphi(x)) & F(\varphi(a) + \eta(\varphi(b), \varphi(a))) \end{vmatrix} \geq 0; \quad \varphi(a) \leq \varphi(x) \leq \varphi(a) + \eta(\varphi(b), \varphi(a)).$$

One can easily show that the following are equivalent:

(1) F is general preinvex function on general invex set.
(2) $F(\varphi(x)) \leq F(\varphi(a)) + \frac{F(\varphi(b)) - F(\varphi(a))}{\eta(\varphi(b), \varphi(a))}(\varphi(x) - \varphi(a))$.
(3) $\frac{F(\varphi(x)) - F(\varphi(a))}{\varphi(x) - \varphi(a)} \leq \frac{F(\varphi(b)) - F(\varphi(a))}{\eta(\varphi(b), \varphi(a))} \leq \frac{F(\varphi(b)) - F(\varphi(x))}{\varphi(a) + \eta(\varphi(b), \varphi(a)) - \varphi(x)}$.
(4) $[\varphi(x) - (\varphi(a) + \eta(\varphi(b), \varphi(a)))]F(\varphi(a)) + \eta(\varphi(b), \varphi(a))F(\varphi(x)) + [\varphi(a) - \varphi(x)]F(\varphi(b)) \geq 0$.
(5) $\frac{F(\varphi(a))}{\eta(\varphi(b), \varphi(a))(\varphi(a) - \varphi(x))} + \frac{F(\varphi(x))}{[\varphi(x) - (\varphi(a) + \eta(\varphi(b), \varphi(a)))][\varphi(a) - \varphi(x)]} + \frac{F(\varphi(b))}{\eta(\varphi(b), \varphi(a))[\varphi(x) - (\varphi(a) + \eta(\varphi(b), \varphi(a)))]} \geq 0$,

where $\varphi(x) = \varphi(a) + t\eta(\varphi(b), \varphi(a))$ and $t \in [0, 1]$.

Remark 2.1. *Note that for $\varphi = I$, where I is the identity function, the above definition reduces to the definition for preinvex functions on an interval.*

Definition 2.10. *Let $I_\eta = [a, a + \eta(b, a)]$. Then F is called preinvex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ a & x & a + \eta(b, a) \\ F(a) & F(x) & F(a + \eta(b, a)) \end{vmatrix} \geq 0; \quad a \leq x \leq a + \eta(b, a).$$

One can easily show that the following are equivalent:

(1) F is preinvex function on invex set.
(2) $F(x) \leq F(a) + \frac{F(b) - F(a)}{\eta(b, a)}(x - a)$.
(3) $\frac{F(x) - F(a)}{x - a} \leq \frac{F(b) - F(a)}{\eta(b, a)} \leq \frac{F(b) - F(x)}{a + \eta(b, a) - x}$.
(4) $[x - (a + \eta(b, a))]F(a) + \eta(b, a)F(x) + (a - x)F(b) \geq 0$.
(5) $\frac{F(a)}{\eta(b, a)(a - x)} + \frac{F(x)}{[x - (a + \eta(b, a))][a - x]} + \frac{F(b)}{\eta(b, a)[x - (a + \eta(b, a))]} \geq 0$,

where $x = a + t\eta(b, a)$ and $t \in [0, 1]$.

Remark 2.2. *If in Definition 2.8, $\eta(\varphi(b), \varphi(a)) = \varphi(b) - \varphi(a)$. Then, we have the definition of general convex functions on interval.*

Definition 2.11. *Let $I_\varphi = [\varphi(a), \varphi(b)]$. Then F is called general convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ \varphi(a) & \varphi(x) & \varphi(b) \\ F(\varphi(a)) & F(\varphi(x)) & F(\varphi(b)) \end{vmatrix} \geq 0; \quad \varphi(a) \leq \varphi(x) \leq \varphi(b).$$

One can easily show that the following are equivalent:

(1) F is general convex function on general convex set.
(2) $F(\varphi(x)) \leq F(\varphi(a)) + \frac{F(\varphi(b)) - F(\varphi(a))}{\varphi(b) - \varphi(a)}(\varphi(x) - \varphi(a))$.
(3) $\frac{F(\varphi(x)) - F(\varphi(a))}{\varphi(x) - \varphi(a)} \leq \frac{F(\varphi(b)) - F(\varphi(a))}{\varphi(b) - \varphi(a)} \leq \frac{F(\varphi(b)) - F(\varphi(x))}{\varphi(b) - \varphi(x)}$.
(4) $(\varphi(x) - \varphi(b))F(\varphi(a)) + (\varphi(b) - \varphi(a))F(\varphi(x)) + (\varphi(a) - \varphi(x))F(\varphi(b)) \geq 0$.
(5) $\frac{F(\varphi(a))}{(\varphi(b) - \varphi(a))(\varphi(a) - \varphi(x))} + \frac{F(\varphi(x))}{(\varphi(x) - \varphi(b))(\varphi(a) - \varphi(x))} + \frac{F(\varphi(b))}{(\varphi(b) - \varphi(a))(\varphi(x) - \varphi(b))} \geq 0$,

where $\varphi(x) = \varphi(a) + t(\varphi(b) - \varphi(a))$ and $t \in [0, 1]$.

Definition 2.12. *A differentiable function F on general invex set $K_{\eta\varphi}$ is said to be general invex function, if there exists arbitrary functions η and φ, such that*

$$F(\varphi(v)) - F(\varphi(u)) \geq \langle F'(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle, \quad u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi},$$

where F' is the differential of F. If $\varphi = I$, our definition of general invexity reduces to the definition of invexity which is mainly due to Hanson [5].

Definition 2.13. *A function F is said to be pseudo general preinvex with respect to η, if there exists a strictly positive bifunction b such that*

$$F(\varphi(v)) < F(\varphi(u)) \Rightarrow F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) \leq F(\varphi(u)) + t(t-1)b(\varphi(v), \varphi(u)),$$

$\forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}, t \in (0,1)$.

Definition 2.14. *A function T is said to be η-monotone, if and only if*

$$\langle T(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle + \langle T(\varphi(v)), \eta(\varphi(v), \varphi(u))\rangle \leq 0, \quad \forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}.$$

3. Results and Discussions

3.1. **Variational-Like Inequalities.** In this section, we derive some general variational-like inequalities.

Theorem 3.1. *If F is a general preinvex function on $K_{\eta\varphi}$, then, the lower level set*

$$L_\alpha = \{u \in H : \varphi(u) \in K_{\eta\varphi} : F(\varphi(u)) \leq \alpha, \alpha \in \mathbb{R}\}$$

is a general invex set.

Theorem 3.2. *A function F is general preinvex function on $K_{\eta\varphi}$ if and only if,*

$$epi(F) = \{(\varphi(u), \alpha) : \varphi(u) \in K_{\eta\varphi}, \alpha \in \mathbb{R}, F(\varphi(u)) \leq \alpha\}$$

is general invex set.

Theorem 3.3. *Let F be a general preinvex function. Suppose $\mu = \inf_{\varphi(u) \in K_{\eta\varphi}} F(\varphi(u))$. Then the set $A_\eta = \{u \in H : \varphi(u) \in K_{\eta\varphi} : F(\varphi(u)) = \mu\}$ is general invex set of $K_{\eta\varphi}$. If F is general strictly preinvex, then A_η is a singelton.*

Proof. Let $\varphi(u), \varphi(v) \in A_\eta$. Then for $0 < t < 1$, we suppose $\varphi(z) = \varphi(u) + t\eta(\varphi(v), \varphi(u))$. Since F is preinvex function. Then

$$F(\varphi(z)) = F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) \leq (1-t)F(\varphi(u)) + tf(\varphi(v)) = \mu,$$

this implies that $\varphi(z) \in A_\eta$ thus A_η is general invex set. Now for other part of the theorem, we assume contrary that $F(\varphi(u)) = F(\varphi(v)) = \mu$. Since $K_{\eta\varphi}$ is general invex set, then for $0 < t < 1$, $\varphi(u) + t\eta(\varphi(v), \varphi(u)) \in K_{\eta\varphi}$. Also, since F is strictly general preinvex function.

$$F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) < (1-t)F(\varphi(u)) + tf(\varphi(v)) = \mu,$$

which is contradiction that $\mu = \inf_{\varphi(u) \in K_{\eta\varphi}} F(\varphi(u))$, this completes the proof. □

Theorem 3.4. *Let F be a general preinvex function on K_η. If ϕ is a nondecreasing convex function, then $\phi \circ F$ is a general preinvex function.*

Proof. Since F is a general preinvex function and ϕ is nondecreasing, then

$$\begin{aligned}
\phi \circ F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) &\leq \phi[F(\varphi(u) + t\eta(\varphi(v), \varphi(u)))] \\
&\leq \phi[(1-t)F(\varphi(u)) + tf(\varphi(v))] \\
&\leq (1-t)\phi \circ F(\varphi(u)) + t\phi \circ F(\varphi(v)).
\end{aligned}$$

This completes the proof. □

Theorem 3.5. *Let F be a semistrictly general preinvex function on K_η. If ϕ is a nondecreasing convex function, then $\phi \circ F$ is a semistrictly general preinvex function.*

Lemma 3.1. *Let $K_{\eta\varphi}$ be general convex set and let F be be general invex function on K. The*

(1) *If*

$$\langle F'(\varphi(u)), \varphi(v)-\varphi(u)\rangle \leq \langle F'(\varphi(u)), \eta(\varphi(v),\varphi(u))\rangle, \qquad \forall u,v \in H : \varphi(u),\varphi(v) \in K_{\eta\varphi},$$

such that $F(\varphi(v)) \leq F(\varphi(u))$*, then* F *is a general pseudo-convex function.*

(2) *If*

$$\langle F'(\varphi(u)), \varphi(v)-\varphi(u)\rangle \leq \langle F'(\varphi(u)), \eta(\varphi(v),\varphi(u))\rangle, \qquad \forall u,v \in H : \varphi(u),\varphi(v) \in K_{\eta\varphi},$$

such that $F(\varphi(v)) < F(\varphi(u))$*, then* F *is strictly general pseudo-convex function.*

Proof. Let $u,v \in H : \varphi(u),\varphi(v) \in K_{\eta\varphi}$ and $F(\varphi(v)) \leq F(\varphi(u))$. Then

$$\begin{aligned}\langle F'(\varphi(u)), \varphi(v)-\varphi(u)\rangle &= \langle F'(\varphi(u)), \varphi(v)-\varphi(u)-\eta(\varphi(v),\varphi(u))\rangle + \langle F'(\varphi(u)), \eta(\varphi(v),\varphi(u))\rangle \\ &\leq \langle F'(\varphi(u)), \varphi(v)-\varphi(u)-\eta(\varphi(v),\varphi(u))\rangle + F(\varphi(v)) - F(\varphi(u)) \\ &\leq \langle F'(\varphi(u)), \varphi(v)-\varphi(u)-\eta(\varphi(v),\varphi(u))\rangle \leq 0.\end{aligned}$$

This completes the proof. The proof of second part is on similar lines. □

Theorem 3.6. *If* F *be a general preinvex function. Then any local minimum of* F *is a global minimum.*

Proof. Let F has a local minimum at $\varphi(u) \in K_{\eta\varphi}$. Assume contrary, that $F(\varphi(v)) < F(\varphi(u))$ for some $\varphi(v) \in K$. Now since F is general preinvex function. Then

$$F(\varphi(u) + t\eta(\varphi(v),\varphi(u))) \leq (1-t)F(\varphi(u)) + tf(\varphi(v)).$$

This implies that

$$F(\varphi(u) + t\eta(\varphi(v),\varphi(u))) - F(\varphi(u)) \leq t(F(\varphi(v)) - F(\varphi(u))) < 0.$$

Thus, we have

$$F(\varphi(u) + t\eta(\varphi(v),\varphi(u))) < F(\varphi(u)),$$

a contradiction. This completes the proof. □

Theorem 3.7. *If* F *be a semistrictly general preinvex function. Then any local minimum of* F *is a global minimum.*

Proof. The proof is similar to previous. □

Theorem 3.8. *Let* F *be general preinvex function with respect to* η*,* $i = 1, 2, \ldots, n$*. Then* $\sum\limits_{i=0}^{n} \mu_i F_i(\varphi(u))$ *is general preinvex with respect to* η*, where* $\mu_i \geq 0$*.*

Proof. Let F_i be general preinvex functions. Then

$$F_i(\varphi(u) + t\eta(\varphi(v),\varphi(u))) \leq (1-t)F(\varphi(u)) + tf(\varphi(v)).$$

Now

$$\begin{aligned}(1-t)\sum_{i=0}^{n} \mu_i F_i(\varphi(u)) + t\sum_{i=0}^{n} \mu_i F_i(\varphi(v)) &= \sum_{i=0}^{n} \mu_i[(1-t)F_i(\varphi(u)) + tf_i(\varphi(v))] \\ &\geq \sum_{i=0}^{n} \mu_i F_i(\varphi(u) + t\eta(\varphi(v),\varphi(u))).\end{aligned}$$

This implies that $\sum\limits_{i=0}^{n} \mu_i F_i(\varphi(u))$ is general preinvex function. □

Theorem 3.9. *If* F *is general preinvex function with respect to* η *such that* $F(\varphi(v)) < F(\varphi(u))$*, then* F *is general pseudo preinvex function with respect to same* η*.*

Proof. Since $F(\varphi(v)) < F(\varphi(u))$ and F is general preinvex function with respect to η, then for all $u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}$ and $t \in (0,1)$, we have

$$\begin{aligned} F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) &\leq F(\varphi(u)) + t(F(\varphi(v)) - F(\varphi(u))) \\ &< F(\varphi(u)) + t(1-t)(F(\varphi(v)) - F(\varphi(u))) \\ &= F(\varphi(u)) + t(t-1)(F(\varphi(u)) - F(\varphi(v))) \\ &= F(\varphi(u)) + t(t-1)b(\varphi(u), \varphi(v)), \end{aligned}$$

where $b(\varphi(u), \varphi(v)) = F(\varphi(u)) - F(\varphi(v)) > 0$. This completes the proof. □

Theorem 3.10. *Let F be a differentiable general preinvex function on $K_{\eta\varphi}$. Then $\varphi(u) \in K_{\eta\varphi}$ is the minimum of F on $K_{\eta\varphi}$ if and only if $\varphi(u) \in K_{eta}$ satisfies the inequality*

$$\langle F'(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle \geq 0, \quad \forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}, \tag{3.1}$$

where F' is the differential of F at $\varphi(u) \in K_{\eta\varphi}$.

The inequality (3.1) is called the general variational-like inequality.

Proof. Let $\varphi(u) \in K_{\eta\varphi}$ be a minimum of general preinvex function F on $K_{\eta\varphi}$. Then by definition of minimum, we have,

$$F(\varphi(u)) \leq F(\varphi(v)), \quad \forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}. \tag{3.2}$$

Since $K_{\eta\varphi}$ is a general invex set, so $\forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}, t \in [0,1]$, we have

$$\varphi(v_t) \equiv \varphi(u) + t\eta(\varphi(v), \varphi(u)) \in K_{\eta\varphi}. \tag{3.3}$$

Replacing $\varphi(v)$ by $\varphi(v_t)$ in (3.2) we get

$$F(\varphi(u)) \leq F(\varphi(v_t)) = F(\varphi(u) + t\eta(\varphi(v), \varphi(u))),$$

which implies that

$$F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) - F(\varphi(u)) \geq 0.$$

Since F is differentiable, so dividing both sides of the above inequality by t and then taking the limit as $t \to 0$, we have

$$0 \leq \lim_{t \to 0} \left(\frac{F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) - F(\varphi(u))}{t} \right) = \langle F'(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle,$$

that is $\varphi(u) \in K_{\eta\varphi}$ satisfies the inequality

$$\langle F'(u), \eta(\varphi(v), \varphi(u))\rangle \geq 0, \quad \forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}.$$

Conversely, let inequality (3.1) holds. We have to show that $\varphi(u) \in K_{\eta\varphi}$, is the minimum of F on the general invex set $K_{\eta\varphi}$. Since F is general preinvex function, then

$$F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) \leq F(\varphi(u)) + t(F(\varphi(v)) - F(\varphi(v))).$$

Now taking limit as $t \to 0$, we have

$$\begin{aligned} F(\varphi(v)) - F(\varphi(u)) &\geq \lim_{t \to 0} \frac{F(\varphi(u) + t\eta(\varphi(v), \varphi(u))) - F(\varphi(u))}{t} \\ &= \langle F'(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle \\ &\geq 0, \end{aligned}$$

thus, it follows that

$$F(\varphi(u)) \leq F(\varphi(v)), \quad \forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi},$$

which completes the proof. □

Theorem 3.11. *Let F be a differentiable function on general invex set $K_{\eta\varphi}$ and suppose condition C holds. Then F is general preinvex function if and only if F is a general invex function.*

Proof. Let F be a differentiable general preinvex function, then

$$F(\varphi(u)+t\eta(\varphi(v),\varphi(u))) \leq F(\varphi(u))+t(F(\varphi(v))-F(\varphi(u))), \quad \forall u,v \in H:\varphi(u),\varphi(v)\in K_{\eta\varphi}, t\in[0,1],$$

since F is differentiable taking limit as $t\to 0$, we have

$$F(\varphi(v))-F(\varphi(u)) \geq \lim_{t\to 0}\frac{F(\varphi(u)+t\eta(\varphi(v),\varphi(u)))-F(\varphi(u))}{t} = \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle.$$

This implies that F is general invex function.
Conversely, suppose that F is general invex function, that is

$$F(\varphi(v))-F(\varphi(u)) \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.4}$$

Since $K_{\eta\varphi}$ is a general invex set. Then $\forall\varphi(u),\varphi(v)\in K_{\eta\varphi}$, $t\in[0,1]$, we have $\varphi(v_t)\equiv\varphi(u)+t\eta(\varphi(v),\varphi(u))\in K_{\eta\varphi}$. Replacing $\varphi(u)$ by $\varphi(v_t)$ in (3.4) and using condition C, we have

$$F(\varphi(v))-F(\varphi(u)+t\eta(\varphi(v),\varphi(u))) \geq (1-t)\langle F'(\varphi(u)+t\eta(\varphi(v),\varphi(u))),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.5}$$

Similarly, we have

$$F(\varphi(u))-F(\varphi(u)+t\eta(\varphi(v),\varphi(u))) \geq -t\langle F'(\varphi(u)+t\eta(\varphi(v),\varphi(u))),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.6}$$

Multiplying (3.5) by t and (3.6) by $(1-t)$, and then adding the resultant, we have

$$F(\varphi(u)+t\eta(\varphi(v),\varphi(u))) \leq (1-t)F(\varphi(u))+tf(\varphi(v)). \tag{3.7}$$

This completes the proof. □

Theorem 3.12. *Let F be a differentiable function on general invex set $K_{\eta\varphi}$ and suppose Condition A holds. Then the differential F' of F is η-monotone if and only if F is a general invex function.*

Proof. Let F be general invex function. Then

$$F(\varphi(v))-F(\varphi(u)) \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.8}$$

Interchanging $\varphi(u)$ and $\varphi(v)$ in above inequality, we have

$$F(\varphi(u))-F(\varphi(v)) \geq \langle F'(\varphi(v)),\eta(\varphi(u),\varphi(v))\rangle. \tag{3.9}$$

Adding (3.8) and (3.9), we have

$$\langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle+\langle F'(\varphi(v)),\eta(\varphi(u),\varphi(v))\rangle \leq 0. \tag{3.10}$$

This implies that F' is η-monotone.
Conversely, suppose that F' is η-monotone, that is F' satisfies inequality (3.10). Then, from (3.10), we have

$$\langle F'(\varphi(v)),\eta(\varphi(u),\varphi(v))\rangle \leq -\langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.11}$$

Now, since $K_{\eta\varphi}$ is general invex set, then, $\forall\varphi(u),\varphi(v)\in K_{\eta\varphi}$, $t\in[0,1]$, we have $\varphi(v_t)\equiv\varphi(u)+t\eta(\varphi(v),\varphi(u))\in K_{\eta\varphi}$. Taking $\varphi(v)$ as $\varphi(v_t)$ in (3.11), and applying condition C, we get

$$\langle F'(\varphi(u)+t\eta(\varphi(v),\varphi(u))),\eta(\varphi(v),\varphi(u))\rangle \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.12}$$

Consider an auxiliary function

$$\varphi(t)=F(\varphi(v_t))\equiv F(\varphi(u)+t\eta(\varphi(v),\varphi(u))), \quad \forall u,v\in H:\varphi(u),\varphi(v)\in K_{\eta\varphi}, t\in[0,1]. \tag{3.13}$$

Now using the fact that F is differentiable, we have

$$\varphi'(t) \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle.$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\varphi(1)-\varphi(0) \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle. \tag{3.14}$$

Using (3.13) and (3.14), we have

$$F(\varphi(u)+\eta(\varphi(v),\varphi(u)))-F(u) \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle.$$

Now using Condition A, we have

$$F(\varphi(v))-F(\varphi(u)) \geq \langle F'(\varphi(u)),\eta(\varphi(v),\varphi(u))\rangle,$$

which shows that F is general invex function. This completes the proof. □

Theorem 3.13. *Let $K_{\eta\varphi}$ be a general invex set in H. Suppose function F be η-pseudomonotone and η-hemicontinuous. If condition C holds, then $\varphi(u) \in K_{\eta\varphi}$ satisfies*

$$\langle F(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle \geq 0, \quad \forall v \in H : \varphi(v) \in K_{\eta\varphi}, \tag{3.15}$$

if and only if $\varphi(u) \in K_{\eta\varphi}$ satisfies

$$\langle F(\varphi(v)), \eta(\varphi(u), \varphi(v))\rangle \leq 0, \quad \forall v \in H : \varphi(v) \in K_{\eta\varphi}. \tag{3.16}$$

Proof. Let $u \in H : \varphi(u) \in K_{\eta\varphi}$ satisfies the following inequality

$$\langle F(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle \geq 0, \quad \forall v \in H : \varphi(v) \in K_{\eta\varphi},$$

which implies that

$$\langle F(\varphi(v)), \eta(\varphi(u), \varphi(v))\rangle \leq 0, \quad \forall v \in H : \varphi(v) \in K_{\eta\varphi},$$

where F is η-pseudomonotone.
Conversely, let (3.16) holds. Since $K_{\eta\varphi}$ is general invex set, then $\forall u, v \in H : \varphi(u), \varphi(v) \in K_{\eta\varphi}, t \in [0,1]$, $\varphi(v_t) \equiv \varphi(u) + t\eta(\varphi(v), \varphi(u)) \in K_{\eta\varphi}$. Taking $\varphi(v) = \varphi(v_t)$ in (3.16) and using condition C, we have

$$\begin{aligned} 0 &\geq \langle F(\varphi(v_t)), \eta(\varphi(u), \varphi(u) + t\eta(\varphi(v), \varphi(u)))\rangle \\ &= -t\langle F(\varphi(v_t)), \eta(\varphi(v), \varphi(u))\rangle, \end{aligned}$$

from which we have

$$\langle F(\varphi(v_t)), \eta(\varphi(v), \varphi(u))\rangle \geq 0, \quad \forall v \in H : \varphi(v) \in K_{\eta\varphi}.$$

Taking limit as $t \to 0$ on both sides of above inequality, we have

$$\langle F(\varphi(u)), \eta(\varphi(v), \varphi(u))\rangle \geq 0, \quad \forall v \in H : \varphi(v) \in K_{\eta\varphi}, \tag{3.17}$$

where we have used the fact that F is η-hemicontinuous. This completes the proof. □

3.2. **Hermite-Hadamard type Inequalities.** Hermite-Hadamard type inequalities provides us necessary and sufficient condition for a function to be convex. In recent years many new generalizations of these inequalities have been obtained via different classes of convex functions. For more information, see [2, 4, 19–21, 23]. In this section, we derive some Hermite-Hadamard type inequalities via general preinvex functions.

Theorem 3.14. *Let $F : I_{\eta\varphi} = [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \to \mathbb{R}$ be a general preinvex function with $\eta(\varphi(b), \varphi(a)) > 0$. If $\eta(.,.)$ satisfies the condition C, then we have*

$$F\left(\frac{2\varphi(a) + \eta(\varphi(b), \varphi(a))}{2}\right) \leq \frac{1}{\eta(\varphi(b), \varphi(a))} \int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))\mathrm{d}\varphi(x) \leq \frac{F(\varphi(a)) + F(\varphi(b))}{2}.$$

Proof. Since F is general preinvex function and $\eta(.,.)$ satisfies the condition C, we have

$$F\left(\frac{2\varphi(a) + \eta(\varphi(b), \varphi(a))}{2}\right) \leq \frac{1}{2}\left[F\left(\varphi(a) + t\eta(\varphi(b), \varphi(a))\right) + F(\varphi(a) + (1-t)\eta(\varphi(b), \varphi(a)))\right].$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$F\left(\frac{2\varphi(a) + \eta(\varphi(b), \varphi(a))}{2}\right) \leq \frac{1}{\eta(\varphi(b), \varphi(a))} \int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))\mathrm{d}\varphi(x). \tag{3.18}$$

Also

$$F(\varphi(a) + t\eta(\varphi(b), \varphi(a))) \leq (1-t)F(\varphi(a)) + tF(\varphi(b)).$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\eta(\varphi(b), \varphi(a))} \int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))\mathrm{d}\varphi(x) \leq \frac{F(\varphi(a)) + F(\varphi(b))}{2}. \tag{3.19}$$

Combining (3.18) and (3.19) completes the proof. □

Theorem 3.15. *Let $F, W : I_{\eta\varphi} = [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \to \mathbb{R}$ be general preinvex functions respectively with $\eta(\varphi(b), \varphi(a)) > 0$. Suppose $\eta(.,.)$ satisfies Condition C, then we have*

$$\begin{aligned} &F\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) W\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \\ &\quad - \frac{1}{2\eta(\varphi(b),\varphi(a))} \int\limits_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))W(\varphi(x))\mathrm{d}\varphi(x) \\ &\leq \frac{1}{6}M(\varphi(a),\varphi(b)) + \frac{1}{3}N(\varphi(a),\varphi(b)), \end{aligned}$$

where

$$M(\varphi(a),\varphi(b)) = F(\varphi(a))W(\varphi(a)) + F(\varphi(b))W(\varphi(b)) \tag{3.20}$$

and

$$N(\varphi(a),\varphi(b)) = F(\varphi(a))W(\varphi(b)) + F(\varphi(b))W(\varphi(a)). \tag{3.21}$$

Proof. Since F and W are general preinvex functions respectively and $\eta(.,.)$ satisfies Condition C, we have

$$\begin{aligned} &F\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) W\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \\ &= F(\varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)) + \frac{1}{2}\eta(\varphi(a)+t\eta(\varphi(b),\varphi(a)), \varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)))) \\ &\quad \times W(\varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)) + \frac{1}{2}\eta(\varphi(a)+t\eta(\varphi(b),\varphi(a)), \varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)))) \\ &\leq \frac{1}{4}[F(\varphi(a)+t\eta(\varphi(b),\varphi(a))) + F(\varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)))] \\ &\qquad \times [W(\varphi(a)+t\eta(\varphi(b),\varphi(a))) + W(\varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)))] \\ &\leq \frac{1}{4}[F(\varphi(a)+t\eta(\varphi(b),\varphi(a)))W(\varphi(a)+t\eta(\varphi(b),\varphi(a))) \\ &\qquad + F(\varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)))W(\varphi(a)+(1-t)\eta(\varphi(b),\varphi(a)))] \\ &\qquad + \frac{1}{2}\left\{2(t-t^2)M(a,b) + [t^2+(1-t)^2]N(a,b)\right\}. \end{aligned}$$

Integrating above inequality with respect to t on [0,1], we have

$$\begin{aligned} &F\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) W\left(\frac{2\varphi(a)+\eta(\varphi(b),\varphi(a))}{2}\right) \\ &\quad - \frac{1}{2\eta(\varphi(b),\varphi(a))} \int\limits_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))W(\varphi(x))\mathrm{d}\varphi(x) \\ &\leq \frac{1}{6}M(\varphi(a),\varphi(b)) + \frac{1}{3}N(\varphi(a),\varphi(b)). \end{aligned}$$

The proof is complete. □

Theorem 3.16. *Let $F, W : I_{\eta\varphi} = [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \to \mathbb{R}$ be general preinvex function with $\eta(b,a) > 0$, then we have*

$$\frac{1}{\eta(\varphi(b),\varphi(a))} \int\limits_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))W(\varphi(x))\mathrm{d}\varphi(x) \leq \frac{1}{3}M(\varphi(a),\varphi(b)) + \frac{1}{6}N(\varphi(a),\varphi(b)).$$

where $M(\varphi(a),\varphi(b))$ and $N(\varphi(a),\varphi(b))$ are given by (3.20) and (3.21) respectively.

Proof. Let F, W be general preinvex functions, then for all $t \in [0,1]$, we have

$$F(\varphi(a)+t\eta(\varphi(b),\varphi(a)))W(\varphi(a)+t\eta(\varphi(b),\varphi(a)))$$
$$\leq [(1-t)F(\varphi(a))+tF(\varphi(b))][(1-t)W(\varphi(a))+tW(\varphi(b))]$$
$$= (1-t)^2F(\varphi(a))W(\varphi(a))+t(1-t)F(\varphi(b))W(\varphi(a))+t(1-t)F(\varphi(a))W(\varphi(b))+t^2F(\varphi(b))W(\varphi(b)).$$

Integrating above inequality with respect to t on $[0,1]$, we have

$$\frac{1}{\eta(\varphi(b),\varphi(a))}\int_{\varphi(a)}^{\varphi(a)+\eta(\varphi(b),\varphi(a))} F(\varphi(x))W(\varphi(x))\mathrm{d}\varphi(x) \leq \frac{1}{3}M(\varphi(a),\varphi(b))+\frac{1}{6}N(\varphi(a),\varphi(b)).$$

This completes the proof. □

4. Conclusion

We have discussed several properties of general preinvex functions. It is shown that the minimum of the differentiable general functions can be characterized by variational-like inequalities, which are called general variational-like inequalities. We have established a necessary and sufficient condition for the minimum of a differential general preinvex functions. In the last section, we have obtained some integral inequalities of Hermite-Hadamard type via general preinvex functions. We would like to mention that the field of general variational-like inequalities is a relatively new one and offer great opportunities for further research. The ideas and techniques of this paper may stimulate further research in the field of mathematical inequalities.

References

[1] G. Cristescu and L. Lupsa, Non-connected Convexities and Applications, Kluwer Academic Publishers, Dordrecht, Holland, (2002).
[2] G. Cristescu, M. A. Noor, M. U. Awan, Bounds of the second degree cumulative frontier gaps of functions with generalized convexity, Carpathian J. Math. 31(2) (2015), 173-180.
[3] Deepmala, A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur 492 010, Chhatisgarh, India (2014).
[4] S. S. Dragomir, C. E. M. Pearce, Selected topics on Hermite-Hadamard inequalities and applications, Victoria University, Australia, 2000.
[5] D. I. Duca and L. Lupsa, Saddle points for vector valued functions: existence, necessary and sufficient theorems, J. Glob. Optim., 53 (2012), 431–440.
[6] C. Fulga and V. Preda, Nonlinear programming with φ-preinvex and local φ-preinvex functions, Eur. J. Oper. Res. 192 (2009), 737–743.
[7] M. A. Hanson, On sufficiency of the Kuhn–Tucker conditions, J. Math. Anal. Appl., 80 (1981) 545–550.
[8] A. Ben-Israel and B. Mond, What is invexity? J. Aust. Math. Soc. Ser. B 28 (1986), 1–9.
[9] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl., 189 (1995), 901–908.
[10] L. N. Mishra, H. M. Srivastava, M. Sen, On existence results for some nonlinear functional-integral equations in Banach algebra with applications, Int. J. Anal. Appl., 11 (1) (2016), 1–10.
[11] L. N. Mishra, R. P. Agarwal, On existence theorems for some nonlinear functional-integral equations, Dynamic Systems and Appl., 25 (2016), 303–320.
[12] L. N. Mishra, On existence and behavior of solutions to some nonlinear integral equations with Applications, Ph.D. Thesis, National Institute of Technology, Silchar 788 010, Assam, India (2017).
[13] M. A. Noor, Differentiable nonconvex functions and general variational inequalities, Appl. Math. Comput., 199 (2008), 623–630.
[14] M. A. Noor, Fuzzy preinvex functions, Fuzzy Sets Syst. 64 (1994), 95–104.
[15] M. A. Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, J. Math. Anal. Approx. Theory 2 (2007), 126–131.
[16] M. A. Noor, New approximation schemes for general variational inequalities, J. Math. Anal. Appl., 251 (2000), 217–229.
[17] M. A. Noor, On Hermite-Hadamard integral inequalities for involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 3 (2007), 75–81.
[18] M. A. Noor, Variational like inequalities, Optimization 30 (1994), 323–330.
[19] M. A. Noor, M.U. Awan, K. I. Noor: On some inequalities for relative semi-convex functions. J. Inequal. Appl. 2013 (2013), Art. ID 332.
[20] M. A. Noor, K. I. Noor, M. U. Awan, Geometrically relative convex functions, Appl. Mathe. Infor. Sci., 8(2) (2014), 607-616.

[21] M. A. Noor, K. I. Noor, M. U. Awan, Hermite-Hadamard inequalities for relative semi-convex functions and applications, Filomat. 28 (2) (2014), 221-230.
[22] M. A. Noor, K. I. Noor, M. U. Awan, J. Li, On Hermite-Hadamard type inequalities for h-preinvex functions. Filomat. 28 (7) (2014), 1463-1474.
[23] M. A. Noor, K. I. Noor, Integral inequalities for differentiable relative preinvex functions(survey), TWMS J. Pure Appl. Math. 7(1)(2016), 3-19
[24] T. Weir and B. Mond, Preinvex functions in multiobjective optimization, J. Math. Anal. Appl., 136 (1988), 29–38.
[25] E. A. Youness, E-convex sets, E-convex functions, and E-convex programming, J. Optim. Theory Appl., 102 (1999), 439–450.

[1]Department of Mathematics, Government College University,, [0pt] Faisalabad, Pakistan.

[2]Mathematics Department, King Saud University, Riyadh, Saudi Arabia.

[3]Department of Mathematics, COMSATS Institute of Information Technology, Islamabad, Pakistan.

[4]Department of Mathematics, Indira Gandhi National Tribal University, Lalpur, Amarkantak, Anuppur, Madhya Pradesh 484 887, India

[5]L. 1627 Awadh Puri Colony Beniganj, Phase - III, Opposite - Industrial Training Institute (I.T.I.), Faizabad 224 001, Uttar Pradesh, India

*Corresponding author: vishnunarayanmishra@gmail.com